On the Lebesgue constant of subperiodic trigonometric interpolation*

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Abstract

We solve a recent conjecture, proving that the Lebesgue constant of Chebyshev-like angular nodes for trigonometric interpolation on a subinterval $[-\omega, \omega]$ of the full period $[-\pi, \pi]$ is attained at $\pm\omega$, its value is independent of ω and coincides with the Lebesgue constant of algebraic interpolation at the classical Chebyshev nodes in $(-1, 1)$.

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1 Introduction

In several recent papers, *subperiodic* trigonometric interpolation and quadrature have been studied, i.e., interpolation and quadrature formulas exact on

$$\mathbb{T}_n([-\omega, \omega]) = \text{span}\{1, \cos(k\theta), \sin(k\theta),\ 1 \leq k \leq n\ ,\ \ \theta \in [-\omega, \omega]\}\ , \qquad (1)$$

where $0 < \omega \leq \pi$; cf. [2, 5, 6]. These are related by a simple nonlinear transformation to interpolation and quadrature on $[-1, 1]$, and have been called "subperiodic" since they concern subintervals of the period of trigonometric polynomials.

For any fixed trigonometric degree $\leq n$, consider the $2n + 1$ angles

$$\theta_j := \theta_j(n, \omega) = 2 \arcsin(\sin(\omega/2)\tau_j) \in (-\omega, \omega)\ ,\ \ j = 0, 1, \ldots, 2n\ , \qquad (2)$$

where $0 < \omega \leq \pi$, and

$$\tau_j := \tau_{j,2n+1} = \cos\left(\frac{(2j + 1)\pi}{2(2n + 1)}\right) \in (-1, 1)\ ,\ \ j = 0, 1, \ldots, 2n \qquad (3)$$

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are the zeros of the $2n+1$-th Chebyshev polynomial $T_{2n+1}(x)$. Denoting by

$$\ell_j(x) = T_{2n+1}(x)/(T'_{2n+1}(\tau_j)(x-\tau_j)) \tag{4}$$

the j-th algebraic Lagrange polynomial (of degree $2n$) for the nodes $\{\tau_j\}$, $\ell_j(\tau_k) = \delta_{jk}$, in [2] it has been proved that the *cardinal functions* for trigonometric interpolation at the angular nodes (2) can be written explicitly as

$$L_n(\theta) = L_n^{\omega}(\theta) = \ell_n(x) \tag{5}$$

and for $j \neq n$

$$\begin{aligned} L_j(\theta) = L_j^{\omega}(\theta) &= \frac{1}{2}\left(\ell_j(x) + \ell_{2n-j}(x)\right)\left(1 + \frac{\tau_j^2}{\sin(\theta_j)}\,\frac{\sin(\theta)}{x^2}\right) \\ &= a_j(\theta)\,\ell_j(x) + b_j(\theta)\,\ell_{2n-j}(x)\ , \end{aligned} \tag{6}$$

where

$$x = x(\theta) = \frac{\sin(\theta/2)}{\sin(\omega/2)} \in [-1,1] \tag{7}$$

with inverse

$$\theta = \theta(x) = 2\arcsin(\sin(\omega/2)x) \in [-\omega,\omega]\ , \tag{8}$$

and

$$a_j(\theta) = \frac{1}{2}\left(1 + \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\right)\ ,\ \ b_j(\theta) = \frac{1}{2}\left(1 - \frac{\cos(\theta/2)}{\cos(\theta_j/2)}\right) = 1 - a_j(\theta)\ . \tag{9}$$

It is worth recalling that the key role played by the transformation (7) on subintervals of the period was also recognized in [1, E.3, p. 235], and more recently in [8], in the context of trigonometric polynomial inequalities.

Moreover, in [2] stability of such Chebyshev-like subperiodic trigonometric interpolation has been studied, proving that its Lebesgue constant increases logarithmically in the degree

$$\sum_{j=0}^{2n} |L_j(\theta)| \leq \frac{1}{\sqrt{1-\alpha^2}} \sum_{j=0}^{2n} |\ell_j(x)| \leq \frac{1}{\sqrt{1-\alpha^2}} \left(1 + \frac{2}{\pi}\log(2n+1)\right)\ ,$$

where $\alpha = \sin(\omega/2)$, $\omega < \pi$. This estimate is useless for $\omega \to \pi$ ($\alpha \to 1$), but in view of numerical evidences (see Figure 1), it has been there conjectured essentially that: *the Lebesgue constant of the angular nodes (2) is attained at $\theta = \pm\omega$, its value is independent of ω and coincides with the Lebesgue constant of algebraic interpolation at the classical Chebyshev nodes (3).*

In this note we prove that the conjecture holds, so that the Lebesgue constant has a logarithmic bound independent of ω.

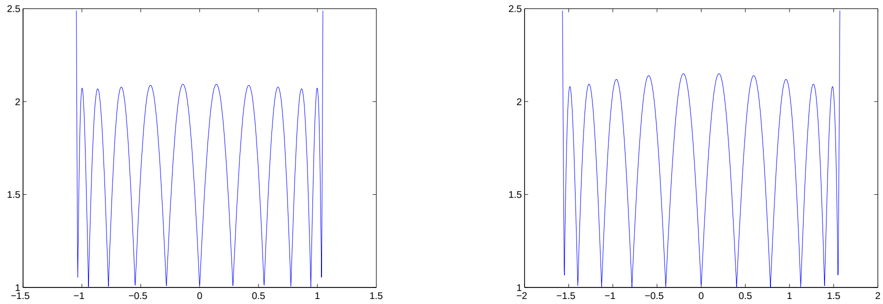

Figure 1: Lebesgue functions for degree $n = 5$ corresponding to the angular nodes (2) for $\omega = \pi/3$ (left) and $\omega = \pi/2$ (right).

2 Bounding the Lebesgue constant

We begin with the following

Lemma 1 *Let us consider the angles $\{\theta_j\}$ in (2) and the corresponding cardinal functions $\{L_j(\theta)\}$ in (6). Moreover, let us consider $\{\phi_j\}$ and $\{L_j^{\pi}(\phi)\}$, i.e., the (equally spaced) angles in $(-\pi, \pi)$*

$$\phi_j = \frac{2(j-n)\pi}{2n+1} = 2\arcsin(\tau_j) \ , \quad j = 0, 1, \dots, 2n \ , \tag{10}$$

and the corresponding cardinal functions for $\omega = \pi$, where

$$\phi = 2\arcsin(x) = 2\arcsin(\sin(\theta/2)/\sin(\omega/2)) \tag{11}$$

with inverses

$$x = \sin(\phi/2) \ , \quad \theta = 2\arcsin(\sin(\omega/2)x) = 2\arcsin(\sin(\omega/2)\sin(\phi/2)) \ , \tag{12}$$

cf. (7)-(8).

Then, for every $\omega \in (0, \pi)$ the following inequality holds

$$|L_j(\theta)| + |L_{2n-j}(\theta)| \le |L_j^{\pi}(\phi)| + |L_{2n-j}^{\pi}(\phi)| \ , \ j = 0, 1, \dots, 2n \ ,$$

and in particular for $|\theta| \ge |\theta_j|$, (i.e., $|x| \ge |\tau_j|$ and $|\phi| \ge |\phi_j|$)

$$|L_j(\theta)| + |L_{2n-j}(\theta)| = |\ell_j(x)| + |\ell_{2n-j}(x)| = |L_j^{\pi}(\phi)| + |L_{2n-j}^{\pi}(\phi)| \ .$$

Proof. First, notice that in view of (4), if $|x| \ge |\tau_j|$ the sign of $\ell_j(x)$ is the same of $\ell_{2n-j}(x)$, whereas if $|x| < |\tau_j|$ the sign of $\ell_j(x)$ is opposite to that of $\ell_{2n-j}(x)$. Indeed, $T_{2n+1}(x)$ is odd, $T'_{2n+1}(x)$ is even, and $\tau_{2n-j} = -\tau_j$.

Then, the sign of $L_j(\theta)$ and $L_j^\pi(\phi)$ is the same of $\ell_j(x)$. Consider the representation (6)-(9), and observe that $a_j(\theta) \geq 0$ since $\theta/2, \theta_j/2 \in (-\pi/2, \pi/2)$. Moreover, for $|x| \geq |\tau_j|$ we have $|\theta| \geq |\theta_j|$ and $b_j(\theta) \geq 0$, whereas for $|x| < |\tau_j|$ we have $|\theta| < |\theta_j|$ and $b_j(\theta) < 0$. It follows that $\ell_j(x)$ has the same sign of $a_j(\theta)\,\ell_j(x)$ and of $b_j(\theta)\,\ell_{2n-j}(x)$, and thus also of $L_j(\theta)$. The case of the sign of $L_j^\pi(\phi)$ is completely analogous.

Consider now $|L_j(\theta)| + |L_{2n-j}(\theta)|$. If $|x| \geq |\tau_j|$ (i.e., $|\theta| \geq |\theta_j|$ and $|\phi| \geq |\phi_j|$) then $L_j(\theta)$ and $L_{2n-j}(\theta)$ have the same sign, along with $L_j^\pi(\phi)$ and $L_{2n-j}^\pi(\phi)$, hence

$$\begin{aligned} |L_j(\theta)| + |L_{2n-j}(\theta)| &= |L_j(\theta) + L_{2n-j}(\theta)| = |\ell_j(x) + \ell_{2n-j}(x)| \\ &= |L_j^\pi(\phi) + L_{2n-j}^\pi(\phi)| = |L_j^\pi(\phi)| + |L_{2n-j}^\pi(\phi)| \,, \end{aligned}$$

but since

$$|\ell_j(x) + \ell_{2n-j}(x)| = |\ell_j(x)| + |\ell_{2n-j}(x)|$$

the equality case follows immediately.

On the contrary, if $|x| < |\tau_j|$ then $L_j(\theta)$ and $L_{2n-j}(\theta)$ have opposite sign and the same holds for $L_j^\pi(\phi)$ and $L_{2n-j}^\pi(\phi)$. Thus

$$|L_j(\theta)| + |L_{2n-j}(\theta)| = |L_j(\theta) - L_{2n-j}(\theta)| = \left| \frac{\cos(\theta/2)}{\cos(\theta_j/2)} (\ell_j(x) - \ell_{2n-j}(x)) \right|$$

$$= \frac{\cos(\theta/2)}{\cos(\theta_j/2)} |\ell_j(x) - \ell_{2n-j}(x)| < \frac{\cos(\phi/2)}{\cos(\phi_j/2)} |\ell_j(x) - \ell_{2n-j}(x)|$$

$$= \left| \frac{\cos(\phi/2)}{\cos(\phi_j/2)} (\ell_j(x) - \ell_{2n-j}(x)) \right| = |L_j^\pi(\phi) - L_{2n-j}^\pi(\phi)| = |L_j^\pi(\phi)| + |L_{2n-j}^\pi(\phi)| \,,$$

as soon as we prove that $\frac{\cos(\theta/2)}{\cos(\theta_j/2)} < \frac{\cos(\phi/2)}{\cos(\phi_j/2)}$ for $|x| < |\tau_j|$.

In fact, the latter becomes

$$\frac{\cos(\arcsin(\sin(\omega/2)x))}{\cos(\arcsin(\sin(\omega/2)\tau_j))} < \frac{\cos(\arcsin(x))}{\cos(\arcsin(\tau_j))} \,,$$

that is

$$\sqrt{\frac{1 - \sin^2(\omega/2)x^2}{1 - \sin^2(\omega/2)\tau_j^2}} < \sqrt{\frac{1 - x^2}{1 - \tau_j^2}}$$

or

$$\frac{1 - \sin^2(\omega/2)x^2}{1 - \sin^2(\omega/2)\tau_j^2} < \frac{1 - x^2}{1 - \tau_j^2} \,.$$

Now, this is equivalent to

$$1 - \sin^2(\omega/2)x^2 - \tau_j^2 + \sin^2(\omega/2)x^2\tau_j^2 < 1 - x^2 - \sin^2(\omega/2)\tau_j^2 + \sin^2(\omega/2)x^2\tau_j^2 \,,$$

that is

$$x^2(1 - \sin^2(\omega/2)) < \tau_j^2(1 - \sin^2(\omega/2))$$

which holds true since we have assumed $|x| < |\tau_j|$. $\square$

We can now state and prove the main result of this note.

Theorem 1 *The maximum of the Lebesgue function of the subperiodic interpolation angles $\{\theta_j\}$ in (2), i.e., their Lebesgue constant, is attained at $\theta = \pm\omega$, its value is independent of ω, and satisfies*

$$\Lambda_n = \max_{\theta\in[-\omega,\omega]} \sum_{j=0}^{2n} |L_j(\theta)| = \max_{\phi\in[-\pi,\pi]} \sum_{j=0}^{2n} |L_j^\pi(\phi)| = \sum_{j=0}^{2n} |L_j(\pm\pi)|$$

$$= \sum_{j=0}^{2n} |L_j(\pm\omega)| = \sum_{j=0}^{2n} |\ell_j(\pm 1)| = \max_{x\in[-1,1]} \sum_{j=0}^{2n} |\ell_j(x)| \,. \tag{13}$$

Proof. From Lemma 1 it follows immediately that

$$\sum_{j=0}^{2n} |L_j(\theta)| \leq \sum_{j=0}^{2n} |L_j^\pi(\phi)|$$

and in particular for $\theta = \pm\omega$

$$\sum_{j=0}^{2n} |L_j(\pm\omega)| = \sum_{j=0}^{2n} |L_j^\pi(\pm\pi)| = \sum_{j=0}^{2n} |\ell_j(\pm 1)| \,.$$

Now, by a classical result by Ehlich and Zeller [7], the maximum of the Lebesgue function of trigonometric interpolation of degree at most n at the $2n+1$ equally spaced angular nodes $\{\phi_j\}$ in (10), is attained (also) at $\phi = \pm\pi$. In fact, their Lebesgue function is periodic with period $2\pi/(2n+1)$, and takes its maximum value at the midpoint of each interval $[\phi_j, \phi_{j+1}]$, $j = -1, 0, \ldots, 2n$. On the other hand, also the maximum of the Lebesgue function of algebraic interpolation at the classical Chebyshev nodes (3) is attained at $x = \pm 1$, cf. [3]. Then

$$\sum_{j=0}^{2n} |L_j(\pm\omega)| \leq \max_{\theta\in[-\omega,\omega]} \sum_{j=0}^{2n} |L_j(\theta)| \leq \max_{\phi\in[-\pi,\pi]} \sum_{j=0}^{2n} |L_j^\pi(\phi)| = \sum_{j=0}^{2n} |L_j^\pi(\pm\pi)|$$

$$= \sum_{j=0}^{2n} |L_j(\pm\omega)| = \sum_{j=0}^{2n} |\ell_j(\pm 1)| = \max_{x\in[-1,1]} \sum_{j=0}^{2n} |\ell_j(x)| \,. \quad \square$$

Remark 1 Observe that in view of well-known estimates (see [3, 4]) and Theorem 1 above

$$\Lambda_n = \max_{\theta\in[-\omega,\omega]} \sum_{j=0}^{2n} |L_j(\theta)| = \max_{x\in[-1,1]} \sum_{j=0}^{2n} |\ell_j(x)|$$

$$= \max_{\phi\in[-\pi,\pi]} \sum_{j=0}^{2n} |L_j^{\pi}(\phi)| \leq \frac{2}{\pi}\log(n) + \delta_n \ ,$$

where the sequence δ_n decreases monotonically from 5/3 to its infimum $(2/\pi)(\log(16/\pi)+\gamma) = 1.40379\ldots$, γ being the Euler-Mascheroni constant.

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