Caratheodory-Tchakaloff Least Squares

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Abstract

We discuss the Caratheodory-Tchakaloff (CATCH) subsampling method, implemented by Linear or Quadratic Programming, for the compression of multivariate discrete measures and polynomial Least Squares

Discrete Tchakaloff Theorem

Theorem: Let μ be a positive finite measure with compact support in $\mathbb{R}^d$. Denote by P^n the space of (the traces on the support of μ of) the algebraic polynomials of total degree not larger than n. There exist $m \leq N := \dim P^n$, $x_1, \ldots, x_m \in \operatorname{supp} \mu$ and $(w_1, \ldots, w_m) \in (\mathbb{R}^+)^m$ such that

$$\int p d\mu = \sum_{i=1}^{m} p(x_i) w_i, \quad \forall p \in P^n.$$

Note that μ is possibly a *finite support* measure, e.g., a quadrature rule itself. The proof relies on the Caratheodory Th. in convex geometry.

Approx. Tchakaloff Points

We can find approximate solution to $V^t c - b = 0$ (V is the Vandermonde matrix at X and b is the moment vector) up to a *moment residual* $\epsilon << 1$. Then for any datum f

$$\left| \sum_{i=1}^{M} f(x_i)\lambda_i - \sum_{i=1}^{m} f(x_i) w_i \right| \leq C_n(\epsilon) E_n(f) + \epsilon \|f\|_{\ell^2_\lambda}.$$

Here

$$C(\epsilon) := 2(\mu(X) + \sqrt{\mu(X)}), \; E_n(f) = \min_{p \in P^n} \|f - p\|_{\ell^\infty(X)}.$$

Compressed Quadrature

Assume $X := \{x_1, \ldots, x_M\}$, $M > N$ and $\mu := \sum_{i=1}^{M} \lambda_i \delta_{x_i}$ is *positive*. Then by Tchakaloff theorem we can find $m < M$ points of X and m positive weights w_i such that (up to re-ordering the sum)

$$\sum_{i=1}^{M} p(x_i)\lambda_i = \sum_{i=1}^{m} p(x_i) w_i, \quad \forall p \in P^n.$$

- Compress-ratio M/m may be large!
- Proof of theorem is not constructive.

Compression of LS

Consider the case $X = \{x_1, \ldots, x_M\}$, $M > \dim P^{2n}$ and $\boldsymbol{\lambda} = (1, 1, \ldots, 1)$. Let $x_1, \ldots x_m$, $w_1, \ldots, w_m$ be Tchakaloff points and weights of degree $2n$ extracted from X, then

$$\|p\|^2_{\ell^2} = \sum_{i=1}^{M} p(x_i)^2 = \sum_{i=1}^{m} p(x_i)^2 w_i =: \|p\|^2_{\ell^2_{\boldsymbol{w}}}, \quad \forall p \in P^n.$$

Denote by $\mathcal{L}_n$ standard LS of degree n and $\mathcal{L}_n^c$ $\boldsymbol{w}$-weighted least squares on ϵ-approximated Tchakaloff points.
Error estimates:

$$\|f - \mathcal{L}_n f\|_{\ell^2(X)} \leq \sqrt{M} E_n(f)$$

$$\|f - \mathcal{L}_n^c f\|_{\ell^2(X)} \leq \left(1 + \sqrt{\frac{1 + \epsilon\sqrt{M}}{1 - \epsilon\sqrt{M}}}\right) \sqrt{M} E_n(f).$$

Linear Programming

Let $V_{i,j} = q_j(x_i)$ be the Vandermonde matrix at X of a basis $\{q_1, \ldots q_N\}$ of P^n. Consider the linear programming problem

$$\begin{cases} \min c^t u & \text{, subject to} \\ V^t u = b, u \in (\mathbb{R}^+)^M \end{cases}. \quad \text{(LP)}$$

Here c is chosen to be linearly independent from the rows of V^t, the feasible region is a polytope and the vertex are *sparse candidate solutions*.

Standard approach $\rightarrow$ *simplex method*.

Quadratic Programming

As an alternative, we may seek for a *sparse, non-negative* solution to the moment problem by minimizing the ℓ^2 norm of the residue

$$\begin{cases} \min \|V^t u - b\|_2 & \text{, subject to} \\ u \in (\mathbb{R}^+)^M \end{cases}. \quad \text{(QP)}$$

Such a problem can be solved by the *lsqnonneg* matlab native function, which implements a variant of the *Lawson-Hanson method*.

Compression of Polynomial Meshes

Admissible polynomial meshes are sequence $\{X_n\}$ of finite subsets of a given polynomial determining compact set $K \subset \mathbb{R}^d$ satisfying

$$\|p\|_K \leq C_n \|p\|_{X_n}, \forall p \in P^n,$$

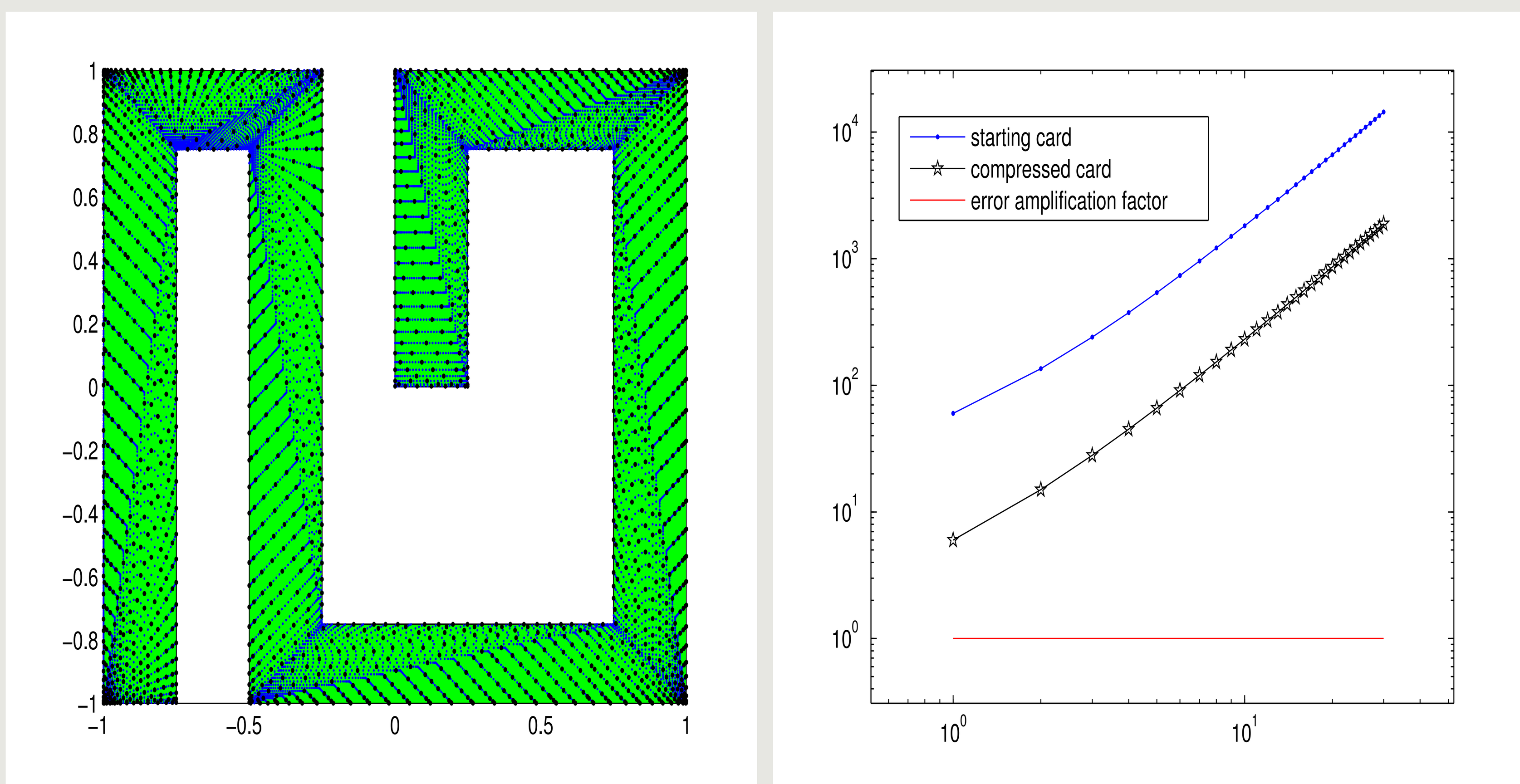
where $C_n = \mathcal{O}(n^\alpha)$ and $M_n := \operatorname{Card} X_n = \mathcal{O}(n^\beta)$. Least squares approximation of $f \in C(K)$ has the nearly optimal property

$$\|L_{X_n}\| := \sup_{f \in C(K)} \|L_{X_n} f\|_K / \|f\|_K \leq C_n \sqrt{M_n}.$$

If $M_n > \dim P^{2n}$ we can use Tchakaloff points T_{2n} extraction as a thinning of X_n and we get the norm estimate

$$\|L_{T_{2n}}\| := \sup_{f \in C(K)} \|L_{T_{2n}} f\|_K / \|f\|_K \leq C_n \sqrt{M_n} \left(1 - \epsilon\sqrt{M_n}\right)^{1/2}.$$

Experiment: Pipe-Mania Domain



Left: the 14415 blue quadrature points are the support of an algebraic quadrature rule of degree 60. The 1891 black points, extracted by quadratic programming, fits the moments up to an absolute error of $1.2215 \cdot 10^{-14}$.
Right: Comparison among the input and output cardinalities and the error amplification factor passing from standard LS to CATCH-LS.

References

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