Subperiodic trigonometric interpolation and quadrature *

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Abstract

We study theoretically and numerically trigonometric interpolation on symmetric subintervals of $[-\pi, \pi]$, based on a family of Chebyshev-like angular nodes (subperiodic interpolation). Their Lebesgue constant increases logarithmically in the degree, and the associated Fejér-like trigonometric quadrature formula has positive weights. Applications are given to the computation of the equilibrium measure of a complex circle arc, and to algebraic cubature over circular sectors.

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1 Introduction

It is well-known that any set of $2n + 1$ equally spaced angular nodes in $[-\pi, \pi)$ is optimal for trigonometric interpolation of degree n, with Lebesgue constant of order $\mathcal{O}(\log(n))$; cf. e.g., [3] and [4, Thm. 3]. On the other hand, explicit near-optimal sets for trigonometric interpolation on *subintervals* of $[-\pi, \pi]$ do not seem to be known, despite the fact that the problem is relevant to real polynomial approximation on circular and spherical sections, such as for example circular arcs, sectors, and lenses, or spherical caps, lunes, and slices. Indeed, these are cases where the polynomials become trigonometric or mixed algebraic-trigonometric in polar/spherical coordinates, with the trigonometric part defined on arcs, or products of arcs.

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In this paper we study interpolation and quadrature by a family of Chebyshev-like angular nodes in $(-\omega, \omega) \subseteq (-\pi, \pi)$, which are the zeros of the trigonometric function $T_{2n+1}(\sin(\theta/2)/\sin(\omega/2))$ (which is not a trigonometric polynomial, as opposed to $T_{2n}(\sin(\theta/2)/\sin(\omega/2))$ which is a trigonometric polynomial of degree n). Here and below, $T_k(\cdot) = \cos(k \arccos(\cdot))$ denotes the Chebyshev polynomial of the first kind of degree k.

We prove that these angular nodes are unisolvent for trigonometric interpolation of degree n on $[-\omega, \omega]$, by providing compact interpolation formulas, and that their Lebesgue constant is of order $\mathcal{O}(\log(n))$. Indeed, the Lebesgue constant turns even out to be experimentally independent of ω, and thus exactly that of trigonometric interpolation at $2n+1$ equally spaced angular nodes in $(-\pi, \pi)$, which is the limit case as $\omega \to \pi$.

Moreover, we prove that the associated Fejér-like trigonometric quadrature formula has positive weights, and we provide a Matlab code for the computation of such weights.

Finally, we give two applications. The first is the computation of the equilibrium measure (in the sense of complex potential theory) of an arc of a circle. The second is the construction of product cubature formulas exact for bivariate algebraic polynomials of degree $\leq n$ over circular sectors, with approximately n^2 nodes and positive weights.

2 Subperiodic trigonometric interpolation

In the sequel we use the change of variables

$$x = \frac{\sin(\theta/2)}{\alpha} \ , \ \ \theta \in [-\omega, \omega] \ , \ \ 0 < \omega \leq \pi \ , \ \ 0 < \alpha = \sin(\omega/2) \leq 1 \ , \qquad (1)$$

and we denote by $\{\xi_j : 1 \leq j \leq 2n+1\}$ the zeros of $T_{2n+1}(x)$, namely

$$\xi_j := \xi_j(n) = \cos\left(\frac{(2j-1)\pi}{2(2n+1)}\right) \in (-1, 1) \ , \ \ j = 1, 2, \ldots, 2n+1 \qquad (2)$$

to which correspond the Chebyshev-like angular nodes

$$\theta_j := \theta_j(n, \omega) = 2 \arcsin(\alpha \xi_j) \in (-\omega, \omega) \ , \ \ j = 1, 2, \ldots, 2n+1 \ , \qquad (3)$$

i.e., the zeros of $T_{2n+1}(\sin(\theta/2)/\alpha)$. By definition we have $\xi_j = \frac{1}{\alpha} \sin(\theta_j/2)$. Note that $\xi_{n+1} = \cos(\pi/2) = 0$ and hence $\theta_{n+1} = 0$, and for $j \neq n+1$, $\xi_j = -\xi_{2n+2-j}$ and hence $\theta_j = -\theta_{2n+2-j}$. Observe that for $\alpha = 1$ ($\omega = \pi$), the $\{\theta_k\}$ turn out to be $2n+1$ equally spaced angular nodes in $(-\pi, \pi)$.

We show now that $\{\theta_j\}$ is a unisolvent set for interpolation in

$$\mathbb{T}_n([-\omega, \omega]) = span\{1, \cos(k\theta), \sin(k\theta) \ , \ 1 \leq k \leq n \ , \ \theta \in [-\omega, \omega]\} \ ,$$

the $2n+1$-dimensional space of trigonometric polynomials of degree not greater than n, restricted to $[-\omega, \omega] \subseteq [-\pi, \pi]$.

Proposition 1 *Denote by*

$$\ell_j(x) = T_{2n+1}(x)/(T'_{2n+1}(\xi_j)(x-\xi_j)) \tag{4}$$

the j-th algebraic Lagrange polynomial for the nodes $\{\xi_j\}$, $\ell_j(\xi_k) = \delta_{jk}$; cf. (2) and [7, Ch.6]. The angular nodes $\{\theta_j\}$ in (3) are unisolvent for interpolation in $\mathbb{T}_n([-\omega,\omega])$ and, with $x = \frac{1}{\alpha}\sin(\theta/2)$ as in (1), the corresponding trigonometric Lagrange polynomials can be written as

$$L_{n+1}(\theta) = \ell_{n+1}(x) \tag{5}$$

and for $j \neq n+1$

$$\begin{aligned} L_j(\theta) &= \frac{1}{2}\left(\ell_j(x) + \ell_{2n+2-j}(x)\right)\left(1 + \frac{\xi_j^2}{\sin(\theta_j)}\,\frac{\sin(\theta)}{x^2}\right) \\ &= a_j(\theta)\ell_j(x) + b_j(\theta)\ell_{2n+2-j}(x) \end{aligned} \tag{6}$$

where

$$a_j(\theta) = \frac{1}{2}\left(1 + \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\right)\ , \quad b_j(\theta) = \frac{1}{2}\left(1 - \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\right) = 1 - a_j(\theta). \tag{7}$$

Moreover, $L_{2n+2-j}(\theta) = b_j(\theta)\ell_j(x) + a_j(\theta)\ell_{2n+2-j}(x)$.

Proof. First note that although $x = \frac{1}{\alpha}\sin(\theta/2)$ is *not* a trigonometric polynomial, $x^2 = \frac{1}{\alpha^2}\sin^2(\theta/2) = \frac{1}{2\alpha^2}(1-\cos(\theta))$ is. Secondly, observe that for $j \neq n+1$, $\ell_j(x) + \ell_{2n+2-j}(x)$ is even in x due to the symmetry of the ξ_j and also equal to 0 at $x = \xi_{n+1} = 0$. Hence $\ell_j(x) + \ell_{2n+2-j}(x)$ is a polynomial in x^2 and also divisible by x^2. Consequently $L_j(\theta)$, as defined by (6) is indeed a trigonometric polynomial.

Further, for $j = n+1$, we also have that $\ell_{n+1}(x)$ is even in x by the symmetry of the ξ_j and hence a polynomial in x^2. It follows that L_{n+1}, as defined by (5), is also a trigonometric polynomial.

Clearly, $L_{n+1}(\theta_k) = \delta_{n+1,k}$. Moreover, for $j \neq n+1$ and $k \neq j, 2n+2-j$, we have $L_j(\theta_k) = 0$ since $\ell_j(\xi_k) = \ell_{2n+2-j}(\xi_k) = 0$. For $k = j$,

$$\begin{aligned} L_j(\theta_j) &= \frac{1}{2}\left(\ell_j(\xi_j) + \ell_{2n+2-j}(\xi_j)\right)\left(1 + \frac{\xi_j^2}{\sin(\theta_j)}\,\frac{\sin(\theta_j)}{\xi_j^2}\right) \\ &= \frac{1}{2}(1+0)\times(1+1) = 1 \end{aligned}$$

and for $k = 2n+2-j$,

$$\begin{aligned} L_j(\theta_{2n+2-j}) &= \frac{1}{2}\left(\ell_j(\xi_{2n+2-j}) + \ell_{2n+2-j}(\xi_{2n+2-j})\right)\left(1 + \frac{\xi_j^2}{\sin(\theta_j)}\,\frac{\sin(-\theta_j)}{(-\xi_j)^2}\right) \\ &= \frac{1}{2}(1+0)\times(1+(-1)) = 0, \end{aligned}$$

since $\xi_{2n+2-j} = -\xi_j$ and $\theta_{2n+2-j} = -\theta_j$.

Concerning the second equality in (6), observe that

$$\begin{aligned} L_j(\theta) &= \frac{1}{2}\left(\ell_j(x) + \ell_{2n+2-j}(x)\right) + \frac{1}{2}\left(\frac{\xi_j}{x}\,\frac{\sin(\theta)}{\sin(\theta_j)}\right)\,\frac{\xi_j}{x}\,\left(\ell_j(x) + \ell_{2n+2-j}(x)\right) \\ &= \frac{1}{2}\left(\ell_j(x) + \ell_{2n+2-j}(x)\right) + \frac{1}{2}\,\frac{\xi_j}{x}\,\frac{\sin(\theta)}{\sin(\theta_j)}\,\left(\ell_j(x) - \ell_{2n+2-j}(x)\right) \end{aligned}$$

as can be easily checked by computing the summands at $x = \xi_k$ ($\theta = \theta_k$), $k = 1, \ldots, 2n+1$. But,

$$\frac{\xi_j}{x}\,\frac{\sin(\theta)}{\sin(\theta_j)} = \frac{\left(\frac{\sin(\theta_j/2)}{\alpha}\right)}{\left(\frac{\sin(\theta/2)}{\alpha}\right)}\,\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta_j/2)\cos(\theta_j/2)} = \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\;.$$

Hence,

$$L_j(\theta) = \frac{1}{2}\left(1 + \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\right)\ell_j(x) + \frac{1}{2}\left(1 - \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\right)\ell_{2n+2-j}(x)\;.$$

The formula for $L_{2n+2-j}(\theta)$ follows similarly. $\square$

Proposition 2 *Define the even and odd parts of a function $f(\theta)$, $\theta \in [-\omega,\omega] \subseteq [-\pi,\pi]$, as $f_e(\theta) = (f(\theta)+f(-\theta))/2$ and $f_o(\theta) = (f(\theta)-f(-\theta))/2$. The trigonometric interpolating polynomial at the angular nodes $\{\theta_j\}$ can be written*

$$\sum_{j=1}^{2n+1} f(\theta_j)\,L_j(\theta) = \sum_{j=1}^{2n+1} f_e(\theta_j)\,\ell_j(x) + \cos(\theta/2)\sum_{j=1}^{2n+1}\frac{f_o(\theta_j)}{\cos(\theta_j/2)}\,\ell_j(x), \quad (8)$$

where, as before, $x = \frac{1}{\alpha}\sin(\theta/2)$.

Proof. First, observe that

$$\begin{aligned} &\sum_{j=1}^{2n+1} f(\theta_j)\,L_j(\theta) \\ &= \sum_{j=1}^{n}\{f(\theta_j)\,L_j(\theta) + f(\theta_{2n+2-j})\,L_{2n+2-j}(\theta)\} + f(\theta_{n+1})\,L_{n+1}(\theta) \\ &= \sum_{j=1}^{n}\{f(\theta_j)\,L_j(\theta) + f(-\theta_j)\,L_{2n+2-j}(\theta)\} + f(\theta_{n+1})\,L_{n+1}(\theta) \\ &= \sum_{j=1}^{n}\{f(\theta_j)(a_j(\theta)\ell_j(x) + b_j(\theta)\ell_{2n+2-j}(x)) + f(-\theta_j)(b_j(\theta)\ell_j(x) + a_j(\theta)\ell_{2n+2-j}(x))\} \\ &\qquad + f(\theta_{n+1})\,\ell_{n+1}(x) \\ &= \sum_{j=1}^{n}\{(a_j(\theta)f(\theta_j) + b_j(\theta)f(-\theta_j))\ell_j(x) \\ &\qquad + (b_j(\theta)f(\theta_j) + a_j(\theta)f(-\theta_j))\ell_{2n+2-j}(x)\} + f(\theta_{n+1})\,\ell_{n+1}(x). \end{aligned}$$

Now, using the facts that $a_j(\theta) = 1/2 + (1/2)\cos(\theta/2)/\cos(\theta_j/2)$ and $b_j(\theta) = 1/2 - (1/2)\cos(\theta/2)/\cos(\theta_j/2)$, we get

$$
\begin{aligned}
&\sum_{j=1}^{2n+1} f(\theta_j)\, L_j(\theta) \\
&= \sum_{j=1}^{n} \left\{ \left(\frac{f(\theta_j) + f(-\theta_j)}{2} + \frac{\cos(\theta/2)}{\cos(\theta_j/2)} \frac{f(\theta_j) - f(-\theta_j)}{2} \right) \ell_j(x) \right. \\
&\quad \left. + \left(\frac{f(\theta_j) + f(-\theta_j)}{2} + \frac{\cos(\theta/2)}{\cos(\theta_j/2)} \frac{f(-\theta_j) - f(\theta_j)}{2} \right) \ell_{2n+2-j}(x) \right\} \\
&\quad + f(\theta_{n+1})\, \ell_{n+1}(x) \\
&= \sum_{j=1}^{2n+1} \frac{f(\theta_j) + f(-\theta_j)}{2}\, \ell_j(x) + \sum_{j=1}^{2n+1} \frac{\cos(\theta/2)}{\cos(\theta_j/2)} \frac{f(\theta_j) + f(-\theta_j)}{2}\, \ell_j(x),
\end{aligned}
$$

i.e., (8). $\square$

Remark 1 The representation (8) allows an efficient and stable computation of the trigonometric interpolant, by resorting to the well-known *barycentric formula* for algebraic Lagrange interpolation (cf., e.g., [1])).

We now turn to estimation of the Lebesgue constant of the angular nodes (3). The first basic step is made by the next Proposition.

Proposition 3 *Consider the angular nodes (3). The following identity holds for $|\theta| \geq |\theta_j|$ (i.e., $|x| \geq |\xi_j|$)*

$$
|L_j(\theta)| + |L_{2n+2-j}(\theta)| = |\ell_j(x)| + |\ell_{2n+2-j}(x)| \ . \tag{9}
$$

Proof. First note that $\theta/2 \in [-\omega/2, \omega/2] \subseteq [-\pi/2, \pi/2]$, hence $\cos(\theta/2) \geq 0$ and $\cos(\theta_j/2) \geq 0$; in particular, $a_j(\theta) \geq 0$ $\forall \theta \in [-\omega, \omega]$. In the case $|\theta| \geq |\theta_j|$ we also have $b_j(\theta) \geq 0$.

Note also that for $|\theta| \geq |\theta_j|$, i.e. $|x| \geq |\xi_j| = |\xi_{2n+2-j}|$, $\ell_j(x)$ and $\ell_{2n+2-j}(x)$ have the *same* sign, by (4) and the fact that $\xi_j = -\xi_{2n+2-j}$. Hence, for $|\theta| \geq |\theta_j|$,

$$
|L_j(\theta)| = a_j(\theta)|\ell_j(x)| + b_j(\theta)|\ell_{2n+2-j}(x)|
$$

and

$$
|L_{2n+2-j}(\theta)| = b_j(\theta)|\ell_j(x)| + a_j(\theta)|\ell_{2n+2-j}(x)| \ .
$$

Thus,

$$
\begin{aligned}
|L_j(\theta)| + |L_{2n+2-j}(\theta)| &= (a_j(\theta) + b_j(\theta))(|\ell_j(x)| + |\ell_{2n+2-j}(x)|) \\
&= |\ell_j(x)| + |\ell_{2n+2-j}(x)| \ . \quad \square
\end{aligned}
$$

Corollary 1 *For $|\theta| \geq |\theta_1|$, i.e., $|x| \geq |\xi_1|$,*

$$\sum_{j=1}^{2n+1} |L_j(\theta)| = \sum_{j=1}^{2n+1} |\ell_j(x)| \ . \tag{10}$$

Proposition 4 *The Lebesgue function of the angular nodes (3) is bounded as*

$$\sum_{j=1}^{2n+1} |L_j(\theta)| \leq \frac{1}{\sqrt{1-\alpha^2}} \sum_{j=1}^{2n+1} |\ell_j(x)| \leq \frac{1}{\sqrt{1-\alpha^2}} \left(1 + \frac{2}{\pi} \log(2n+1)\right) \ . \tag{11}$$

Proof. We will show that

$$|L_j(\theta)| + |L_{2n+2-j}(\theta)| \leq \frac{1}{\sqrt{1-\alpha^2}} \left(|\ell_j(x)| + |\ell_{2n+2-j}(x)|\right), \ j = 1, \ldots, n+1,$$

from which the Proposition follows directly. For $j = n+1$ this is trivial as $\sqrt{1-\alpha^2} \leq 1$. Hence suppose that $j \neq n+1$. From Proposition 3, we need only to consider $|\theta| \leq |\theta_j|$, i.e., $|x| \leq |\xi_j|$. In this case, note that $\ell_j(x)$ and $\ell_{2n+2-j}(x)$ have *opposite* signs, by (4).

Consider first the case when $\ell_j(x) \geq 0$ and $\ell_{2n+2-j}(x) \leq 0$. Note also that $b_j(\theta) \leq 0$ for $|\theta| \leq |\theta_j|$ (whereas $a_j(\theta) \geq 0 \ \forall \theta \in [-\omega, \omega]$. Hence

$$L_j(\theta) = a_j(\theta)\ell_j(x) + (-b_j(\theta))(-\ell_{2n+2-j}(x))$$

and thus

$$|L_j(\theta)| = a_j(\theta)|\ell_j(x)| - b_j(\theta)|\ell_{2n+2-j}(x)| \ .$$

Similarly,

$$L_{2n+2-j}(\theta) = -((-b_j(\theta))\ell_j(x) + a_j(\theta)(-\ell_{2n+2-j}(x)))$$

and so

$$|L_{2n+2-j}(\theta)| = (-b_j(\theta))|\ell_j(x)| + a_j(\theta)|\ell_{2n+2-j}(x)| \ .$$

Consequently,

$$\begin{aligned} |L_j(\theta)| + |L_{2n+2-j}(\theta)| &= (a_j(\theta) - b_j(\theta))(|\ell_j(x)| + |\ell_{2n+2-j}(x)|) \\ &= \frac{\cos(\theta/2)}{\cos(\theta_j/2)} (|\ell_j(x)| + |\ell_{2n+2-j}(x)|). \end{aligned}$$

Note now that $\cos(\theta_j/2) = \sqrt{1-\sin^2(\theta_j/2)} = \sqrt{1-\alpha^2\xi_j^2} \geq \sqrt{1-\alpha^2}$ and the results follows.

The case $\ell_j(x) \leq 0$ and $\ell_{2n+2-j}(x) \geq 0$ is entirely similar. $\square$

Estimate (11) is not useful for $\omega \to \pi$ ($\alpha \to 1$). On the other hand, we have numerical evidence (see, e.g., Figure 1) that:

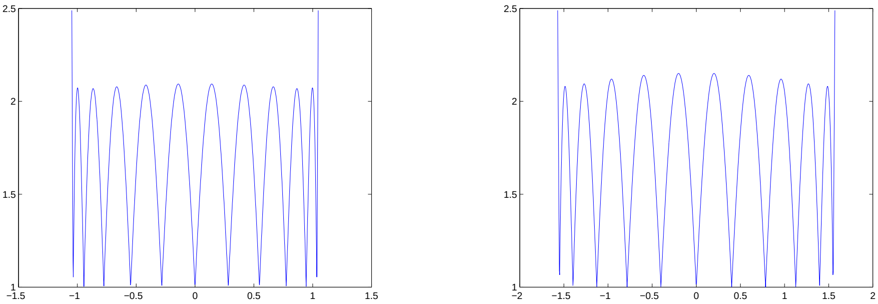

Figure 1: Lebesgue functions for degree $n = 5$ corresponding to the angular nodes (3) for $\omega = \pi/3$ (left) and $\omega = \pi/2$ (right).

Conjecture 1 *The maximum of the Lebesgue function of the angular nodes (3) (i.e., their Lebesgue constant) is attained at* $\theta = \pm\omega$.

In view of the well-known fact that the global maximum of the Lebesgue function of the Chebyshev nodes (2) is attained at ± 1 (cf. [5]), by Conjecture 1, Corollary 1 and Proposition 4 it would follow immediately that the Lebesgue function of the angular nodes (3) is *independent of* ω, and hence (cf. [8, Thm. 1.2]) that

$$\Lambda_n = \max_{\theta \in [-\omega,\omega]} \sum_{j=1}^{2n+1} |L_j(\theta)| \leq 1 + \frac{2}{\pi} \log(2n+1) \; . \tag{12}$$

In Figure 2 we compare the numerically evaluated Lebesgue constant of subperiodic trigonometric interpolation with $1+\frac{2}{\pi} \log(2n+1)$, $n = 1, \ldots, 50$.

3 Subperiodic trigonometric quadrature

Once we have at our disposal a set of unisolvent angular nodes, such as (3), we can study the corresponding quadrature formula. Our main result is the following:

Proposition 5 *The interpolatory trigonometric quadrature formula based on the angular nodes (3) has positive weights*

$$0 < w_j^T = w_j^T(n,\omega) = 2\alpha \int_{-1}^{1} \frac{\ell_j(x)}{\sqrt{1-\alpha^2 x^2}}\, dx \; , \quad j = 1, \ldots, 2n+1 \; . \tag{13}$$

The weights can be computed by the even generalized Chebyshev moments as

$$w_j^T = \frac{2\alpha}{2n+1} \left(m_0 + 2 \sum_{k=1}^{n} m_{2k}\, T_{2k}(\xi_j) \right) \; , \quad j = 1, \ldots, 2n+1 \; , \tag{14}$$

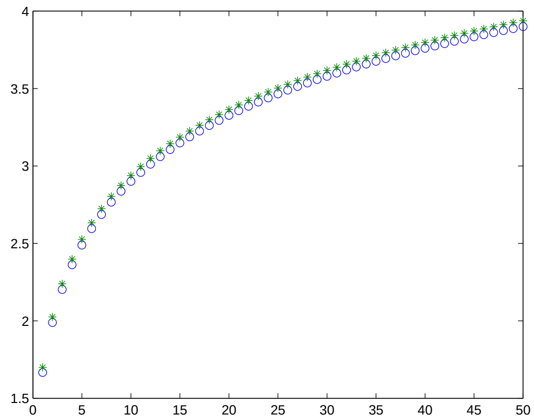

Figure 2: numerically evaluated Lebesgue constant ($\circ$) compared with the upper bound (12) ($*$).

where

$$m_{2k} = \int_{-1}^{1} \frac{T_{2k}(x)}{\sqrt{1-\alpha^2 x^2}}\, dx \;, \quad k = 0, 1, \ldots, n \;.$$

Proof. First, observe that by the change of variables $\theta = 2\arcsin(\alpha x)$

$$w_j^T = \int_{-\omega}^{\omega} L_j(\theta)\, d\theta = 2\alpha \int_{-1}^{1} \frac{\ell_j(x)}{\sqrt{1-\alpha^2 x^2}}\, dx \;, \quad j = 1, \ldots, 2n+1 \;.$$

Hence we must show that *the weights* $\{\lambda_j\}$ *of the algebraic quadrature formula for the weight function* $\lambda(x) = 1/\sqrt{1-\alpha^2 x^2}$, $x \in (-1, 1)$, *based on the zeros of* $T_n(x)$, *are positive* $(0 \le \alpha \le 1)$.

Now, setting $x = \cos(\phi)$,

$$\lambda_j = \int_{-1}^{1} \frac{\ell_j(x)}{\sqrt{1-\alpha^2 x^2}}\, dx = \int_{0}^{\pi} \frac{\sin(\phi)}{\sqrt{1-\alpha^2\cos^2(\phi)}}\, \ell_j(\cos(\phi))\, d\phi \;.$$

But, as is well known,

$$\ell_j(\cos(\phi)) = \frac{1}{n} + \frac{2}{n} \sum_{r=1}^{n-1} \cos(r\phi_j)\cos(r\phi) \;,$$

where $\cos(\phi_j)$ are the zeros of $T_n(x)$, and so

$$\lambda_j = \frac{\pi}{n} \left\{ \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin(\phi)}{\sqrt{1-\alpha^2\cos^2(\phi)}}\, d\phi \right.$$

$$+\sum_{r=1}^{n-1}\left(\frac{2}{\pi}\int_0^\pi \frac{\sin(\phi)}{\sqrt{1-\alpha^2\cos^2(\phi)}}\cos(r\phi)\,d\phi\right)\cos(r\phi_j)\Bigg\} = \frac{\pi}{n}\,S_{n-1}(\phi_j) \quad (15)$$

where $S_{n-1}(\phi)$ is the Fourier cosine series of degree $n-1$ for

$$F_\alpha(\phi) := \frac{\sin(\phi)}{\sqrt{1-\alpha^2\cos^2(\phi)}}\ .$$

Let A_j denote the $j-th$ Fourier coefficient of $F_\alpha(\phi)$. Note that

$$A_{2m+1} = \frac{2}{\pi}\int_0^\pi \frac{\sin(\phi)}{\sqrt{1-\alpha^2\cos^2(\phi)}}\cos((2m+1)\phi)\,d\phi = 0$$

since $\cos((2m+1)(\pi-\phi)) = (-1)^{2m+1}\cos((2m+1)\phi) = -\cos((2m+1)\phi)$. Hence

$$S(\phi) = A_0 + \sum_{k=1}^{\infty} A_{2k}\cos(2k\phi).$$

We calculate

$$A_0 = \frac{1}{\pi}\int_0^\pi \frac{\sin(\phi)}{\sqrt{1-\alpha^2\cos^2(\phi)}}\,d\phi = \frac{1}{\alpha\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = \frac{2}{\alpha\pi}\arcsin(\alpha)\ .$$

Next, we show that $A_{2k} < 0$ for all k. First, note that

$$F'_\alpha(\phi) = (1-\alpha^2)\,\frac{\cos(\phi)}{(1-\alpha^2\cos^2(\phi))^{3/2}}$$

and

$$F''_\alpha(\phi) = -(1-\alpha^2)\,\frac{\sin(\phi)(1+2\alpha^2\cos^2(\phi))}{(1-\alpha^2\cos^2(\phi))^{5/2}}\ .$$

Hence

$$\begin{aligned}
A_{2k} &= \frac{2}{\pi}\int_0^\pi F_\alpha(\phi)\,\cos(2k\phi)\,d\phi \\
&= \frac{4}{\pi}\int_0^{\pi/2} F_\alpha(\phi)\,\cos(2k\phi)\,d\phi \\
&= \frac{4}{\pi}\left(F_\alpha(\phi)\,\frac{\sin(2k\phi)}{2k}\Bigg|_{\phi=0}^{\phi=\pi/2} - \frac{1}{2k}\int_0^{\pi/2} F'_\alpha(\phi)\,\sin(2k\phi)\,d\phi\right) \\
&= -\frac{2}{k\pi}\int_0^{\pi/2} F'_\alpha(\phi)\,\sin(2k\phi)\,d\phi
\end{aligned}$$

where $F'_\alpha(\phi) \geq 0$ and is strictly decreasing on $[0,\pi/2]$.

Observe that $\sin(2k\phi)$ has alternating positive and negative zones all of them symmetric and beginning with a positive zone. Figure 3 gives an illustration for $\omega = \pi/5$ and $k = 6$.

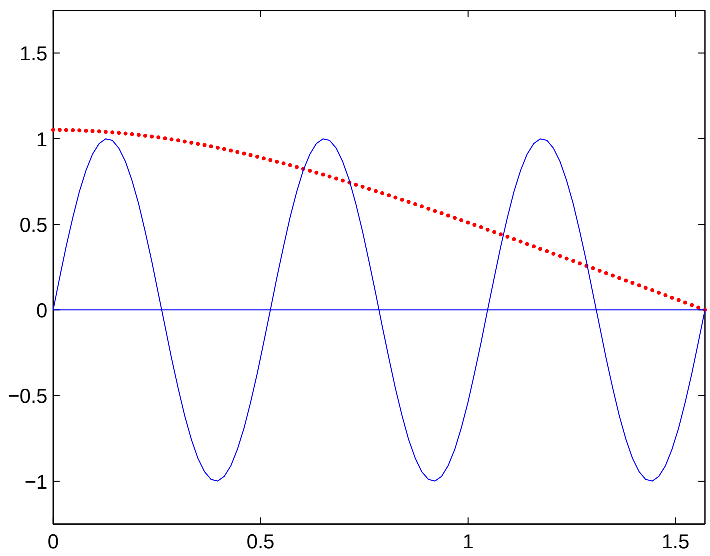

Figure 3: F'_α for $\omega = \pi/5$ and $\sin(2k\phi)$ for $k = 6$.

Hence we may match each negative zone with its preceeding positive zone. However, since $F'_\alpha(\phi)$ is decreasing, the *positive* zone is always multiplied by a greater weight than the corresponding negative one. It follows that

$$\int_0^{\pi/2} F'_\alpha(\phi)\, \sin(2k\phi)\, d\phi > 0$$

and thus $A_{2k} < 0$.

¿From (15) we have

$$\begin{aligned}
\lambda_j &= \frac{\pi}{n}\, S_{n-1}(\phi_j) \\
&= \frac{\pi}{n}\left(A_0 - \sum_{k=1}^{n-1} |A_k| \cos(k\phi_j)\right) \\
&\geq \frac{\pi}{n}\left(A_0 - \sum_{k=1}^{n-1} |A_k|\right) \\
&= \frac{\pi}{n}\, S_{n-1}(0).
\end{aligned}$$

But since $|\sin(\phi)|/\sqrt{1-\alpha^2\cos^2(\phi)}$ is continuous and piecewise smooth on $[-\pi, \pi]$

$$\lim_{n\to\infty} S_n(0) = F_\alpha(0) = 0\ ,$$

or, in other words,

$$\lim_{n\to\infty}\left(A_0-\sum_{k=1}^{n-1}|A_k|\right)=0\ .$$

Now, $A_0-\sum_{k=1}^{n-1}|A_k|$ is a decreasing sequence in n and so $A_0-\sum_{k=1}^{n-1}|A_k|\downarrow 0$, which implies that $A_0-\sum_{k=1}^{n-1}|A_k|>0$ for every n, and thus $\lambda_j>0$ for every j, which shows that $w_j^T>0$ for every j.

Concerning computation of the weights, by the Christoffel-Darboux formula for the Chebyshev polynomials (cf., e.g., [6, §1.3.3])

$$1+2\sum_{k=1}^{2n+1}T_k(x)\,T_k(y)=\frac{T_{2n+2}(x)T_{2n+1}(y)-T_{2n+1}(x)T_{2n+2}(y)}{x-y}$$

and the fact that $T_{2n+1}(\xi_j)=0$, $T'_{2n+1}(\xi_j)=(-1)^{j-1}(2n{+}1)/\sin(\arccos(\xi_j))$ and $T_{2n+2}(\xi_j)=(-1)^j\sin(\arccos(\xi_j))$, we get

$$\ell_j(x)=\frac{1}{2n+1}\left(1+2\sum_{k=1}^{2n}T_k(x)\,T_k(\xi_j)\right)\ ,$$

from which (14) immediately follows by observing that the odd generalized Chebyshev moments vanish. $\square$

We have also numerical evidence that:

Conjecture 2 *The weights of the interpolatory trigonometric quadrature formula based on the angular nodes (3) satisfy the inequality*

$$0<w_j^F<\frac{w_j^T}{2\alpha}\le w_j^{GC}\ ,\quad j=1,\dots,2n+1 \tag{16}$$

where $\{w_j^F\}$ and $\{w_j^{GC}\}$ are the weights of the algebraic Fejér and Gauss-Chebyshev quadrature formulas in $(-1,1)$, respectively.

For the reader's convenience, a Matlab function (named `trigquad.m`) that computes the angular nodes and weights of subperiodic trigonometric quadrature, is available at: http://www.math.unipd.it/~marcov/CAAsoft.html.

4 Applications

4.1 The equilibrium measure of an arc of a circle in $\mathbb{C}^1$

As a first application, on the basis of the results above we compute in an elementary way the potential theoretic *equilibrium measure* (cf. [2, p. 444]) of an arc of the complex unit circle

$$\Gamma_\omega=\left\{e^{i\theta}\,:\,-\omega\le\theta\le\omega\right\}$$

with angle 2ω. In what follows we set $z = e^{i\theta}$ and $z_k = e^{i\theta_k}$.

If $L_k(\theta)$ is the *trigonometric* Lagrange polynomial of degree n for the angular nodes (3), then substituting $\cos(j\theta) = (z^j + z^{-j})/2$, $\sin(j\theta) = (z^j + z^{-j})/(2i)$ we see that

$$P_k(z) = z^n L_k(\theta)$$

is a complex algebraic polynomial of degree $2n$. Further

$$P_k(z_j) = z_j^n L_k(\theta_j) = z_j^n \delta_{jk}$$

so that

$$\ell_k(z) = z_k^{-n} P_k(z)$$

is the complex Lagrange polynomial of degree $2n$ for the points $z_1, z_2, \dots, z_{2n+1}$.

Also, the Lebesgue function is

$$\lambda_{2n}(z) = \sum_{k=1}^{2n+1} |\ell_k(z)| = \sum_{k=1}^{2n+1} |L_k(\theta)| = \mathcal{O}(\log n)\ , \quad z \in \Gamma_\omega$$

and hence the equally weighted discrete measure based on the $\{z_k\}$ (or $\{\theta_k\}$) tends weak-$*$ to the equilibrium measure for Γ_ω (cf., e.g., [2, Thm. 1.5]).

To find this measure we evaluate

$$\frac{1}{2n+1} \sum_{k=1}^{2n+1} f(\theta_k) = \frac{1}{2n+1} \sum_{k=1}^{2n+1} f(2\arcsin(\alpha\xi_k)) = \frac{1}{2n+1} \sum_{k=1}^{2n+1} g(\xi_k)$$

where $g(x) = f(2\arcsin(\alpha x))$; cf. (1)–(3). But

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=1}^{2n+1} g(\xi_k) = \frac{1}{\pi} \int_{-1}^{1} g(x)\, \frac{1}{\sqrt{1-x^2}}\, dx$$

since the $\{\xi_k\}$ are the zeros of $T_{2n+1}(x)$.

Finally, by the change of variables $\theta = 2\arcsin(\alpha x)$ we get

$$\begin{aligned}
\frac{1}{\pi} \int_{-1}^{1} g(x)\, \frac{1}{\sqrt{1-x^2}}\, dx &= \frac{1}{\pi} \int_{-1}^{1} f(2\arcsin(\alpha x))\, \frac{1}{\sqrt{1-x^2}}\, dx \\
&= \frac{1}{2\alpha\pi} \int_{-\omega}^{\omega} f(\theta)\, \frac{1}{\sqrt{1-\left(\frac{\sin(\theta/2}{\alpha}\right)^2}}\, \cos(\theta/2)\, d\theta \\
&= \frac{1}{2\pi} \int_{-\omega}^{\omega} f(\theta)\, \frac{\cos(\theta/2)}{\sqrt{\alpha^2 - \sin^2(\theta/2)}}\, d\theta.
\end{aligned}$$

Hence, we have shown the following

Theorem 1 *The* equilibrium measure *for* Γ_ω *is*

$$d\mu(\theta) = \frac{1}{2\pi}\, \frac{\cos(\theta/2)}{\sqrt{\alpha^2 - \sin^2(\theta/2)}}\, d\theta = \frac{1}{2\pi}\, \frac{\cos(\theta/2)}{\sqrt{\sin^2(\omega/2) - \sin^2(\theta/2)}}\, d\theta\ . \quad (17)$$

4.2 Algebraic cubature over circular sectors

The subperiodic trigonometric quadrature formula of Proposition 5 allows the construction of algebraic quadrature formulas over multivariate domains or surfaces for which an algebraic polynomial, by a suitable change of variables (e.g., polar or spherical coordinates), becomes a tensor product trigonometric or mixed algebraic/trigonometric polynomial on domains related to circular arcs. Examples are circular sectors and lenses, or spherical lat-long rectangles, spherical caps and lunes, spherical slices, and even surface/solid sections of the cylinder or of the torus.

One of the simplest cases is that of a *circular sector*, which, with no loss of generality, can be taken as

$$S_\omega = \{(x,y) = (\rho\cos(\theta), \rho\sin(\theta)) \ , \ 0 \le \rho \le 1 \ , \ -\omega \le \theta \le \omega\} \ , \ \omega \in (0,\pi) \ .$$

Algebraic cubature formulas, i.e., formulas exact for bivariate polynomials of degree $\le n$, over circular sectors do not seem to be known in the literature. Having at disposal the subperiodic trigonometric quadrature of Proposition 5, it is now simple to construct a product-like cubature formula for a sector.

Indeed, any bivariate polynomial $P(x,y) \in \mathbb{P}_n^2$ becomes in polar coordinates a tensor product polynomial in $\mathbb{P}_n \bigotimes \mathbb{T}_n$. Hence

$$\iint_{S_\omega} P(x,y)\,dx\,dy = \int_0^1 \int_{-\omega}^{\omega} P(\rho\cos(\theta), \rho\sin(\theta))\,\rho\,d\theta\,d\rho$$

$$= \sum_{i=1}^{\lceil \frac{n}{2} \rceil + 1} \sum_{j=1}^{2n+1} \rho_i^{GL}\, w_i^{GL}\, w_j^T\, P(\rho_i^{GL}\cos(\theta_j), \rho_i^{GL}\sin(\theta_j)) \ , \ \ \forall P \in \mathbb{P}_n^2 \qquad (18)$$

where $\{\rho_i^{GL}\}$ and $\{w_i^{GL}\}$ are the nodes and weights of the Gauss-Legendre formula of degree of exactness $n+1$ on $[0,1]$ (cf. [6]). Notice that the cubature formula (18) has approximately n^2 nodes, and that it is *stable* since its weights $\rho_i^{GL}\, w_i^{GL}\, w_j^T$ are all positive.

In Figure 4 we show the cubature nodes corresponding to different degrees of exactness in two circular sectors.

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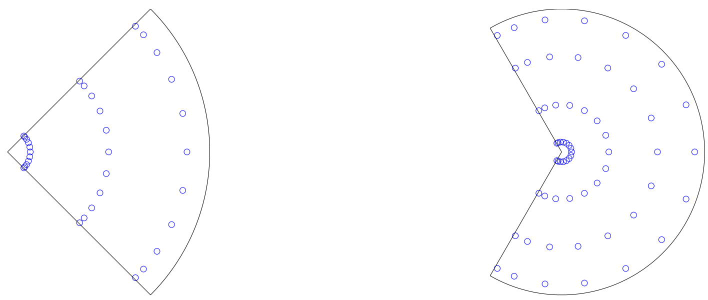

Figure 4: Left: $33 = 11 \times 3$ algebraic cubature nodes of exactness degree 5 on a sector with $\omega = \pi/4$; Right: $60 = 15 \times 4$ algebraic cubature nodes of exactness degree 7 on a sector with $\omega = 2\pi/3$.

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