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Polynomial interpolation and quadrature on subregions of the sphere

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Abstract

Using some recent results on trigonometric interpolation and quadrature on subintervals of the period, we present a rule for numerical integration over some regions of the sphere, that include spherical caps and zones.
The rule has positive weights and is exact for all spherical polynomials of degree less or equal than n.
We also construct Weakly Admissible Meshes and approximate Fekete points for polynomial interpolation on such regions of the sphere.
All the algorithms have been implemented in Matlab.

Weakly Admissible Meshes

WAM [4]: sequence of discrete subsets $\mathcal{A}_n \subset K$, with K compact set or manifold in $\mathbb{R}^d$, with the polynomial inequality

$$\|p\|_K \leq C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n}, \quad \forall p \in \mathbb{P}_n^d(K), \; dim(\mathbb{P}_n^d(K)) = N$$

where $\mathrm{card}(\mathcal{A}_n) \geq N$ and $C(\mathcal{A}_n)$ grow polynomially with n;
when $C(\mathcal{A}_n) \equiv C$ we have an Admissible Mesh (AM)

Some properties:

- unisolvent interpolation sets with polynomially growing Lebesgue constant are WAMs with $C(\mathcal{A}_n) = \Lambda_n$
- finite unions or products of (W)AMs are (W)AMs

Approximate Fekete Points

$K \subset \mathbb{R}^d$ compact set or manifold,
$\boldsymbol{p} = \{p_j\}_{1\leq j\leq N}$ polynomial basis,
$\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_N\} \subset K$ interpolation points
$V(\boldsymbol{\xi}, \boldsymbol{p}) = [p_j(\xi_i)]$ Vandermonde matrix, $\det(V) \neq 0$

$\Pi_n f(x) = \sum_{j=1}^N f(\xi_j)\,\ell_j(x)$ determinantal Lagrange formula

$$\ell_j(x) = \frac{\det(V(\xi_1, \ldots, \xi_{j-1}, x, \xi_{j+1}, \ldots, \xi_N))}{\det(V(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_N))}, \quad \ell_j(\xi_i) = \delta_{ij}$$

Fekete points: $|\det(V(\xi_1, \ldots, \xi_N))|$ is max in K^N

$$\Longrightarrow \text{Lebesgue constant} = \Lambda_n = \max_{x\in K} \sum_{j=1}^{N} |\ell_j(x)| \leq N$$

Fekete points are analytically known only in a few cases [3]: interval, complex circle, cube (for tensor-product interpolation)

• IDEA: extract Fekete points from a discretization of K: Approximate Fekete Points!
• Fekete points extracted from a WAM have a Lebesgue constant $\Lambda_n \leq NC(\mathcal{A}_n)$ (but often much smaller!)
• It is possible to extract Approximate Fekete points with simple instruments of numerical linear algebra [1]

Subperiodic Trigonometric Formulas

$$\mathbb{T}_n([-\omega, \omega]) := span\{1, \cos(k\theta), \sin(k\theta), 1 \leq k \leq n, \theta \in [-\omega, \omega]\}$$

Trigonometric interpolation
Let $\xi_j = \cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right) \in (-1, 1)$, $j = 1, 2, \ldots, 2n+1$
zeros of T_{2n+1}, then the zeros of $T_{2n+1}(\sin(\theta/2)/\sin(\omega/2))$

$$\theta_j = 2\arcsin(\sin(\omega/2)\xi_j) \in (-\omega, \omega), \; j = 1, 2, \ldots, 2n+1$$

are unisolvent for interpolation in $\mathbb{T}_n([-\omega, \omega])$ [2]
with $\Lambda_n = \mathcal{O}(\log(n))$ [6]
(they are a trigonometric WAM)

Gaussian quadrature
Let $w : (-\omega, \omega) \to \mathbb{R}$ symmetric weight function
$\{\xi_j, \lambda_j\}_{j=1,\ldots,n+1}$ nodes and weights of an algebraic gaussian rule relatively to the symmetric weight function

$$s(x) := w(2\arcsin(\sin(\omega/2)x)) \cdot \frac{2\sin(\omega/2)}{\sqrt{1 - \sin^2(\omega/2)x^2}}, \quad x \in (-1, 1)$$

Then [5], [7]

$$\int_{-\omega}^{\omega} f(\theta)w(\theta)d\theta = \sum_{j=1}^{n+1} \lambda_j f(\theta_j), \quad f \in \mathbb{T}_n([-\omega, \omega])$$

where $\theta_j := 2\arcsin(sin(\omega/2)\xi_j) \in (-\omega, \omega)$, $j = 1, 2, \ldots, n+1$

Regions of interests $\mathfrak{R}$

$$\mathbb{P}_n(\mathbb{S}^2) := \{p(\mathbf{x}) \in \mathbb{P}_n(\mathbb{R}^3) \mid \mathbf{x} \in \mathbb{S}^2\}$$

Spherical Coordinates
$\phi \in [0, 2\pi]$
$\theta \in [0, \pi]$

$$\begin{cases} x_1 = \cos(\phi)\sin(\theta) \\ x_2 = \sin(\phi)\sin(\theta) \\ x_3 = \cos(\theta) \end{cases}$$

Geographic Rectangle
(in colatitude and longitude)

$$\mathfrak{R}_{\gamma_s,\gamma_f,\tau_s,\tau_f} = \mathfrak{R} = \{\mathbf{x} \in \mathbb{S}^2 : \tau_s \leq \theta \leq \tau_f, \gamma_s \leq \phi \leq \gamma_f\}$$

special cases: spherical caps, zones, slices

Algebraic Quadrature

THEOREM 1 *Let*

- $p \in \mathbb{P}_n(\mathfrak{R})$, $\mathfrak{R} \subset \mathbb{S}^2$
- $\{\lambda_h, \theta_h\}, \{\lambda_j, \phi_j\}$ *weights and angular nodes of subperiodic Gaussian quadrature*
  - *degree respectively* $n+1$ *and* n
  - *weight function* $w := 1$
  - *intervals* $[\tau_s, \tau_f]$ *and* $[\gamma_s, \gamma_f]$
- *points with spherical coordinates* $\xi_{h,j} := \xi(\theta_h, \phi_j)$

Then the quadrature formula:

$$\mathcal{Q}_{\mathfrak{R},n}(p) := \sum_{h=1}^{n+2}\sum_{j=1}^{n+1} \lambda_{h,j}\, p(\xi_{h,j})$$

with $(n+1) \times (n+2) = n^2 + \mathcal{O}(n)$ *nodes and where the weights are defined as*

$$\lambda_{h,j} := \lambda_h \lambda_j \sin(\theta_h)$$

has algebraic degree of precision n

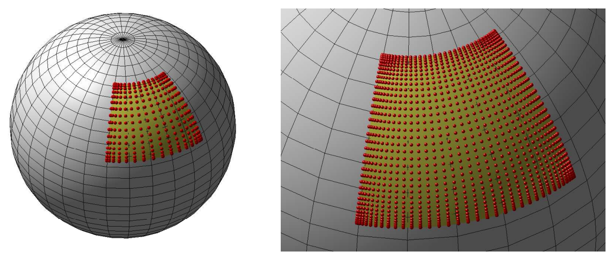

FIGURE 1. Algebraic degree of precision $N = 15$ (272 nodes) on the left and $N = 35$ (1332 nodes) on the right.

Particular Cases

In some particular cases, thanks to some symmetries of the domain, we are able to halve the number of nodes

Caps
if n odd: $(n+1) \times (n+1)/2 = n^2/2 + \mathcal{O}(n)$ nodes
if n even: $(n+1) \times (n+2)/2 - (n/2) = n^2/2 + \mathcal{O}(n)$ nodes

Zones
if n odd: $(n+1) \times (n+1)/2 = n^2/2 + \mathcal{O}(n)$ nodes
if n even: $(n+2) \times (n+2)/2 = n^2/2 + \mathcal{O}(n)$ nodes

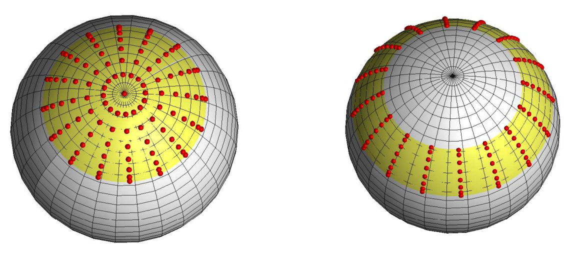

FIGURE 2. Cap and Zone with algebraic degree of precision even $N = 14$ with respectively 113 and 128 nodes.

Quadrature tests

$\mathfrak{R} = \{\mathbf{x} \in \mathbb{S}^2 : \pi/6 \leq \theta \leq \pi/3, 0 \leq \phi \leq \pi/2\}$
$f_1(\mathbf{x}) = \exp(-x^2 - 100\,y^2 - 0.5\,z^2)$,
$f_2(\mathbf{x}) = \sin(-x^2 - 100\,y^2 - 0.5\,z^2)$,
$f_3(\mathbf{x}) = \max(1/4 - ((x - 1/\sqrt{5})^2 + (y - 2/\sqrt{5})^2 + (z - 2/\sqrt{5})^2), 0))^3$

Deg.	f_1	f_2	f_3
5	$1.50e-02$	$1.57e+00$	$2.49e-02$
15	$4.10e-07$	$1.09e-01$	$2.23e-04$
35	$1.56e-15$	$5.50e-04$	$1.28e-05$
50	$9.36e-16$	$2.83e-11$	$3.01e-07$

TABLE 1. Relative errors for degrees $5, \ldots, 50$, w.r.t. some test functions.

Interpolation

First Step: Construction of a WAM

THEOREM 2 *The following grid of points*

$$\mathcal{S}_n := (\phi_j, \theta_k)_{j=1,\ldots,2n+1,\; k=1,\ldots,2n+1} =$$

$$\left\{2\arcsin\left(\sin\left(\frac{(\gamma_f - \gamma_s)/2}{2}\right)\cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right)\right) + \frac{\gamma_s + \gamma_f}{2}\right\}_j$$

$$\times \left\{2\arcsin\left(\sin\left(\frac{(\tau_f - \tau_s)/2}{2}\right)\cos\left(\frac{(2k-1)\pi}{2(2n+1)}\right)\right) + \frac{\tau_s + \tau_f}{2}\right\}_k$$

is a WAM on $\mathfrak{R}$ *with cardinality* $(2n+1)^2 = 4n^2 + \mathcal{O}(n)$ *and constant* $C(\mathcal{S}_n) = \mathcal{O}(\log^2(n))$

Second Step: Extract from the WAM the Approximate Fekete Points

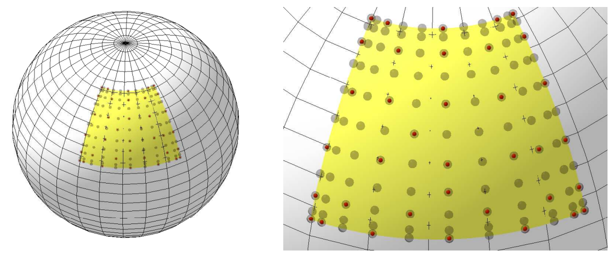

FIGURE 4. Points of a WAM of degree 5 (121 points) and the corresponding Approximate Fekete Points (36 points).

Interpolation over a Cap

As in the quadrature, also here we have some symmetries that let us halve the number of WAM's points

Caps
$card(\mathcal{S}_n)$: $(2n+1) \times (n+1) - n = 2n^2 + \mathcal{O}(n)$ points
$C(\mathcal{S}_n) = \mathcal{O}(\log^2(n))$

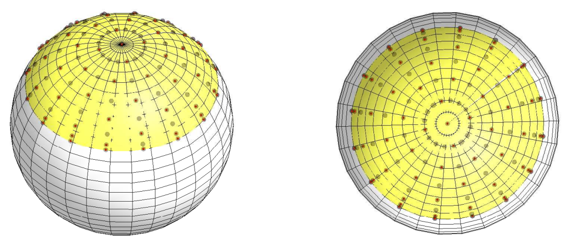

FIGURE 5. Points of a WAM of degree 7 (113 points) and the related Approximate Fekete Points (64 points).

Numerical Experiments over a Cap

$$\mathfrak{R} = \{\mathbf{x} \in \mathbb{S}^2 : 0 \leq \theta \leq \pi/3, 0 \leq \phi \leq 2\pi\}$$

Deg.	WAM Card	AFP Card	$\Lambda_n^{(AFP)}$
5	61	36	9.9
10	221	121	27.5
15	481	256	51.1
20	841	441	103.7
25	1301	676	191.3
30	1861	961	314.1

TABLE 2. WAM and AFP cardinality, and Lebesgue constants $\Lambda_n^{(AFP)}$ of extracted pointset for degrees $5, 10, \ldots, 30$.

References

[1] L. BOS, S. DE MARCHI, A. SOMMARIVA AND M. VIANELLO, *Computing multivariate Fekete and Leja points by numerical linear algebra*, SIAM J. Numer. Anal. 48 (2010), 1984–1999.

[2] L. BOS AND M. VIANELLO, *Subperiodic trigonometric interpolation and quadrature*, Appl. Math. Comput. 218 (2012), 10630–10638.

[3] M. BRIANI, A. SOMMARIVA AND M. VIANELLO, *Computing Fekete and Lebesgue points: simplex, square, disk*, J. Comput. Appl. Math. 236 (2012), 2477–2486.

[4] J.P. CALVI AND N. LEVENBERG, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory 152 (2008).

[5] G. DA FIES AND M. VIANELLO, *Trigonometric Gaussian quadrature on subintervals of the period*, Electron. Trans. Numer. Anal. 39 (2012), 102–112.

[6] G. DA FIES AND M. VIANELLO, *On the Lebesgue constant of subperiodic trigonometric interpolation*, J. Approx. Theory 167 (2013), 59–64.

[7] M. GENTILE, A. SOMMARIVA AND M. VIANELLO, *Polynomial approximation and quadrature on geographic rectangles*, in preparation.

[8] K. HESSE AND R. S. WOMERSLEY, *Numerical integration with polynomial exactness over a spherical cap*, Adv. Comput. Math. 36 (2012), 451–483.

[9] I. H. SLOAN AND R. S. WOMERSLEY, *Constructive Polynomial Approximation on the Sphere*, J. Approx. Theory 103 (2000), 91–118.