Subperiodic Dubiner distance, norming meshes and trigonometric polynomial optimization*

Marco Vianello[1]

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Abstract

We extend the notion of Dubiner distance from algebraic to trigonometric polynomials on subintervals of the period, and we obtain its explicit form by the Szegő variant of Videnskii inequality. This allows to improve previous estimates for Chebyshev-like trigonometric norming meshes, and suggests a possible use of such meshes in the framework of multivariate polynomial optimization on regions defined by circular arcs.

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1 Introduction

The notion of Dubiner distance generated by algebraic polynomials on a compact set $K \subset \mathbb{R}^d$, introduced in the seminal paper [9], namely

$$dub(x,y) = \sup_{deg(p)\geq 1,\, \|p\|_K \leq 1} \left\{ \frac{|\arccos(p(x)) - \arccos(p(y))|}{deg(p)} \right\} \;,\quad x,y \in K \;, \quad (1)$$

plays a deep role in the framework of polynomial approximation. Here $\|\cdot\|_K$ denotes the sup-norm on K, and $deg(p)$ the total degree.

For example it can be proved that good interpolation points for degree n on some standard compact sets are spaced proportionally to $1/n$ in such a distance. This happens with the Morrow-Patterson and the Padua interpolation points on the square [5], or the Fekete points on the cube, ball or simplex (in any dimension), cf. [3]. Indeed, up to now the Dubiner distance is explicitly known

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[1]Department of Mathematics, University of Padova, Italy
corresponding author: marcov@math.unipd.it

only on such compact sets, and on the sphere S^{d-1}; cf. [3]. In particular, on the cube it turns out that

$$dub(x,y) = \max_{1\le i\le d} \{|\arccos(x_i) - \arccos(y_i)|\} \ , \ \ x,y \in [-1,1]^d \ , \qquad (2)$$

due to the fact that, by the Van der Corput-Schaake inequality, on $[-1,1]$ the Dubiner distance is simply the arccos distance. On the other hand, on the sphere it turns out to be just the standard geodesic distance.

The following proposition shows that the Dubiner distance is also intimately related with the notions of norming set and polynomial mesh. It is proved in [15] and the proof is essentially outlined also in [1].

Proposition 1 *Let $K \subset \mathbb{R}^d$ be a compact set and $X \subset K$ such that*

$$\rho = \max_{x\in K} \min_{y\in X} dub(x,y) \le \theta/n \ , \ \ \theta \in (0,\pi/2) \ , \ \ n \in \mathbb{N} \ , \ n \ge 1 \ , \qquad (3)$$

where ρ denotes the Dubiner fill distance of X in K. Then for every algebraic polynomial p with $deg(p) \le n$ the following estimate holds

$$\|p\|_K \le \frac{1}{\cos(\theta)} \, \|p\|_X \ . \qquad (4)$$

Proof. We sketch the simple proof, with the only purpose to make clear the role of the Dubiner distance. Assume without loss of generality that $\|p\|_K = p(x_0) = 1$ for some $x_0 \in K$ (possibly normalizing and/or multiplying p by -1). Since the Dubiner fill distance of X satisfies (3), there exists $y_0 \in X$ such that

$$|\arccos(p(x_0)) - \arccos(p(y_0))| = |\arccos(p(y_0))| \le \frac{\theta \, deg(p)}{n} \le \theta < \frac{\pi}{2} \ .$$

Then, the arccos function being monotonically decreasing and nonnegative, we get $p(y_0) \ge \cos(\theta) > 0$, and finally

$$\|p\|_K = 1 \le \frac{p(y_0)}{\cos\theta} \le \frac{1}{\cos\theta} \, \|p\|_X \ . \quad \square$$

Observe that the subset X in Proposition 1, that does not need in general to be discrete, is usually called a *norming set* for the subspace of polynomials of maximum degree n (for the sup-norm). If (4) holds for a sequence $\{X_n\}$ of finite norming set with $card(X_n) = \mathcal{O}(\mathcal{V}^s)$, where $\mathcal{V} = dim\{p|_K : deg(p) \le n\}$, then this is known as a *polynomial mesh* on K, namely

$$\|p\|_K \le C \, \|p\|_{X_n} \ , \ \ \forall p : \ deg(p) \le n \qquad (5)$$

(with constant $C = 1/\cos(\theta)$), which may be termed *optimal* if $s = 1$, cf. [12, 13]; observe indeed that necessarily $card(X_n) \ge \mathcal{V}$, since X_n is determing for the restriction to K of polynomials with $deg(p) \le n$. We recall that $\mathcal{V} = \binom{n+d}{d} = \mathcal{O}(n^d)$ if K is polynomial determining (i.e., a polynomial vanishing on K vanishes everywhere), which happens for example when K has nonempty interior. But we may have a lower dimension, for example on an algebraic variety: the dimension of trivariate polynomials restricted to the sphere S^2 is $(n+1)^2$.

The notion of polynomial mesh has been playing an emerging role in the field of polynomial approximation during the last decade, from both the theoretical and the computational point of view. Among their properties, we may report that polynomial meshes are invariant under affine transformations, and that they can be extended by algebraic transformations and by finite unions and products.

On the other hand, they turn out to be *nearly optimal for least squares approximation* in the sup-norm, and to contain *Fekete-like discrete extremal subsets* which are conveniently computable and well-suited for polynomial interpolation (having a Lebesgue constant with slow growth in the degree). Moreover, polynomial meshes have also begun to be used in the framework of fully discrete approaches for *polynomial optimization*. We refer the reader, e.g., to [2, 6, 12, 15] and the references therein.

2 Subperiodic trigonometric Dubiner distance

In this paper we extend the notion of Dubiner distance to univariate trigonometric polynomials on subintervals of the period. The distance can be explicitly computed, as we prove with the following

Proposition 2 *Consider definition (1) with p a real trigonometric polynomial and $K = [-\omega, \omega]$, $0 < \omega \leq \pi$. Then*

$$dub(x, y) = |F_\omega(x) - F_\omega(y)| \; , \;\; x, y \in [-\omega, \omega] \; , \tag{6}$$

where

$$F_\omega(x) = 2 \arcsin\left(\frac{\sin(x/2)}{\sin(\omega/2)}\right) \; . \tag{7}$$

Proof. Let p be a real trigonometric polynomial. From the classical Videnskii inequality (a subperiodic trigonometric analogue of the Bernstein polynomial inequality, cf. [19])

$$|p'(x)| \leq deg(p)\, f_\omega(x)\, \|p\|_{[-\omega,\omega]} \; , \;\; x \in (-\omega, \omega) \; , \tag{8}$$

where

$$f_\omega(x) = \frac{\cos(x/2)}{\sqrt{\sin^2(\omega/2) - \sin^2(x/2)}} \; , \tag{9}$$

one can obtain its Szegő variant

$$\left(\frac{p'(x)}{f_\omega(x)}\right)^2 + (deg(p))^2\, (p(x))^2 \leq (deg(p))^2\, \|p\|^2_{[-\omega,\omega]} \; , \;\; x \in (-\omega, \omega) \; , \tag{10}$$

as proved e.g. in [17] with a general approach valid for Bernstein-like inequalities in algebraic and trigonometric polynomial spaces. For $\|p\|_{[-\omega,\omega]} \leq 1$, (10) reads

$$|p'(x)| \leq deg(p)\, f_\omega(x) \sqrt{1 - p^2(x)}\; \|p\|_{[-\omega,\omega]} \; , \tag{11}$$

and assuming without loss of generality that $x \leq y$, we can write (observe that the set of zeros of $1 - p^2$ has zero Lebesgue measure)

$$|\arccos(p(x)) - \arccos(p(y))| = \left| \int_x^y \frac{p'(t)}{\sqrt{1 - p^2(t)}}\, dt \right| \leq \int_x^y \frac{|p'(t)|}{\sqrt{1 - p^2(t)}}\, dt$$

$$\leq deg(p) \int_x^y f_\omega(t)\, dt = deg(p)\, |F_\omega(x) - F_\omega(y)| \ , \tag{12}$$

since

$$F'_\omega(x) = \frac{\cos(x/2)/\sin(\omega/2)}{\sqrt{1 - \sin^2(x/2)/\sin^2(\omega/2)}} = f_\omega(x) \ ,$$

i.e., F_ω is a primitive of f_ω. From (12) we get immediately

$$dub(x, y) \leq |F_\omega(x) - F_\omega(y)| \ .$$

To prove the equality, consider the trigonometric polynomial $T_2(\xi(x)) = \cos(2 \arccos(\xi(x)),\ \xi(x) = \sin(x/2)/\sin(\omega/2)$, which has degree 1. Indeed,

$$\cos(2 \arccos(\xi(x)) = 2 \cos^2(\arccos(\xi(x)) - 1 = 2\xi^2(x) - 1$$

$$= -1 + 2\, \frac{\sin^2(x/2)}{\sin^2(\omega/2)} = -1 + 2\, \frac{1 - \cos(x)}{2}\, \frac{1}{\sin^2(\omega/2)} \ .$$

Then we get $\arccos(T_2(\xi(x))) = 2 \arccos(\xi(x)) = \pi - 2 \arcsin(\xi(x)) = \pi - F_\omega(x)$, from which it follows that $|\arccos(T_2(\xi(x))) - \arccos(T_2(\xi(y)))| = |F_\omega(x) - F_\omega(y)|$ and thus $dub(x, y) = |F_\omega(x) - F_\omega(y)|$. $\square$

3 Subperiodic trigonometric norming meshes

Now, Proposition 1 is still valid for the trigonometric Dubiner distance, because the proof does not resort to the fact that the polynomials are algebraic, and "mutatis mutandis" can be applied to trigonometric polynomials.

Notice that for $\omega = \pi$ we get $dub(x, y) = |x - y|$. This implies a well-known inequality proved by Ehlich and Zeller in [10] for trigonometric polynomials, namely that $2mn$ equally spaced angles in (a, b) with spacing $\pi/(2mn)$, where $b - a = 2\pi$, $m > 1$, $n \geq 1$, form a trigonometric polynomial mesh in n with constant $C = 1/\cos(\theta)$, $\theta = \pi/(2m)$ (we stress that trigonometric norming inequalities are translation invariant).

On the other hand, we can now apply Proposition 1 also for $\omega < \pi$, since we have at hand the explicit expression of the subperiodic Dubiner distance. This allows us to refine recent estimates obtained for Chebyshev-like trigonometric norming meshes on $[-\omega, \omega]$, cf. [18].

We shall use the following nonlinear transformation

$$\sigma_\omega : [-1, 1] \to [-\omega, \omega] \ , \ \ \sigma_\omega(x) = 2 \arcsin(x \sin(\omega/2)) \ , \tag{13}$$

that turned out to be a key tool in recent studies on "subperiodic" trigonometric approximation, see, e.g., [7, 18].

Proposition 3 *Let p be a trigonometric polynomial such that $deg(p) \leq n$, $0 < \omega \leq \pi$ and $N = \lceil 2mn \rceil$. Let $\mathcal{T}_N = \{\cos(\phi_i)\}$ be either the set of N Chebyshev zeros, $\phi_i = (2i - 1)\pi/(2N)$, $1 \leq i \leq N$, or the set of $N + 1$ Chebyshev extrema, $\phi_i = i\pi/N$, $0 \leq i \leq N$. Then the following estimate holds*

$$\|p\|_{[-\omega,\omega]} \leq \frac{1}{\cos(\pi/(2m))}\, \|p\|_{\sigma_\omega(\mathcal{T}_N)} \tag{14}$$

for every $m > 1$.

Proof. Define the Chebyshev-like angles in $[-\omega,\omega]$ as

$$\xi_i = \sigma_\omega(\cos(\phi_i)) \;, \tag{15}$$

that is $\{\xi_i\} = \sigma_\omega(\mathcal{T}_N)$, and observe that

$$F_\omega(\sigma_\omega(\cos(\phi))) = 2\arcsin(\cos(\phi)) = \pi - 2\phi \;,$$

and thus

$$dub(\xi,\eta) = 2|\phi-\psi| \;\;,\;\; \xi = \sigma_\omega(\cos(\phi)) \;,\; \eta = \sigma_\omega(\cos(\psi)) \;.$$

Then, the Dubiner fill distance of the discrete set $\{\xi_i\}$ is

$$\rho = \max\left\{dub(\xi_1,-\omega), dub(\xi_N,\omega), \frac{1}{2}\max_i dub(\xi_{i+1},\xi_i)\right\}$$

$$= \max\left\{2\phi_1, 2(\pi-\phi_N), \max_i |\Delta\phi_i|\right\} \;. \tag{16}$$

In the case of the Chebyshev zeros we have $|\Delta\phi_i| = \pi/N$ and $\phi_1 = (\pi-\phi_N) = \pi/(2N)$, whereas for the Chebyshev extrema $|\Delta\phi_i| = \pi/N$ and $\phi_1, (\pi-\phi_N) = 0$, from which we get the following estimate of the Dubiner fill distance

$$\rho \le \frac{\pi}{N} \le \frac{\pi}{2mn} \;, \tag{17}$$

that by the trigonometric version of Proposition 1 gives (14) for $m > 1$. $\square$

Remark 1 By angle translation, Proposition 3 can be extended to any interval $J = [\alpha,\beta]$ with $\omega = (\beta-\alpha)/2 \le \pi$, where the relevant mesh is

$$\sigma_J(\mathcal{T}_N) = \sigma_\omega(\mathcal{T}_N) + (\alpha+\beta)/2 \;. \tag{18}$$

Remark 2 We observe that Proposition 3 improves the results in [18], where the constant of the norming mesh in (14) is

$$C_1 = \frac{1}{1-\pi/(2m)} > C_* = \frac{1}{\cos(\pi/(2m))} \;. \tag{19}$$

In particular, $C_1 = 1 + \mathcal{O}(m^{-1})$ whereas $C_* = 1 + \mathcal{O}(m^{-2})$. Using the trigonometric Dubiner distance, a similar improvement can be obtained also for the constants of the general Jacobi-like norming meshes in ([15]).

4 Trigonometric polynomial optimization

The improvement (19) appears to be of little impact at a first glance, and indeed it is for applications where the closeness of C to 1 has a little importance (such as least squares approximation on polynomial meshes, cf. [2, 6]). Nonetheless, it becomes relevant, for example, whenever the notion of polynomial mesh is used for approximate continuous optimization. The reason is summarized by the following elementary proposition, where the polynomial can be algebraic as well as trigonometric. With no loss of generality, we may refer to a minimization problem.

Proposition 4 *Assume that (5) holds. Then, for every polynomial p with $deg(p) \leq n$, we can write the range-relative minimization error estimate*

$$\min_{X_n} p - \min_K p \leq (C-1) \left(\max_K p - \min_K p \right) . \qquad (20)$$

Proof. Consider the nonnegative polynomial $q(x) = \max_K p - p(x)$. We have that $\|q\|_K = \max_K p - \min_K p$ and $\|q\|_{X_n} = \max_K p - \min_{X_n} p$. Then by (5) applied to q

$$\|q\|_K - \|q\|_{X_n} = \min_{X_n} p - \min_K p \leq (C-1)\,\|q\|_{X_n} \leq (C-1)\,\|q\|_K , \qquad (21)$$

that is (20). $\square$

Remark 3 The error estimate (20) is relative to the function range, as usual in polynomial optimization, and $\min_{X_n} p$ is termed a $(1-\varepsilon)$-approximation to $\min_K p$, with $\varepsilon = C - 1$ (cf., e.g., [8]). Since (21) holds for an arbitrary polynomial, then we are also approximating its maximum modulus with a relative error (standard notion) bounded by $C - 1$.

Now, we can apply Proposition 4 directly to trigonometric polynomials with $K = [\alpha, \beta]$, $\beta - \alpha \leq 2\pi$ (see Remark 1). In view of (19), using the new constant C_* instead of C_1 entails an error estimate

$$C_* - 1 \sim (\pi^2/8)m^{-2} , \qquad (22)$$

by discrete optimization of a trigonometric polynomial of degree n on approximately $2mn$ subperiodic Chebyshev-like angles. This is exactly the error bound obtainable for algebraic polynomial optimization on a real interval by approximately mn Chebyshev nodes, in view of the classical Ehlich-Zeller estimates in [10] (see also [4] and [15, 20] with the references therein).

Starting from univariate algebraic and subperiodic trigonometric polynomial optimization, we have at hand the tools for grid-based multivariate algebraic polynomial optimization on several regions defined by circular arcs, such as sections of disk, sphere, ball, torus. We give a brief description of the technique, whose base is essentially the following fact: *a multivariate algebraic polynomial restricted to an arc of a circle is a univariate trigonometric polynomial of the same degree*.

Consider for example a torus in three-dimensional space with external radius R and internal radius r, whose parametric equations are

$$(x, y, z) = \tau(\theta_1, \theta_2) = ((R + r\cos(\theta_1))\cos(\theta_2), (R + r\cos(\theta_1))\sin(\theta_2), r\sin(\theta_1)) , \qquad (23)$$

$(\theta_1, \theta_2) \in [-\pi, \pi] \times [-\pi, \pi]$, where θ_1 and θ_2 are usually termed the *poloidal* and *toroidal* (angular) coordinates, respectively. Consider a "poloidal-toroidal rectangle" of the torus

$$K = \tau(J_1 \times J_2) , \quad J_1, J_2 \subseteq [-\pi, \pi] , \qquad (24)$$

where J_1 and J_2 are two closed subintervals. Denote by $\mathbb{T}_n(J)$ the univariate trigonometric polynomials of degree $\leq n$, restricted to an interval J. Then, for every trivariate algebraic polynomial p such that $deg(p) \leq n$ we have that

$p(\tau(\theta_1, \theta_2))$ belongs to the tensor-product space $\mathbb{T}_n(J_1) \bigotimes \mathbb{T}_n(J_2)$, thus we can write the polynomial mesh estimate

$$\|p\|_K = \|p\|_{\tau(J_1 \times J_2)} = \|p \circ \tau\|_{J_1 \times J_2} \le C_*^2 \, \|p \circ \tau\|_{G_N} = C_*^2 \, \|p\|_{\tau(G_N)} \ , \quad (25)$$

where $G_N = \sigma_{J_1}(\mathcal{T}_N) \times \sigma_{J_2}(\mathcal{T}_N)$ (see Remark 1 for the definition of σ_J). By Proposition 4, discrete optimization on this subperiodic Chebyshev-like grid approximates the continuous extremal values with a range-relative error bound

$$C_*^2 - 1 = \frac{1}{\cos^2(\pi/(2m))} - 1 = \tan^2(\pi/(2m)) \sim (\pi^2/4)m^{-2} \ . \quad (26)$$

The same approach can be used for polynomial optimization on a generalized "latitude-longitude rectangle" of a sphere of radius r, that corresponds to $R = 0$ in (23)-(24) (the usual spherical coordinates). See Figure 1 for two examples on the sphere and on the torus.

More generally, working by the appropriate geometric transformation and coordinates we can apply the discrete optimization method on standard sections of disk, sphere and ball, such as caps, lenses, lunes, sectors, slices; see [11, 16] for several instances of this kind with the relevant geometric transformations. The corresponding polynomial meshes have constant $C = C_*^2$ (planar and surface instances) or $C = C_*^3$ (solid instances), and cardinality $\mathcal{O}((mn)^2)$ or $\mathcal{O}((mn)^3)$, respectively. By Proposition 4, the polynomial optimization error bound is (26) for planar and surface regions, or

$$C_*^3 - 1 = (C_* - 1)(C_*^2 + C_* + 1) \sim (\pi^2/8)m^{-2}(3 + 3(\pi^2/8)m^{-2}) \sim (3\pi^2/8)m^{-2} \quad (27)$$

for solid regions. This means that to compute a $(1 - \varepsilon)$-approximation to its extremal values on the region, it is then sufficient to sample a polynomial of degree n on $\mathcal{O}(\varepsilon^{-1}n^2)$ points, or $\mathcal{O}(\varepsilon^{-3/2}n^3)$ points, respectively.

Clearly, the procedure is not restricted to dimension 2 and 3. For example, it can be applied to regions of the hypersphere S^{d-1} defined by angular subintervals of the period in generalized spherical coordinates, producing Chebyshev-like grids that are norming meshes with constant C_*^{d-1} and cardinality $\mathcal{O}((mn)^{d-1})$. Discrete polynomial optimization on such meshes gives a range-relative error bounded by

$$C_*^{d-1} - 1 = (C_* - 1)(C_*^{d-2} + C_*^{d-3} + \cdots + C_* + 1) \sim ((d-1)\pi^2/8)m^{-2} \ . \quad (28)$$

As already observed in [15, 20] for mesh-based polynomial optimization on cubes, this is a sort of brute-force approach, that could be useful (in low dimension and with relatively small degrees) when a rough estimate of the extremal values is sought without resorting to more sophisticated optimization methods, or conversely as a starting guess for such methods.

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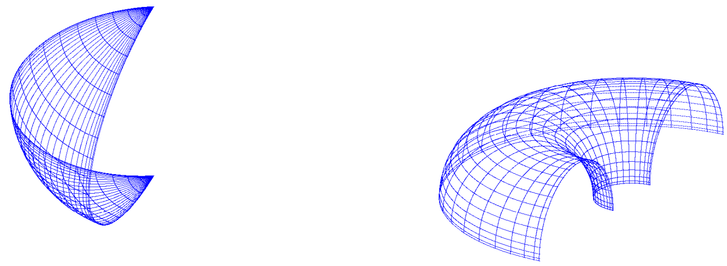

Figure 1: 25×25 subperiodic Chebyshev-like grids on a spherical lune and on a torus section ($mn = 12$); both grids guarantee a range-relative polynomial optimization error of $\tan^2(\pi/(2m))$ for degree n, that is approximately 0.17 for $n = 3$ ($m = 4$) or 0.07 for $n = 2$ ($m = 6$).

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