Constructing optimal polynomial meshes on planar starlike domains *

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Abstract

We construct polynomial norming meshes with optimal cardinality growth, on planar compact starlike domains that satisfy a Uniform Interior Ball Condition (UIBC).

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1 Introduction

In the recent literature on multivariate polynomial approximation, the concept of *polynomial mesh* (also called *admissible mesh*) has begun to play an important role; cf., e.g., the seminal paper [8] and [6, 13] and the references therein. Polynomial meshes are sequences $\{\mathcal{A}_n\}$ of *finite norming sets* (in the uniform norm) on a multidimensional polynomial determining compact $K \subset \mathbb{R}^d$ or $K \subset \mathbb{C}^d$ (i.e., a polynomial vanishing there vanishes everywhere), such that the following *polynomial inequality* holds

$$\|p\|_K \le C \, \|p\|_{\mathcal{A}_n} \, , \quad \forall p \in \mathbb{P}^d_n \, , \tag{1}$$

with a cardinality increasing at most like $\mathcal{O}(n^s)$, $s \ge d$ (here and below, $\|f\|_X$ denotes the sup-norm of a function f bounded on the set X). Among their properties, we recall that admissible meshes are preserved by affine transformations, and can be easily extended by finite union and product

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[8]. In the present note, we restrict our attention to the real case, i.e., real polynomials and $K \subset \mathbb{R}^d$.

Polynomial meshes provide a "good discrete model" of a compact set for many practical purposes. For example, they are nearly optimal for uniform least squares approximation [8], and contain Fekete-like interpolation subsets with the same asymptotic behavior as the continuous Fekete points of K, that can be computed by numerical linear algebra techniques (cf., e.g., [4, 5, 18]). Such approximate Fekete points have been used within spectral element and collocation methods for PDEs (cf. [14, 20]). For a recent and deep survey on polynomial approximation and interpolation in several variables, we refer the reader to [2].

In [8, Thm.5], it has been shown that any (real) compact set which satisfies a Markov polynomial inequality with exponent r

$$\|\nabla p(x)\|_2 \leq M n^r \|p\|_K \ , \quad \forall x \in K, \, p \in \mathbb{P}_n^d \ , \tag{2}$$

has an admissible mesh with $\mathcal{O}(n^{rd})$ cardinality (for example, $r = 2$ for compact sets which satisfy a Uniform Interior Cone Condition).

On the other hand, in applications it is important to control the cardinality of such discrete models. Indeed, some attention has been devoted to the construction of *optimal* and *near optimal* polynomial meshes, which have cardinality $\mathcal{O}(n^d)$ and $\mathcal{O}((n \log n)^d)$, respectively, in compact sets with special geometries (observe that in (1) necessarily $\mathrm{card}(\mathcal{A}_n) \geq \dim(\mathbb{P}_n^d) \sim n^d/d!$); cf., e.g., [7, 13, 15, 16, 17]. Moreover, the polynomial inequality (1) can be relaxed, asking that it holds with $C = C_n$, a sequence of constants increasing at most polynomially with n: in such a case, we speak of *weakly* admissible meshes. Weakly admissible meshes with $\mathcal{O}(n^d)$ cardinality and constants $C_n = \mathcal{O}((\log n)^d)$ are known in several instances, cf., e.g., [4, 11].

In the present note we prove constructively the existence of optimal polynomial meshes, i.e., with cardinality $\mathcal{O}(n^2)$, on a planar compact starlike domain (that is not restrictive to consider to be centered in the origin) assuming that it satisfies a classical *Uniform Interior Ball Condition* (UIBC, cf., e.g., [1] and references therein). Special instances are $C^{1,1}$ planar starlike domains, thus generalizing in the planar case a recent result by A. Kroó on C^2 starlike domains [13].

2 Optimal polynomial meshes by a UIBC

In the sequel, the notion of *tangential Markov inequality* on a rectifiable curve Γ *with respect to a compact* K will play a key role. Given a *rectifiable* curve $\Gamma \subset K \subset \mathbb{R}^2$ (i.e., it has a continuous parametrization and finite length), this curve has a canonical Lipschitz continuous parametrization with respect to the arclength, which is almost everywhere differentiable. Given the tangent unit vector $\boldsymbol{\tau}$ at a regular point $x \in \Gamma$ (a point where

the canonical parametrization is differentiable with nonzero derivative), a tangential Markov inequality with exponent r for Γ w.r.t. K has the form

$$|\langle \nabla p(x), \boldsymbol{\tau} \rangle| \leq M n^r \|p\|_K \ , \ \ \forall p \in \mathbb{P}_n^d \ , \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product and M, r are independent of the (regular) point x.

The fulfillement of a tangential Markov inequality like (3) allows us to construct a suitable polynomial inequality on the curve.

Lemma 1 *Let $K \subset \mathbb{R}^2$ be a compact set, and let $\Gamma \subset K$ be a rectifiable curve that satisfies (3) at its regular points. Then, for any $\alpha \in (0,1)$, there exists a mesh of equally spaced points (with respect to the arclength) on Γ, say $X_n = X_n(\alpha)$, such that*

$$\|p\|_\Gamma \leq \|p\|_{X_n} + \alpha \|p\|_K \ , \tag{4}$$

and $\operatorname{card}(X_n) = \mathcal{O}(n^r)$.

Proof. Let us term $\boldsymbol{\omega}(s)$, $s \in [0, L]$ the canonical parametrization of Γ with respect to the arclength (which is Lipschitz continuous and thus almost everywhere differentiable in $[a, b]$), where L denotes the total length of Γ. For any pair $\boldsymbol{x}, \boldsymbol{y} \in \Gamma = \boldsymbol{\omega}([0, L])$, take two values of the parameter, say $s_1, s_2 \in [0, L]$, such that $\boldsymbol{x} = \boldsymbol{\omega}(s_1)$, $\boldsymbol{y} = \boldsymbol{\omega}(s_2)$ (the curve being not necessarily simple). Then, for every polynomial $p \in \mathbb{P}_n^2$, we can write

$$|p(\boldsymbol{x}) - p(\boldsymbol{y})| = |p(\boldsymbol{\omega}(s_2)) - p(\boldsymbol{\omega}(s_1))| = \left| \int_{s_1}^{s_2} \frac{d}{ds} p(\boldsymbol{\omega}(s))\, ds \right|$$

$$\leq \int_{s_1}^{s_2} \left| \frac{d}{ds} p(\boldsymbol{\omega}(s)) \right| ds = \int_{s_1}^{s_2} |\langle \nabla p(\boldsymbol{\omega}(s)), \boldsymbol{\omega}'(s) \rangle|\, ds \leq M n^r \, \|p\|_K \, \ell(\boldsymbol{x}, \boldsymbol{y}) \ ,$$

where $\ell(\boldsymbol{x}, \boldsymbol{y}) = s_2 - s_1$ denotes the length of the corresponding arc of Γ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. Observe that the integral representation above holds since $p(\omega(\cdot))$ is absolutely continuous, as p is a polynomial and ω is Lipschitz continuous.

Fix $\alpha \in (0, 1)$. Taking $N + 1$ equally spaced points on Γ with respect to the arclength

$$X_n = \{ \boldsymbol{y}_k = \boldsymbol{\omega}(kL/N) \, , \ k = 0, \ldots, N \} \ , \ \ N = \left\lceil \frac{M n^r L}{2\alpha} \right\rceil \ , \tag{5}$$

and observing that for every $\boldsymbol{x} \in \Gamma$ there is a point $\boldsymbol{y}_{k(x)} \in X_n$ such that $\ell(\boldsymbol{x}, \boldsymbol{y}_{k(x)}) \leq \frac{1}{2} L/N$, we can write the inequality

$$|p(\boldsymbol{x})| \leq |p(\boldsymbol{y}_{k(x)})| + |p(\boldsymbol{x}) - p(\boldsymbol{y}_{k(x)})| \leq |p(\boldsymbol{y}_{k(x)})| + \alpha \|p\|_K \ ,$$

that implies (4). $\square$

Observe that if the curve is open, then the cardinality of X_n is $N+1$, whereas it is N if the curve is closed, $\boldsymbol{\omega}(0) = \boldsymbol{\omega}(L)$.

We can now state and prove the following:

Theorem 1 *Let $K \subset \mathbb{R}^2$ be a planar compact starlike domain. Assume that K satisfies a* Uniform Interior Ball Condition *(UIBC), i.e., every point of ∂K belongs to the boundary of a disk with radius $\rho > 0$, contained in K (geometrically, there is a fixed disk that can roll along the boundary remaining inside K; cf., e.g., [1]).*

Then, for every fixed $\alpha \in (0, 1/\sqrt{2})$, K possesses a sequence of finite norming sets $\{\mathcal{A}_n\}$ such that

$$\|p\|_K \leq \frac{\sqrt{2}}{1-\alpha\sqrt{2}} \|p\|_{\mathcal{A}_n} , \quad \forall p \in \mathbb{P}_n^2 , \tag{6}$$

with

$$\mathrm{card}(\mathcal{A}_n) \leq 2n \left\lceil \frac{n \,\mathrm{length}(\partial K)}{2\alpha\rho} \right\rceil + 1 = \mathcal{O}(n^2) , \tag{7}$$

i.e., an optimal admissible mesh.

Proof. We recall that a compact set $K \subset \mathbb{R}^2$ is termed *starlike* if there exists $\boldsymbol{x}_0 \in K$ (the "star center") such that for every $\boldsymbol{x} \in K$ the segment $[\boldsymbol{x}_0, \boldsymbol{x}]$ is contained in K. Without loss of generality, by making a translation if necessary, we can assume that the star center is the origin.

First, we show that K has a norming set formed by $n+1$ curves, that are suitable scalings of the boundary. Being compact and starlike with respect to the origin, K is the union of the rays $[\mathbf{0}, \boldsymbol{x}]$, $\boldsymbol{x} \in \partial K$. On each ray, a polynomial $p \in \mathbb{P}_n^2$ becomes a univariate polynomial of degree not greater than n, and thus by a well-known result of Ehlich and Zeller (cf. [9] and [7, 19]), there is an admissible mesh for the ray with constant $\sqrt{2}$, given by $2n+1$ Chebyshev-Lobatto points of $[\mathbf{0}, \boldsymbol{x}]$, namely

$$\boldsymbol{u}_j(\boldsymbol{x}) = a_j \, \boldsymbol{x} , \quad a_j = \frac{1+\xi_j}{2} , \tag{8}$$

where $\xi_j = \cos(j\pi/(2n))$, $j = 0, \ldots, 2n$, are the Chebyshev-Lobatto points in $[-1, 1]$. Then, the $2n+1$ curves

$$\Gamma_j = \{\boldsymbol{u}_j(\boldsymbol{x}) , \ \boldsymbol{x} \in \partial K\} = a_j \partial K$$

form a norming set for K, i.e.,

$$\|p\|_K \leq \sqrt{2} \, \|p\|_{\bigcup \Gamma_j} \leq \sqrt{2} \max_j \|p\|_{\Gamma_j} , \quad \forall p \in \mathbb{P}_n^2 . \tag{9}$$

Notice that Γ_{2n} degenerates into a singleton, $\Gamma_{2n} = \{\mathbf{0}\}$.

Second, we show that on each curve Γ_j an inequality like (4) is satisfied, with $r = 1$. Now, observe that the UIBC condition implies the (weaker) Uniform Interior Cone Condition, which in turn ensures that the boundary is a rectifiable curve (cf. [10, Thm. 4.5.11]). Moreover, the UIBC condition implies that a tangential Markov inequality w.r.t. K such as (3) holds with exponent $r = 1$, at the regular points of the boundary. Indeed, for any $\boldsymbol{x} \in \partial K$ there is $\boldsymbol{x}^* \in \mathrm{int}(K)$ such $\boldsymbol{x} \in \partial D$, $D \subset K$ being the disk centered at $\boldsymbol{x}^*$ with radius ρ. If we take the parametrization of ∂D in polar coordinates centered at $\boldsymbol{x}^*$, say $\boldsymbol{z}(\phi) = \boldsymbol{x}^* + (\rho\cos(\phi), \rho\sin(\phi))$, $\phi \in [0, 2\pi]$, then every $p \in \mathbb{P}_n^2$ restricted to ∂D becomes a univariate trigonometric polynomial $t(\phi) = p(\boldsymbol{z}(\phi)) \in \mathbb{T}_n$.

Consider a regular point $\boldsymbol{x} \in \partial K$, i.e., a point where the canonical parametrization with respect to the arclength is differentiable with nonzero derivative: then, the disk D is tangent to ∂K at $\boldsymbol{x}$. By the classical Markov inequality for trigonometric polynomials (cf., e.g., [3])

$$|t'(\phi)| \leq n\|t\|_{[0,2\pi]} ,$$

and the fact that $|t'(\phi)| = |\langle \nabla p(\boldsymbol{z}(\phi)), \boldsymbol{z}'(\phi)\rangle|$, $|\boldsymbol{z}'(\phi))| = \rho$, and $\boldsymbol{x} = \boldsymbol{z}(\phi^*)$ for a certain ϕ^*, we get immediately

$$|\langle \nabla p(\boldsymbol{x}), \boldsymbol{\tau}\rangle| = |\langle \nabla p(\boldsymbol{z}(\phi^*)), \boldsymbol{\tau}\rangle| \leq \frac{n}{\rho}\,\|p\|_{\partial D} \leq \frac{n}{\rho}\,\|p\|_K , \tag{10}$$

where $\boldsymbol{\tau} = \pm\boldsymbol{z}'(\phi^*)/\rho$ are the common unit tangent vectors to ∂K and ∂D at $\boldsymbol{x}$. This shows that (3) holds with $M = 1/\rho$.

By (10) and Lemma 1 with $\Gamma = \partial K$

$$\|p\|_{\partial K} \leq \|p\|_{X_n} + \alpha\|p\|_K , \quad \forall p \in \mathbb{P}_n^2 , \tag{11}$$

from which it follows that, setting $q(\boldsymbol{x}) = p(a_j\boldsymbol{x})$,

$$\|p\|_{\Gamma_j} = \|q\|_{\partial K} \leq \|q\|_{X_n} + \alpha\|q\|_K$$

$$= \|p\|_{a_jX_n} + \alpha\|p\|_{a_jK} \leq \|p\|_{a_jX_n} + \alpha\|p\|_K , \quad \forall p \in \mathbb{P}_n^2 , \tag{12}$$

since each curve $\Gamma_j = a_j\partial K$ is an affine transformation (scaling) of the boundary.

Fix α such that $0 < \alpha < 1/\sqrt{2}$. By (12) and (9) we can write for every $p \in \mathbb{P}_n^2$

$$\|p\|_K \leq \sqrt{2}\,\max_j \left\{\|p\|_{a_jX_n}\right\} + \alpha\sqrt{2}\|p\|_K$$

$$= \sqrt{2}\,\|p\|_{\mathcal{A}_n} + \alpha\sqrt{2}\|p\|_K , \quad \mathcal{A}_n = \bigcup_{j=0}^{2n} a_jX_n = \{\mathbf{0}\} \cup \bigcup_{j=0}^{2n-1} a_jX_n , \tag{13}$$

from which we finally get

$$\|p\|_K \leq \frac{\sqrt{2}}{1-\alpha\sqrt{2}} \|p\|_{\mathcal{A}_n} , \quad \forall p \in \mathbb{P}_n^2 , \tag{14}$$

where by (5) $\operatorname{card}(\mathcal{A}_n) \leq 2n \operatorname{card}(X_n) + 1 = 2nN + 1 = \mathcal{O}(n^2)$. Observe, in fact, that $\operatorname{card}(\mathcal{A}_n) = 2n(N-1)+1$ if $\mathbf{0} \in X_n$, otherwise $\operatorname{card}(\mathcal{A}_n) = 2nN+1$. □

The theorem above generalizes, in the planar case, a recent result proved by A. Kroó in arbitrary dimension for C^2 starlike domains; cf. [13]. Indeed,

Corollary 1 *Let $K \subset \mathbb{R}^2$ be the closure of an open, bounded, starlike, and $C^{1,1}$ subset. Then, K has an optimal admissible mesh.*

Proof. We recall that a closed domain $K \subset \mathbb{R}^d$ (the closure of an open connected subset) is termed $C^{1,1}$ if there are a fixed radius, say $R > 0$, and a constant $L > 0$ such that for each point $\boldsymbol{\xi} \in \partial K$ there exists a $C^{1,1}$ function $f : I \to \mathbb{R}$ (I compact interval) such that after a suitable rotation, $K \cap B(\boldsymbol{\xi}, R) = \{\boldsymbol{x} = (x_1, x_2) \in K : x_2 \leq f(x_1)\}$, where $\|f'\|_I$ and the Lipschitz constant of f' are uniformly bounded by L.

Now, it is known that a closed domain is $C^{1,1}$ if and only if it satisfies a uniform two-sided (interior and exterior) ball condition; cf., e.g., [1, Cor. 3.14]. Then, all the assumptions of Theorem 1 are satisfied, and the conclusion follows. □

Remark 1 The assumptions of Theorem 1 are much weaker than those of Corollary 1. In fact, the boundary of a $C^{1,1}$ domain is a regular curve, whereas Theorem 1 allows singular points, for example inward (but not outward) corners and cusps (the domain does not even need to be Lipschitz).

Remark 2 In the special case of C^2 *convex domains*, the maximal ρ in Theorem 1 is equal to the *minimal ray of curvature*, in view of the so called *Blaschke's rolling ball theorem*, cf., e.g., [12]. This fact could be used as the basis of an algorithm for the computation of optimal polynomial meshes on C^2 convex domains.

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