Weakly Admissible Meshes and Discrete Extremal Sets *

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Abstract

We present a brief survey on (Weakly) Admissible Meshes and corresponding Discrete Extremal Sets, namely Approximate Fekete Points and Discrete Leja Points. These provide new computational tools for polynomial least squares and interpolation on multidimensional compact sets, with different applications such as numerical cubature, digital filtering, spectral and high-order methods for PDEs.

1 Introduction.

Locating good points for multivariate polynomial approximation, in particular interpolation, is an open challenging problem, even in standard domains. One set of points that is always good, in theory, is the so-called *Fekete points*. They are defined to be those points that maximize the (absolute value of the) Vandermonde determinant on the given compact set. However, these are known analytically in only a few instances (the interval and the complex circle for univariate interpolation, the cube for tensor product interpolation), and are very difficult to compute, requiring an expensive and numerically challenging nonlinear multivariate optimization.

Recently, a new insight has been given by the theory of "Admissible Meshes" of Calvi and Levenberg [9], which are nearly optimal for least-squares approximation and contain interpolation sets (Discrete Extremal Sets) nearly as good as Fekete points of the domain. Such sets, termed Approximate Fekete Points and Discrete Leja Points, are computed using only basic tools of numerical linear algebra, namely QR and LU factorizations of Vandermonde matrices. Admissible Meshes and Discrete Extremal Sets allow us to replace a continuous compact set by a discrete version, that is "just as good" for all practical purposes.

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2 Weakly Admissible Meshes (WAMs).

Given a *polynomial determining* compact set $K \subset \mathbb{R}^d$ or $K \subset \mathbb{C}^d$ (i.e., polynomials vanishing there are identically zero), a Weakly Admissible Mesh (WAM) is defined in [9] to be a sequence of discrete subsets $\mathcal{A}_n \subset K$ such that

$$\|p\|_K \leq C(\mathcal{A}_n)\|p\|_{\mathcal{A}_n} \; , \quad \forall p \in \mathbb{P}_n^d(K) \tag{1}$$

where both $\mathrm{card}(\mathcal{A}_n) \geq N$ and $C(\mathcal{A}_n)$ grow at most *polynomially* with n. When $C(\mathcal{A}_n)$ is bounded we speak of an Admissible Mesh (AM). Here and below, we use the notation $\|f\|_X = \sup_{x \in X} |f(x)|$, where f is a bounded function on the compact X.

We sketch below the main features of WAMs in terms of ten properties (cf. [4, 9]):

P1: $C(\mathcal{A}_n)$ is invariant under affine mapping
P2: any sequence of unisolvent interpolation sets whose Lebesgue constant grows at most polynomially with n is a WAM, $C(\mathcal{A}_n)$ being the Lebesgue constant itself
P3: any sequence of supersets of a WAM whose cardinalities grow polynomially with n is a WAM with the same constant $C(\mathcal{A}_n)$
P4: a finite union of WAMs is a WAM for the corresponding union of compacts, $C(\mathcal{A}_n)$ being the maximum of the corresponding constants
P5: a finite cartesian product of WAMs is a WAM for the corresponding product of compacts, $C(\mathcal{A}_n)$ being the product of the corresponding constants
P6: in $\mathbb{C}^d$ a WAM of the boundary ∂K is a WAM of K (by the maximum principle)
P7: given a polynomial mapping π_s of degree s, then $\pi_s(\mathcal{A}_{ns})$ is a WAM for $\pi_s(K)$ with constants $C(\mathcal{A}_{ns})$ (cf. [4, Prop.2])
P8: any K satisfying a Markov polynomial inequality like $\|\nabla p\|_K \leq Mn^r\|p\|_K$ has an AM with $\mathcal{O}(n^{rd})$ points (cf. [9, Thm.5])
P9: least-squares polynomial approximation of $f \in C(K)$: the least-squares polynomial $\mathcal{L}_{\mathcal{A}_n} f$ on a WAM is such that

$$\|f - \mathcal{L}_{\mathcal{A}_n} f\|_K \lessapprox C(\mathcal{A}_n)\sqrt{\mathrm{card}(\mathcal{A}_n)} \, \min\{\|f - p\|_K, \, p \in \mathbb{P}_n^d(K)\}$$

(cf. [9, Thm.1])
P10: Fekete points: the Lebesgue constant of Fekete points extracted from a WAM can be bounded like $\Lambda_n \leq NC(\mathcal{A}_n)$ (that is the elementary classical bound of the continuum Fekete points times a factor $C(\mathcal{A}_n)$); moreover, their asymptotic distibution is the same of the continuum Fekete points, in the sense that the corresponding discrete probability measures converge weak-$*$ to the pluripotential equilibrium measure of K (cf. [4, Thm.1])

The properties above give the basic tools for the construction and application of WAMs in the framework of polynomial interpolation and approximation.

For illustrative purposes we focus briefly on the real bivariate case, i.e., $K \subset \mathbb{R}^2$. Property **P8**, applied for example to convex compacts with nonempty interior where a Markov inequality with exponent $r = 2$ always holds, says that it is possible to obtain an Admissible Mesh with $\mathcal{O}(n^4)$ points. The construction works as follows (see [9, Thm.5], also for a generalization to the nonconvex case).

First, we recall that every *convex* compact set of $\mathbb{R}^2$ with nonempty interior admits the Markov inequality $\max_{x\in K}\|\nabla p(x)\|_2 \leq M\,n^2\,\|p\|_K$, $M = \frac{\alpha(K)}{w(K)}$, for every $p \in \mathbb{P}_n^2(K)$, where $\alpha(K) \leq 4$, and $w(K)$ is the minimal distance between two parallel supporting lines for K; cf. [19]. Consider a cartesian grid $\{(ih, jh)\,,\, i,j \in \mathbb{Z}\}$ with constant stepsize h: for every square of the grid that has nonempty intersection with K, take a point in this intersection. Let $\mathcal{A}_n$ be the mesh formed by such points. For every $x \in K$, let $a \in \mathcal{A}_n$ be the point closest to x: by construction, both belong to the same square of the grid. Using the mean value theorem, the Cauchy-Schwarz inequality and the Markov inequality, we can write $|p(x)-p(a)| \leq \|\nabla p(y)\|_2\|x-a\|_2 \leq M\sqrt{2}hn^2\,\|p\|_K$, since y belongs to the open segment connecting x and a, which lies in K. Then, from $|p(x)| \leq |p(x)-p(a)|+|p(a)| \leq M\sqrt{2}hn^2\,\|p\|_K+|p(a)|$, the polynomial inequality $\|p\|_K \leq \frac{1}{1-\mu}\,\|p\|_{\mathcal{A}_n}$ follows, provided that $h = h_n :\ M\sqrt{2}h_n n^2 \leq \mu < 1$, i.e., $\mathcal{A}_n$ is an AM with constant $C = 1/(1-\mu)$. Taking for example $\mu = 1/\sqrt{2}$ and using the minimal rectangle including the compact, it is not difficult to show that the cardinality of the mesh is approximately n^4 times $area(K) \times (\alpha(K)/w(K))^2$, that is, e.g., approximately πn^4 for the unit disk (cf. [4, Table 1]).

In order to avoid such a large cardinality, which has severe computational drawbacks, we can turn to WAMs, which can have a much lower cardinality, typically $\mathcal{O}(n^2)$ points.

In [4] a WAM on the disk with approximately $2n^2$ points and $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$ has been constructed with standard polar coordinates, using essentially property **P2** for univariate Chebyshev and trigonometric interpolation. Moreover, using property **P2** and **P7**, WAMs for the triangle and for linear trapezoids, again with approximately $2n^2$ points and $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$, have been obtained simply by mapping the so-called Padua points of degree $2n$ from the square by standard quadratic transformations (the first known optimal points for bivariate polynomial interpolation, with a Lebesgue constant growing like log-squared of the degree, cf. [3]).

In [8] these results have been improved, showing that there are WAMs for the disk and the triangle with approximately n^2 points and still the same growth of the relevant constants. In particular, a symmetric polar WAM of the unit disk is made by equally spaced angles and Chebyshev-Lobatto points on the corresponding diameters

$$\mathcal{A}_n = \{(r_j\cos\theta_k, r_j\sin\theta_k)\}$$

$$\{(r_j,\theta_k)\}_{j,k} = \left\{\cos\frac{j\pi}{n}\,,\ 0 \leq j \leq n\right\} \times \left\{\frac{k\pi}{m}\,,\ 0 \leq k \leq m-1\right\}$$

where $m = n+2$ for even n even and $m = n+1$ for odd n (see Figure 1).

A WAM of the unit simplex is obtained from a WAM of the disk for polynomials of degree $2n$ containing only even powers, by the standard quadratic transformation from the (quadrant of the) disk to the simplex $(u,v) \mapsto (x_1,x_2) = (u^2,v^2)$,

$$\mathcal{A}_n = \{(r_j^2\cos^2\theta_k, r_j^2\sin^2\theta_k)\}$$

$$\{(r_j,\theta_k)\}_{j,k} = \left\{\cos\frac{j\pi}{2n}\,,\ 0 \leq j \leq n\right\} \times \left\{\frac{k\pi}{2n}\,,\ 0 \leq k \leq n\right\}$$

see Figure 2. This mesh can then be mapped to any triangle by the standard affine transformation, in view of Property **P1**. The resulting mesh points lie on a grid of intersecting straight lines, namely a pencil from one vertex (image of the point $(0,0)$ of the simplex) cut by a pencil parallel to the opposite side (image of the hypothenuse of the simplex). The points on each segment of the pencils, and in particular the points on each side, are the corresponding Chebyshev-Lobatto points.

Property **P4** allows to obtain WAMs for any polygon that can be subdivided into triangles by standard algorithms of computational geometry (see for example the underlying meshes in Figure 3).

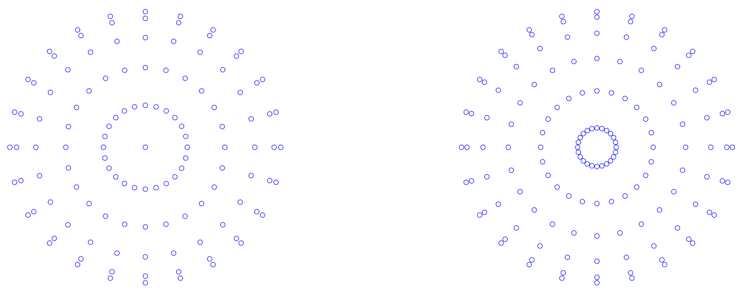

Figure 1: Symmetric polar WAMs of the disk for degree $n = 10$ (left) and $n = 11$ (right).

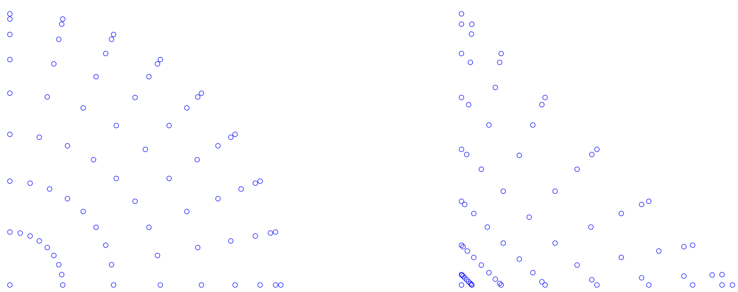

Figure 2: A WAM of the quadrant for even polynomials of degree $n = 16$ (left), and the corresponding WAM of the simplex for degree $n = 8$ (right).

2.1 Discrete Least Squares Approximation on WAMs.

Consider a WAM $\{\mathcal{A}_n\}$ of a polynomial determining compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$), say $\mathcal{A}_n = \{a_1, \ldots, a_M\}$, $M \geq N = \dim(\mathbb{P}_n^d)$, and the associated rectangular Vandermonde-like matrix

$$V(\boldsymbol{a};\boldsymbol{p}) = V(a_1,\ldots,a_M;p_1,\ldots,p_N) = [p_j(a_i)] \ , \ \ 1 \leq i \leq M \ , \ 1 \leq j \leq N \quad (2)$$

where $\boldsymbol{a} = (a_i)$ is the array of mesh points, and $\boldsymbol{p} = (p_j)$ is the array of basis polynomials for $\mathbb{P}_n^d$ (both ordered in some manner). For convenience, we shall consider $\boldsymbol{p}$ as a column vector $\boldsymbol{p} = (p_1, \ldots, p_N)^t$.

The least-squares projection operator at the WAM can be constructed by the following algorithm
iterated orthogonalization:

- $V(\boldsymbol{a};\boldsymbol{p}) = Q_1R_1$, $Q_1 = Q_2R_2$
- $Q = Q_2$, $T = R_1^{-1}R_2^{-1}$

which amounts to a change of basis from $\boldsymbol{p}$ to the discrete orthonormal basis $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)^t = T^t\boldsymbol{p}$ with respect to the inner product $\langle f, g\rangle = \sum_{i=1}^{M} f(a_i)\overline{g(a_i)}$ (we use here the QR factorization with Q rectangular $M \times N$ and R upper triangular $N \times N$). Observe that the Vandermonde matrix in the new basis, namely $V(\boldsymbol{a};\boldsymbol{\varphi}) = V(\boldsymbol{a};\boldsymbol{p})T = Q$, is a numerically orthogonal (unitary) matrix, i.e. $\overline{Q^t}Q = I$. The reason for iterating the QR factorization is to cope with ill-conditioning which is typical of Vandermonde-like matrices. Two orthogonalization iterations generally suffice, unless the original matrix $V(\boldsymbol{a};\boldsymbol{p})$ is so severely ill-conditioned (rule of thumb: condition number greater than the reciprocal of machine precision) that the algorithm fails (cf. [15]). This phenomenon of "twice is enough", is well-known in numerical Gram-Schmidt orthogonalization.

Denoting by $\mathcal{L}_{\mathcal{A}_n}$ the discrete least-squares projection operator, we can write

$$\mathcal{L}_{\mathcal{A}_n} f(x) = \sum_{j=1}^{N} \left(\sum_{i=1}^{M} f(a_i)\overline{\varphi_j(a_i)} \right) \varphi_j(x) = \sum_{i=1}^{M} f(a_i)\, g_i(x) \tag{3}$$

where $g_i(x) = K_n(x, a_i)$, $i = 1, \ldots, M$, $K_n(x, y) = \sum_{j=1}^{N} \varphi_j(x)\overline{\varphi_j(y)}$ being the *reproducing kernel* corresponding to the discrete inner product. In matrix terms, the relevant set of generators of $\mathbb{P}_n^d$ (which is not a basis when $M > N$), becomes simply $\boldsymbol{g} = (g_1, \ldots, g_M)^t = QT^t\boldsymbol{p}$, where the transformation matrix T and the orthogonal (unitary) matrix Q are computed once and for all for a fixed mesh. Moreover, the norm of the least-squares operator is given by $\|\mathcal{L}_{\mathcal{A}_n}\| = \max_{x\in K} \sum_{i=1}^{M} |g_i(x)| = \max_{x\in K} \|QT^t\boldsymbol{p}(x)\|_1$.

Property **P9** ensures that the WAMs described in the previous section can be directly used for least-squares approximation of continuous functions with an error which is near-optimal, up to a factor $\mathcal{O}(n \log^2 n)$. The latter, however, turns out to be a rough overestimate (see [8]). Concerning the triangle, for example, for $n > 5$ the (numerically evaluated) norms turn out to be lower than the Lebesgue constants of the best known points for polynomial interpolation, which have been obtained by various authors with different techniques up to degree $n = 19$, in view of the relevance to spectral and high-order methods for PDEs (cf. [18]); see Table 1. We stress that the triangle WAM can be explicitly computed at any degree and used via the iterated orthogonalization process, provided that the Vandermonde conditioning is not too severe.

3 Fekete Points and Discrete Extremal Sets.

The concept of Fekete points for interpolation can be described in a very general functional, not necessarily polynomial, setting. It is worth observing that such Fekete points should not be confused with the "minimum energy" Fekete points, the two concepts being equivalent only in the univariate complex case (cf. [13]).

Given a compact set $K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$), a finite-dimensional space of linearly independent continuous functions, $S_N = \text{span}(p_j)_{1\leq j\leq N}$, and a finite set

Table 1: Comparison of the Lebesgue constants of the best known points for interpolation in the triangle with the uniform norms of least-squares projection operators at the WAM.

deg	8	9	10	11	12	13	14	15	16	17	18	19
intp	5.0	5.7	6.7	7.3	7.6	9.3	9.0	9.3	12.1	13.3	13.5	14.2
LS	4.2	4.4	4.7	4.9	5.0	5.2	5.3	5.5	5.7	5.9	6.1	6.2

$\{\xi_1, \ldots, \xi_N\} \subset K$, ordering in some manner the points and the basis we can construct the Vandermonde-like matrix $V(\boldsymbol{\xi}; \boldsymbol{p}) = [p_j(\xi_i)]$, $1 \leq i, j \leq N$. If $\det V(\boldsymbol{\xi}; \boldsymbol{p}) \neq 0$ the set $\{\xi_1, \ldots, \xi_N\}$ is unisolvent for interpolation in S_N, and

$$\ell_j(x) = \frac{\det V(\xi_1, \ldots, \xi_{j-1}, x, \xi_{j+1}, \ldots, \xi_N; \boldsymbol{p})}{\det V(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_N; \boldsymbol{p})}, \quad j = 1, \ldots, N, \qquad (4)$$

is a cardinal basis, i.e. $\ell_j(\xi_k) = \delta_{jk}$ and $L_{S_N} f(x) = \sum_{j=1}^{N} f(\xi_j)\, \ell_j(x)$ interpolates any function f at $\{\xi_1, \ldots, \xi_N\}$. In matrix terms, the cardinal basis $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_N)^t$ is obtained from the original basis $\boldsymbol{p} = (p_1, \ldots, p_N)^t$ as $\boldsymbol{\ell} = L\boldsymbol{p}$, $L := (V(\boldsymbol{\xi}; \boldsymbol{p}))^{-t}$.

In the case that such points maximize the (absolute value of the) denominator of (4) in K^N (Fekete points), then $\|\ell_j\|_\infty \leq 1$ for every j, and thus the norm of the interpolation operator $L_{S_N} : C(K) \to S_N$ is bounded by the dimension of the interpolation space,

$$\|L_{S_N}\| = \max_{x \in K} \sum_{j=1}^{N} |\ell_j(x)| = \max_{x \in K} \|L\boldsymbol{p}(x)\|_1 \leq N. \qquad (5)$$

Clearly, Fekete points as well as $\|L_{S_N}\|$ are independent of the choice of the basis in S_N, since the determinant of the Vandermonde-like matrices changes by a factor independent of the points (namely the determinant of the transformation matrix between the bases).

In the polynomial framework, $\Lambda_n = \|L_{S_N}\|$ is the so-called Lebesgue constant of interpolation at the point set $\{\xi_j\}$; moreover, Fekete points and Lebesgue constants are preserved under affine mapping of the domain. It is also worth recalling that (5) is often a rather pessimistic overestimate of the actual growth.

There are several open problems about Fekete points, whose properties have been studied till now mainly in the univariate complex case in view of their deep connection with potential theory. They are analytically known only in few cases: the interval (Gauss-Lobatto points), the complex circle (equispaced points), and the cube (tensor-product of Gauss-Lobatto points for tensor interpolation). An important qualitative result has been proved only recently, namely that Fekete points are asymptotically equidistributed with respect the pluripotential equilibrium measure of K, cf. [1]. Their asymptotic spacing is known only in few instances, cf. the recent paper [7]. Moreover, the numerical computation of Fekete points becomes rapidly a very large scale problem, namely a nonlinear optimization problem in $N \times d$ variables. It has been solved numerically only in very special cases, like the triangle and the sphere, for a fixed limited range of degrees.

A reasonable approach for the computation of Fekete points is to use a discretization of the domain, moving from the continuum to nonlinear combinatorial optimization. Property **P10** gives a first guideline on the fact that WAMs are good candidates as starting meshes. Since the rows of the Vandermonde matrix $V = V(\boldsymbol{a}, \boldsymbol{p})$, cf. (2), correspond to the mesh points and the columns to the basis elements, computing the Fekete points of a WAM amounts to selecting N rows of V such that the volume generated by these rows, i.e., the absolute value of the determinant of the resulting $N \times N$ submatrix, is maximum. This problem, however, is known to be NP-hard, so heuristic or stochastic algorithms are mandatory; cf. [10] for the notion of volume generated by a set of vectors (which generalizes the geometric concept related to parallelograms and parallelepipeds), and an analysis of the problem from a computational complexity point of view.

An approximate solution can be given by one of the following two *greedy* algorithms, given as pseudo-codes in a Matlab-like notation, which compute what we call "Discrete Extremal Sets"; cf. [4, 5].

algorithm greedy 1 (Approximate Fekete Points):

- $V = V(\boldsymbol{a}, \boldsymbol{p})$; $ind = [\,]$;
- `for` $k = 1 : N$ "select i_k: $\mathrm{vol}\, V([ind, i_k]\,, 1 : N)$ is max"; $ind = [ind, i_k]$; `end`
- $\boldsymbol{\xi} = \boldsymbol{a}(i_1, \dots, i_N)$

algorithm greedy 2 (Discrete Leja Points):

- $V = V(\boldsymbol{a}, \boldsymbol{p})$; $ind = [\,]$;
- `for` $k = 1 : N$ "select i_k: $|\det V([ind, i_k]\,, 1 : k)|$ is max"; $ind = [ind, i_k]$; `end`
- $\boldsymbol{\xi} = \boldsymbol{a}(i_1, \dots, i_N)$

These algorithms are genuinely different. In both, the selected points (as opposed to for the continuum Fekete points) depend on the choice of the polynomial basis. But in the second algorithm, which is based on the notion of determinant, the selected points depend also on the ordering of the basis. In the univariate case with the standard monomial basis, it is not difficult to recognize that the selected points are indeed the Leja points extracted from the mesh (cf. [2, 13] and references therein). On the contrary, dependence on the ordering of the basis does not occur with the first algorithm, which is based on the notion of volume generated by the rows of a rectangular matrix (the two notions being eventually equivalent on the final square submatrix). On the other hand, Discrete Leja Points form a sequence, in the sense that, if $\{p_1, \dots, p_k\}$ is a basis of $\mathbb{P}_k^d$ for every k, then on a fixed mesh the first $N_k = \dim(\mathbb{P}_k^d)$ selected points are exactly the Discrete Leja Points for degree k. This gives an important computational feature to Discrete Leja Points: once we have computed the points for degree n, we have automatically at hand (nested) interpolation sets for all lower degrees.

The two greedy algorithms above correspond to basic procedures of numerical linear algebra. Indeed, in algorithm 1 the core "select i_k: $\mathrm{vol}\, V([ind, i_k]\,, 1 : N)$ is maximum" can be implemented as "select the largest norm row $row_{i_k}(V)$ and remove from every row of V its orthogonal projection onto row_{i_k}", since the corresponding orthogonalization process does not affect volumes (as can be understood geometrically applying the method to a collection of 3-dimensional vectors and thinking in terms of parallelograms and parallelepipeds). Working for convenience with the transposed Vandermonde matrix, this process is equivalent to the *QR factorization with column pivoting* proposed by Businger and Golub in 1965, and exploited for example by Matlab in the solution of underdetermined systems by the standard "backslash" operator; cf. [6] for a full discussion of the equivalence.

On the other hand, it is clear that in algorithm 2 the core "select i_k: $|\det V([ind, i_k]\,,\, 1:k)|$ is maximum" can be implemented by one column elimination step of the *Gaussian elimination* process with standard *row pivoting*, since such process automatically seeks the maximum keeping invariant the absolute value of the relevant subdeterminants.

As a consequence of the considerations above (see [5] for a more detailed discussion), the computation of Discrete Extremal Sets can be done by few basic linear algebra operations, corresponding to the the LU factorization with row pivoting of the Vandermonde matrix (cf. [14]), and to the QR factorization with column pivoting of the transposed Vandermonde matrix (cf. [15]). This is summarized in the following Matlab-like scripts:

algorithm AFP (Approximate Fekete Points):
- $W = (V(\boldsymbol{a},\boldsymbol{p}))^t$; $\boldsymbol{w} = W\backslash(1,\dots,1)^t$; $ind = \texttt{find}(\boldsymbol{w} \neq \boldsymbol{0})$; $\boldsymbol{\xi} = \boldsymbol{a}(ind)$

algorithm DLP (Discrete Leja Points):
- $V = V(\boldsymbol{a},\boldsymbol{p})$; $[L, U, \boldsymbol{\sigma}] = \texttt{LU}(V, \text{"}\texttt{vector}\text{"})$; $ind = \boldsymbol{\sigma}(1,\dots,N)$; $\boldsymbol{\xi} = \boldsymbol{a}(ind)$

Notice that in algorithm AFP we could replace $(1,\dots,1)$ by any nonzero vector, and that in algorithm DLP we are using the version of the LU factorization with row pivoting that produces a row permutation vector. When the conditioning of the Vandermonde matrices is too high, the algorithms can still be used provided that a preliminary iterated orthogonalization, that is a change to a discrete orthogonal basis, is performed as in Section 2.1; cf. [4, 5, 15].

As already pointed out, once the underlying extraction WAM has been fixed, Approximate Fekete Points depend on the choice of the basis, whereas Discrete Leja Points depend also on its order. Nevertheless, such Discrete Extremal Sets share the same asymptotic behavior, which is exactly that of the continuum Fekete points, as is stated in the following theorem (cf. [4, 5] for a proof and [12] for the definition of the relevant terms).

Theorem 1 *Suppose that $K \subset \mathbb{C}^d$ is compact, non-pluripolar, polynomially convex and regular (in the sense of Pluripotential theory) and that for $n = 1, 2, \cdots$, $\mathcal{A}_n \subset K$ is a WAM. Let $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_N\}$ be the Approximate Fekete Points selected from $\mathcal{A}_n$ by the greedy algorithm AFP using any basis $\boldsymbol{p}$, or the Discrete Leja Points selected from $\mathcal{A}_n$ by the greedy algorithm DLP using any basis of the form $\boldsymbol{p} = L\boldsymbol{e}$, where $\boldsymbol{e} = \{e_1, \dots, e_N\}$ is any ordering of the standard monomials x^α such that $deg(e_j) \leq deg(e_k)$ for $j \leq k$ that is consistent with the degree and $L \in \mathbb{C}^N \times \mathbb{C}^N$ is lower triangular. Then*

- $\lim_{n\to\infty} |det\, V(\boldsymbol{\xi}; \boldsymbol{p})|^{1/m_n} = \tau(K)$*, the transfinite diameter of K, where $m_n = dnN/(d+1)$ (the sum of the degrees of the N monomials of degree at most n);*
- *the sequence of discrete probability measures $\mu_n := \frac{1}{N}\sum_{j=1}^N \delta_{\xi_j}$ converge to the pluripotential-theoretic equilibrium measure $d\mu_K$ of K, in the sense that $\lim_{n\to\infty} \int_K f(x)d\mu_n = \int_K f(x)d\mu_K$ for every $f \in C(K)$.*

To give an example of computation of Discrete Extremal Sets, we consider the nonregular convex hexagon in Figure 3, with the WAMs generated by two different triangulations, and the Chebyshev product basis of the minimal including rectangle. From Table 2 we see that, concerning Lebesgue constants, DLP are of lower quality than AFP: this is not surprising, since the same phenomenon is well-known concerning continuous Fekete and Leja points. Nevertheless, both provide reasonably good interpolation points, as it is seen from the interpolation errors on two test functions of different regularity in Table 3.

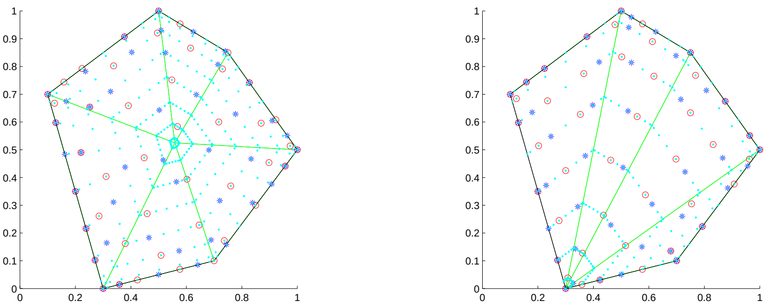

Figure 3: $N = 45$ Approximate Fekete Points (circles) and Discrete Leja Points (asterisks) for degree $n = 8$ extracted from two WAMs of a nonregular convex hexagon (dots).

Table 2: Lebesgue constants for AFP and DLP extracted from two WAMs of a nonregular convex hexagon (WAM1: barycentric triangulation, WAM2: minimal triangulation; see Fig. 3).

mesh	points	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$	$n = 30$
WAM1	AFP	6.5	18.9	20.4	40.8	73.3	73.0
	DLP	7.1	19.6	49.8	58.3	108.0	167.0
WAM2	AFP	6.8	12.3	34.2	52.3	49.0	80.4
	DLP	10.7	48.4	62.0	91.6	86.6	203.0

Table 3: Max-norm of the interpolation errors with AFP and DLP extracted from WAM2 for two test functions: $f_1 = \cos(x_1 + x_2)$; $f_2 = ((x_1 - 0.5)^2 + (x_2 - 0.5)^2)^{3/2}$.

function	points	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$	$n = 30$
f_1	AFP	6E-06	5E-13	3E-15	3E-15	3E-15	4E-15
	DLP	8E-06	2E-12	2E-15	4E-15	3E-15	4E-15
f_2	AFP	3E-03	2E-04	1E-04	4E-05	2E-05	1E-05
	DLP	3E-03	3E-04	1E-04	3E-05	2E-05	5E-06

3.1 Some applications.

Polynomial interpolation and approximation are at the core of many important numerical techniques, and we are of the opinion that Weakly Admissible Meshes and Discrete Extremal Sets could give new useful tools in several applications. We mention here three of them.

As a first natural application, we can consider *numerical cubature*. In fact, if in algorithm AFP we take as right-hand side $\boldsymbol{b} = \boldsymbol{m} = \int_K \boldsymbol{p}(x)\, d\mu$ (the *moments* of the polynomials basis with respect to a given measure), the vector $\boldsymbol{w}(ind)$ gives directly the weights of an algebraic cubature formula at the corresponding Approximate Fekete Points. When the boundary of K is approximated by splines, for $d\mu = dx$ such moments can be computed by the formulas developed in [16]. The same can clearly be done with Discrete Leja Points, solving the system $V(\boldsymbol{\xi}; \boldsymbol{p})\boldsymbol{w} = \boldsymbol{m}$ by the LU factorization, with the additional feature that, if the polynomial basis is as in Theorem 1, we get a nested family of cubature formulas (Leja points being a sequence).

In [17], Approximate Fekete Points have been applied to *weighted polynomial interpolation*, where one considers a basis $w\boldsymbol{p}$ for the Vandermonde matrix, w being a suitable weight function. There are two different but related frameworks: approximation of functions with *prescribed singularities* (e.g., poles) where singularities are absorbed in the weight function, and approximation with *weighted norms*, which is relevant for example to the construction of real and complex, uni- and multivariate *digital filters*.

The possibility of locating good points for polynomial approximation on polygonal regions/elements, could also be useful in the numerical solution of PDEs by *spectral and high-order methods*, for example in the emerging field of discontinuous Galerkin methods (see, e.g., [11]).

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