Near-algebraic Tchakaloff-like quadrature on spherical triangles *

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Abstract

We present a numerical code for the computation of nodes and weights of a low-cardinality positive quadrature formula on spherical triangles, nearly exact for polynomials of a given degree. The algorithm is based on subperiodic trigonometric gaussian quadrature for planar elliptical sectors and on Caratheodory-Tchakaloff quadrature compression via NNLS.

1 Introduction

In this note we develop an algorithm for the computation of nodes and *positive* weights of a quadrature formula on spherical triangles, which is nearly exact for algebraic polynomials of a given degree n on the 2-sphere, and whose cardinality does not exceed the dimension of the corresponding polynomial space, namely $dim(\mathbb{P}_n^3(S^2)) = (n+1)^2$. One of our main goals is also to provide easily usable numerical codes.

Indeed, despite of the relevance of spherical triangles in the field of Geomathematics, the topic of numerical quadrature on spherical triangles, starting from a classical paper by K. Atkinson in the '80s [3], has received some attention in the literature of the last decades, with however a substantial lack of easily available numerical software (at least to our knowledge); cf. [4, 5, 6, 7] and [16, §7.2] for an overview. Some of the methods have been developed in the framework of numerical PDEs on the sphere, cf. e.g. among others [15, 17]. Notable exceptions are [19, 20], where however the algorithms are not tailored to polynomial spaces, the problem being discussed in the framework of local RBF interpolation on scattered data.

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In Section 2 we describe our algorithm (implemented in Matlab) for the construction of the quadrature formula, based on subperiodic trigonometric gaussian quadrature for planar elliptical sectors [8] and on Caratheodory-Tchakaloff quadrature compression via NNLS [18], and in Section 3 we present some numerical tests.

2 Quadrature on spherical triangles

We shall concentrate on spherical triangles $\mathcal{T} = \widehat{ABC}$ with centroid $(A + B + C)/\|A + B + C\|_2$ at the north pole (with no loss of generality, up to a suitable rotation) that are not "too large" in the sense that they are completely contained in the northern hemisphere, and do not touch the equator. Then we can write the surface integral of a continuous function $f(x, y, z)$ on such a spherical triangle in cartesian coordinates (cf. e.g. [1])

$$\int_{\mathcal{T}} f(x,y,z)\, d\sigma = \int_{\Omega} f(x,y,g(x,y))\, \frac{1}{g(x,y)}\, dx\, dy \;, \qquad (1)$$

where $g(x, y) = \sqrt{1 - x^2 - y^2}$ and Ω is the projection of $\mathcal{T}$ onto the xy-plane, that is the curvilinear triangle whose vertices, say $\hat{A}, \hat{B}, \hat{C}$, are the xy-coordinates of A, B, C; see Fig. 3.

Notice that the sides of Ω are *arcs of ellipses* centered at the origin, being the projections (and thus transformations by an affine mapping) of great circle arcs (the sides of $\mathcal{T}$). Then we can split the planar integral into the sum of the integrals on three elliptical sectors, obtained by joining the origin with the vertices $\hat{A}, \hat{B}, \hat{C}$, namely

$$\int_{\Omega} f(x,y,g(x,y))\, \frac{1}{g(x,y)}\, dx\, dy = \sum_{i=1}^{3} \int_{S_i} f(x,y,g(x,y))\, \frac{1}{g(x,y)}\, dx\, dy \;. \qquad (2)$$

Now, we seek a quadrature formula which as close as possible to an algebraic formula (at machine precision), when f is a polynomial in $\mathbb{P}_n^3$, that is $f(x, y, g)$ is a spherical polynomial. This is possible since we have at hand algebraic quadrature formulas on circular and elliptical sectors, that have been constructed in [8] by means of arc blending and *subperiodic* trigonometric gaussian quadrature [9]. We have already used such formulas, via subdivision into sectors, in quite different applications, for example algebraic quadrature on geographic rectangles [14], on the intersection and union of planar disks [22], on curvilinear elements obtained by intersection/difference of a polygonal element with a disk in VEM methods for PDEs [2].

Clearly $f(x, y, g)/g$ in general is not a polynomial in (x, y), but let us focus on the monomial basis

$$f(x,y,z) = x^{\alpha} y^{\beta} z^{\gamma} = x^{\alpha} y^{\beta} g^{\gamma} \;, \;\; 0 \leq \alpha + \beta + \gamma \leq n \qquad (3)$$

(which on $\mathcal{T}$ is not a basis but a set of generators for the spherical polynomials). We have two distinct situations:

- if γ is *odd*, then $f(x,y,g)/g$ is a polynomial in (x,y) of degree at most $n-1$, namely $f(x,y,g)/g \in \mathbb{P}^2_{\alpha+\beta+\gamma-1} \subseteq \mathbb{P}^2_{n-1}$;

- on the other hand, if γ is *even* (including $\gamma = 0$), then $g^\gamma = (g^2)^{\gamma/2}$ is a polynomial of degree γ and $f(x,y,g)/g \in \frac{1}{g}\,\mathbb{P}^2_{\alpha+\beta+\gamma} \subseteq \frac{1}{g}\,\mathbb{P}^2_n$.

In the second instance, let $p_\varepsilon(x,y)$ be a polynomial of degree $m = m(\varepsilon)$ such that $|p_\varepsilon - 1/g| \leq \varepsilon\,(1/|g|)$, then $fp_\varepsilon \in \mathbb{P}^2_{n+m}$ approximates f/g up to ε. This entails that a quadrature formula with nodes $\{(x_j,y_j)\}$ and positive weights $\{w_j\}$ of exactness degree $n+m$ on Ω will be nearly exact for $f(x,y,g)/g$ if $f \in \mathbb{P}^3_n$, and thus a formula with nodes $\{(x_j,y_j,g(x_j,y_j))\}$ and weights $\{w_j/g(x_j,y_j)\}$ will be near-algebraic (nearly exact) in $\mathbb{P}^3_n(\mathcal{T})$, i.e. for spherical polynomials restricted to the spherical triangle $\mathcal{T}$.

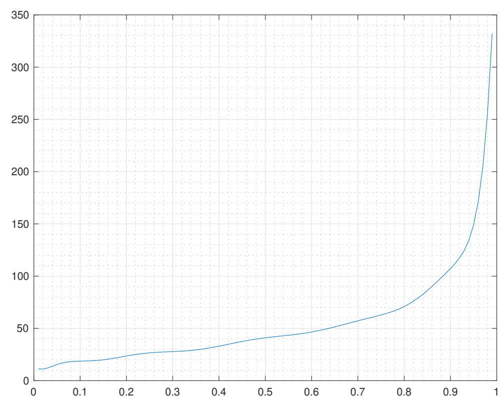

Figure 1: The degree $m(\varepsilon)$ as a function of ρ.

In order to find $m = m(\varepsilon)$, recalling that $g(x,y) = \sqrt{1-(x^2+y^2)}$ and

$$0 \leq x^2+y^2 \leq \rho = \max\left\{\|\hat{A}\|_2^2, \|\hat{B}\|_2^2, \|\hat{C}\|_2^2\right\} < 1\ ,\quad (x,y) \in \Omega\ , \tag{4}$$

it is sufficient to find the degree of a (near) optimal univariate polynomial approximation (up to ε) to the function $1/\sqrt{1-t}$ for $t \in [0,\rho]$. Though a theoretical analysis could be done, for example by means of Jackson theorem, this would typically produce overestimates of the actual degree. We have then chosen a numerical approach. To this purpose, we have used the powerful Chebfun package [12] (which eventually works with Chebyshev-like approximation), tabulating the degrees $\nu = \nu(\varepsilon,\rho)$ at machine precision ε on a fine discretization of $\rho \in (0, 0.99]$. Once we have found the appropriate ν, which is the degree (the "length" minus 1) of the Chebfun representing $1/\sqrt{1-t}$, we simply take $m = 2\nu$, since the underlying univariate polynomial has to be composed with $t = x^2+y^2$. To give an idea of the size increase, in Fig. 1 we have plotted a least squares fit of the values of m computed by Chebfun as a function of ρ.

Concerning the quadrature formula of exactness degree $n+m$ on Ω, we observe that each elliptical sector S_i, where $\cup_{i=1}^{3} S_i = \Omega$, can be seen as an affine transformation of a circular sector of the unit disk with arclength equal to the length of the great circle arc projected onto the elliptical arc. Such an affine transformation is completely determined by mapping the planar triangle with the same vertices of the circular sector, onto the planar triangle given by the vertices of the elliptical sector (keeping the origin fixed); see Fig. 2.

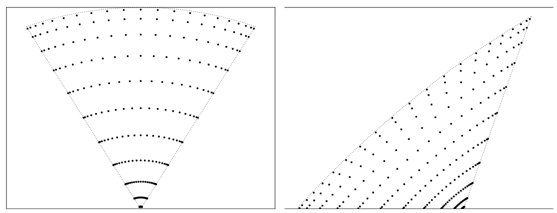

Figure 2: Circular and elliptical sectors with quadrature nodes.

We can then take the algebraic quadrature formula of exactness degree $m+n$ for the circular sector, whose $N_{n+m} = (n+m+1)\lceil \frac{n+m+2}{2} \rceil$ nodes are mapped to each elliptical sector and whose positive weights have to be multiplied by the absolute value of the transformation matrix determinant. Finally, the collection of all nodes, say $\{(x_j, y_j)\}$, $1 \leq j \leq 3N_{n+m}$, is lifted to the overhanging spherical triangle and the corresponding weights, say $\{w_j\}$, are multiplied by $1/g(x_j, y_j)$, to get the resulting quadrature formula on the spherical triangle

$$I_{\mathcal{T}}(f) = \int_{\mathcal{T}} f(x,y,z)\, d\sigma \approx \sum_{j=1}^{3N_{n+m}} \frac{w_j}{\sqrt{1-x_j^2-y_j^2}}\, f\left(x_j, y_j, \sqrt{1-x_j^2-y_j^2}\right), \tag{5}$$

which is nearly exact for spherical polynomials of degree not exceeding n; see Fig. 3.

Our Matlab codes implementing all the steps of the procedure just described are available at [23]. Moreover, we have performed a further compression step of the resulting quadrature formula on spherical triangles, since its cardinality $3N_{n+m} \approx \frac{3}{2}(n+m)^2$ can be very high, especially for spherical triangles with long sides, where ρ in (4) approaches 1. To give an idea, already for $\rho = 0.5$ we have $m = 40$ (see Fig. 1), so that the overall number of nodes is more than 2000 even for small n. For example, for $n = 5$ it is about 3200, while the dimension of the spherical polynomials is only $6^2 = 36$.

Indeed, starting from [21], in a series of papers we have implemented what we have called CATCH (Caratheodory-Tchakaloff) compression of multivariate discrete measures, in particular multivariate quadrature formulas. Based on the Tchakaloff theorem on positive quadrature, that in the framework of discrete measures can be proved via the Caratheodory theorem on conical linear combinations of finite-dimensional vectors (applied to the columns of the Vandermonde-like matrix of a moment matching system, cf. [18]), we can compress the quadrature formula obtained above to another one with positive weights, having the same moments up to degree n and as support a subset of the nodes with cardinality not exceeding $dim(\mathbb{P}_n^3(\mathcal{T})) = dim(\mathbb{P}_n^3(S^2)) = (n+1)^2$. The compression technique adopts an accelerated version of the Lawson-Hanson active-set method for NonNegative Least Squares (NNLS) to solve the underdetermined moment matching system, automatically adapting to the appropriate polynomial space on algebraic varieties such as the sphere; cf. [11] and the routine dCATCH in the recent d-variate package [10].

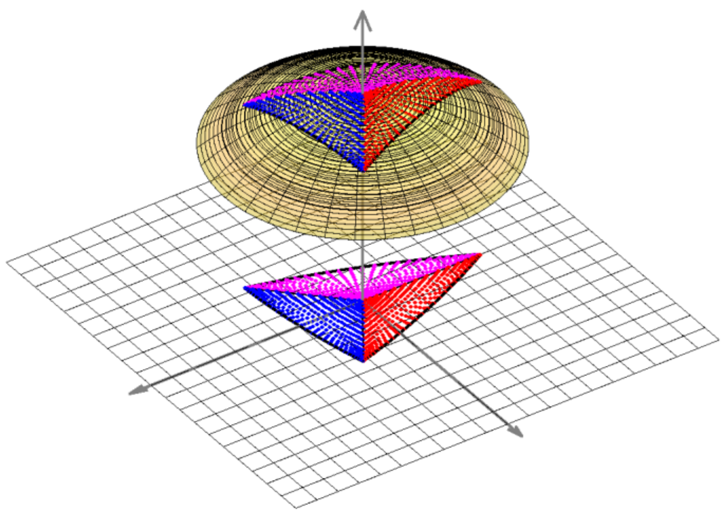

Figure 3: Quadrature nodes on a spherical triangle lifted from the projected elliptical triangle, before compression.

3 Numerical examples

In this section we present several numerical tests, in order to assess the quality of our compressed near-algebraic quadrature formulas on three spherical triangles with different extension (the key parameter being ρ in (4)). In particular, the triangle of Table 3 is the sphere octant with vertices $(1,0,0), (0,1,0), (0,0,1)$. In Tables 1-3 we show the cardinalities of the basic and of the compressed formula, the corresponding compression ratio and the CPU time needed for the computation of the compressed formula, on a sequence of polynomial degrees. All the numerical results have been obtained in Matlab R2018a, on a 2,7GHz Intel Core i5 CPU with 16GB of RAM. It is worth noticing that using the accelerated version [11] of the Lawson-Hanson active-set method for NNLS in the compression stage (which is the computational bulk), gives a speed-up by a factor 3-4 with respect to the standard `lsqnonneg` Matlab implementation.

Moreover, we measure the quality of the formulas by computing the *average relative errors* in integrating the Spherical Harmonic basis, after having filtered out the integrals that are null or tiny (below a given tolerance, say e.g. 10^{-12}); the reference values have been obtained by the basic formula at higher degree, with an additional check by an adaptive code implemented along the lines of [3].

We see that the compression ratios are remarkable, especially at low degrees and "large" ρ, and that both the basic and the compressed formula have a very good quality, the compressed exhibiting a limited loss of precision with respect to the basic one (within one order of magnitude in this degree range).

n	# basic	# compr	Cratio	CPU	$E_{\text{basic}}(SPH)$	$E_{\text{compr}}(SPH)$
5	936	36	26:1	0.03s	1e-15	3e-15
10	1305	121	11:1	0.04s	2e-15	1e-14
15	1836	256	7:1	0.1s	3e-15	3e-14
20	2340	441	5:1	0.8s	3e-15	2e-14
25	3036	676	4:1	5s	4e-15	1e-14
30	3675	961	4:1	22s	5e-15	2e-14

Table 1: Cardinalities, compression ratio, CPU time in seconds and average relative errors on Spherical Harmonics (SPH) for a spherical triangle with $\rho \approx 0.077$.

n	# basic	# compr	Cratio	CPU	$E_{\text{basic}}(SPH)$	$E_{\text{compr}}(SPH)$
5	1632	36	45:1	0.03s	2e-15	2e-15
10	2109	121	17:1	0.04s	2e-15	2e-15
15	2772	256	11:1	0.3s	5e-15	1e-14
20	3384	441	8:1	0.7s	4e-15	1e-14
25	4212	676	6:1	8s	1e-14	1e-14
30	4959	961	5:1	31s	3e-14	4e-14

Table 2: As in Table 1 for a spherical triangle with $\rho \approx 0.27$.

n	# basic	# compr	Cratio	CPU	$E_{\text{basic}}(SPH)$	$E_{\text{compr}}(SPH)$
5	5580	36	155:1	0.02s	3e-15	4e-15
10	6435	121	53:1	0.1s	1e-14	1e-14
15	7560	256	30:1	0.3s	1e-14	3e-14
20	8550	441	19:1	1s	1e-14	5e-14
25	9840	676	15:1	4s	1e-14	6e-14
30	10965	961	11:1	19s	3e-14	1e-13

Table 3: As in Table 1 for a sphere octant ($\rho \approx 0.67$).

In Table 4 we show the quadrature errors on five test functions:

$$f_1(x, y, z) = 1 + x + y^2 + x^2y + x^4 + y^5 + x^2y^2z^2 \ ,$$

$$f_2(x,y,z) = \cos(10(x+y+z)) \ ,$$
$$f_3(x,y,z) = 0.75\exp\left(-(9x-4)^2/4 - (9y-2)^2/4 - (9z-2)^2/4\right)$$
$$+0.75\exp\left(-(9x+1)^2/49 - (9y+1)/10 - (9z+1)/10\right)$$
$$+0.5\exp\left(-(9x-7)^2/4 - (9y-3)^2/4 - (9z-5)^2/4\right)$$
$$-0.2\exp\left(-(9x-2)^2 - (9y-7)^2 - (9z-5)^2\right) \ ,$$
$$f_4(x,y,z) = (1+\tanh(9x-9y+9z))/9 \ ,$$
$$f_5(x,y,z) = (1+\mathrm{sign}(9x-9y+9z))/9 \ ,$$

on the sphere octant of Table 3 ($\rho \approx 0.67$). Functions of this kind have been used in the literature on numerical integration on the sphere, cf. [13, 24]. In particular, f_1 is a polynomial of degree 6, f_2, f_3 and f_4 are smooth, but f_4 has a steep gradient and f_5 is discontinuous, both at a great circle arc (the intersection of the plane $x-y+z=0$ with the octant). For completeness we give also the values of the integrals at machine precision, say $I_j = I_{\mathcal{T}}(f_j)$, computed by an adaptive method, that are: $I_1 = 3.66706142481523$, $I_2 = -0.492762315715176$, $I_3 = 0.265883813176965$, $I_4 = 0.273012443544125$, $I_5 = 0.273546537186839$.

n	$E_{\text{compr}}(f_1)$	$E_{\text{compr}}(f_2)$	$E_{\text{compr}}(f_3)$	$E_{\text{compr}}(f_4)$	$E_{\text{compr}}(f_5)$
5	1e-05	4e-03	6e-02	3e-02	9e-02
10	2e-15	3e-06	9e-04	2e-03	1e-02
15	2e-15	2e-11	2e-04	1e-04	3e-03
20	1e-15	5e-15	2e-05	4e-04	6e-03
25	1e-15	3e-15	2e-07	5e-05	5e-03
30	6e-16	4e-15	4e-08	3e-05	2e-03

Table 4: Relative errors in the integration of the test functions defined above by the compressed formula, on the sphere octant of Table 3.

The results generally agree with expectations for these functions by an algebraic formula, in particular slower convergence for the rapidly varying functions, since the quadrature errors are substantially ruled only by the best uniform approximation with spherical polynomials of degree not exceeding n. We do not report the errors of the basic formula, that are several orders of magnitude below those of the compressed formula (except for the discontinuous function), which is however not surprising due to the huge number of nodes before compression.

All the codes and the demos are available at [23].

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