Random sampling and unisolvent interpolation by almost everywhere analytic functions

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Abstract

We prove *a.s.* (almost sure) unisolvency of interpolation by continuous random sampling with respect to any given density, in spaces of multivariate *a.e.* (almost everywhere) analytic functions. Examples are given concerning polynomial and RBF approximation.

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1. The main result

This note is aimed at proving a general result on unisolvency of multivariate interpolation in spaces of *a.e.* (almost everywhere) analytic functions, by continuous random sampling with any density. Such a result might already be known in some specific function space, and we do not even attempt to give a partial overview of the vast literature on random sampling and its connections with approximation theory, quoting for example [1, 3, 14, 21] among recent contributions, with the references therein. On the contrary, we wish to emphasize that in the random setting there is a common framework for apparently quite different approaches, such as for example polynomial approximation and RBF approximation.

Such a common framework is given by real analytic functions. We recall that a function is real analytic on an open subset $\Omega \subset \mathbb{R}^d$ if for each $x \in \Omega$ the function may be represented by a convergent power series in some neighborhood of x; cf. [13, Ch.2].

Theorem 1.1. *Let Ω be an open connected subset of $\mathbb{R}^d$ and $\{f_j\}_{j\geq 1}$ a set of functions defined on Ω, such that*

i) each f_j is real analytic up to a set of null Lebesgue measure, say $I_j \subset \Omega$;

ii) the $\{f_j\}_{j\geq 1}$ are linearly independent on every connected component of $\Omega \setminus I$, where $I = \bigcup_{j=1}^{m} I_j$.

Moreover, let $\{x_i\}_{i\geq 1}$ be a randomly distributed sequence on Ω with respect to any given probability density $\sigma(x)$, i.e. a point sequence produced by sampling a sequence of continuous random variables $\{X_i\}_{i\geq 1}$ which are independent and identically distributed in Ω with density $\sigma(x)$. Then, for every $m \geq 1$ the matrix $V_m = [f_j(x_i)]$, $1 \leq i, j \leq m$, is a.s. *(almost surely) nonsingular.*

Proof. We proceed by induction on m.

First, we prove the assertion for $m = 1$. In fact, $det(V_1) = f_1(x_1) = 0$ iff x_1 falls on the zero set of $det(f_1(x)) = f_1(x)$, $x \in \Omega$. Now, the zero set of f_1 in Ω, say $\mathcal{Z}(f_1)$, is the disjoint union of the zero set in

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$\Omega \setminus I$, say $\mathcal{Z}_1(f_1)$, with the possible zero set in I, say $\mathcal{Z}_2(f_1)$. Observe that f_1 is real analytic on the set $\Omega \setminus I$, which is open since I is closed in the induced topology of Ω, each f_j being analytic in $\Omega \setminus I_j$. Moreover, by assumption f_1 is not identically zero on each connected component of $\Omega \setminus I$. Notice that such connected components are at most countable.

Hence the zero set $\mathcal{Z}_1(f_1)$, which is the union of the zero sets of the connected components of $\Omega \setminus I$, has null Lebesgue measure by a well-known basic result of measure theory, asserting that the zero set of a not identically zero real analytic function on an open connected set in $\mathbb{R}^d$ has null Lebesgue measure (cf. [16] for an elementary proof).

In turn, the zero set $\mathcal{Z}_2(f_1)$ has null Lebesgue measure being a subset of I, and thus $\mathcal{Z}(f_1)$ has null Lebesgue measure. Consequently, $\mathcal{Z}(f_1)$ has null measure also with respect to the density $\sigma(x)$, since $\int_{\mathcal{Z}(f_1)} \sigma(x)\, dx = 0$, i.e. $f_1(x_1)$ is *a.s.* nonzero (i.e., with probability 1).

Now, let us assume that V_m is nonsingular with probability 1, and consider the $(m+1) \times (m+1)$ matrix $U_{m+1}(x)$ obtained by adding to V_m the $(m+1)$-th column

$$[f_{m+1}(x_1), \dots, f_{m+1}(x_m), f_{m+1}(x)]^t$$

and the $(m+1)$-th row

$$[f_1(x), \dots, f_m(x), f_{m+1}(x)] \; .$$

Applying Laplace rule to the last row, we get that

$$det(U_{m+1}(x)) = det(V_m) f_{m+1}(x) + \alpha_m f_m(x) + \cdots + \alpha_1 f_1(x)$$

where $\alpha_1, \dots, \alpha_m$ are the other corresponding minors with the appropriate sign. Notice that $det(U_{m+1}(x))$ is not identically zero on each connected component of $\Omega \setminus I$ with probability 1, since the $\{f_j\}$ are linearly independent there, and $det(V_m) \neq 0$ with probability 1, by inductive hypothesis.

Thus, by the same arguments of the $m = 1$ instance with x_{m+1} and $det(U_{m+1}(x))$ substituting x_1 and $f_1(x)$, respectively, we get that the probability that x_{m+1} falls on the zero set in Ω of $det(U_{m+1}(x))$ is null, i.e. $V_{m+1} = U_{m+1}(x_{m+1})$ is nonsingular with probability 1. To be more precise,

$$prob\{det(V_{m+1}) = 0\} = prob\{det(V_{m+1}) = 0 \;\&\; det(V_m) = 0\}$$

$$+ prob\{det(V_{m+1}) = 0 \;\&\; det(V_m) \neq 0\} = 0 + 0 = 0 \; ,$$

since the events are disjoint and both their probabilities are null (notice that the first one has null probability since it is a subevent of the event $det(V_m) = 0$, which has null probability by inductive hypothesis). □

Remark 1.1. We stress that the above result is valid with *any distribution density.* For example, in a box, points can have a uniform distribution, but also a normal or a product Chebyshev distribution. It is also worth noticing that the case of everywhere analytic functions has been recently considered in [21], within an abstract context named "μZC sequences" of finite-dimensional function spaces (see also the references therein for previous univariate results).

Remark 1.2. Theorem 1.1 concerns in principle multivariate interpolation by random sampling. On the other hand, *a.s.* (almost sure) unisolvency at random nodes is relevant also in the least-squares framework, since the corresponding rectangular Vandermonde-like matrix V has *a.s.* maximal rank, because it contains an *a.s.* nonsingular interpolation matrix, and the Gram matrix V^tV becomes *a.s.* positive definite.

2. Application to polynomial and RBF interpolation

Below, for the purpose of illustration, we give some applications of the above result in different function spaces. As a first application, we observe that *point sequences of the appropriate length, randomly distributed with respect to any given probability density on an open connected set, are* a.s. *unisolvent for multivariate* *interpolation by polynomials, trigonometric polynomials, and rational functions whose denominator does not vanish on the domain.* Indeed, it is sufficient to observe that both polynomials and trigonometric polynomials are entire functions, whereas rational functions are real analytic. To quote some examples among many others in polynomial approximation, unisolvency at random nodes is useful within different topics such as numerical differentiation by local polynomial interpolation, multinode Shepard-like interpolation, compressed (Quasi)MonteCarlo integration, randomized weakly admissible meshes and polynomial least-squares; cf. e.g. [4, 6, 7, 8, 21] with the references therein.

We can now give a relevant Corollary of Theorem 1.1, concerning RBF interpolation.

Corollary 2.1. *Sequences $x_1, \dots, x_m$ of randomly distributed points with respect to any given probability density on an open connected set $\Omega \subset \mathbb{R}^d$, are* a.s. *unisolvent for multivariate RBF interpolation with fixed distinct centers $\{\xi_1, \dots, \xi_m\} \subset \Omega$ by Gaussians, Multiquadrics (MQ), Inverse Multiquadrics (IMQ), and Thin-Plate Splines (TPS) for $d \geq 1$, and also by Radial Powers (RP) for $d \geq 2$.*

Proof. The quoted RBF are of the form $f_j(x) = \phi(\|x - \xi_j\|_2)$ with univariate radial functions, respectively, $\phi(r) = e^{-r^2}$ (Gaussians), $\phi(r) = (1 + r^2)^{1/2}$ (MQ), $\phi(r) = (1 + r^2)^{-1/2}$ (IMQ), $\phi(r) = r^k$, k odd (RP), and $\phi(r) = r^k \log(r)$, k even (TPS); cf. e.g. [10].

Gaussians, MQ and IMQ are real analytic in $\mathbb{R}^d$, since the corresponding $\phi(r)$ is real analytic in $\mathbb{R}$ and the composition of real analytic functions is real analytic [13, Prop.1.4.2, p.19]. Moreover, they are linearly independent on Ω. Indeed, Gaussians and IMQ are strictly positive definite and the corresponding standard interpolation matrix at the centers, $U = [\phi(\|\xi_i - \xi_j\|_2]$, $1 \leq i, j \leq m$, is positive definite; cf. e.g. [10]. On the other hand, MQ are linearly independent in view of a classical result of Micchelli on the invertibility of U for conditionally positive definite RBF of order 1; cf. [15]. Then Theorem 1.1 applies.

On the other hand, RP and TPS are everywhere continuous, and real analytic with the exception of the center ξ_j, since in both cases $\phi(\sqrt{\cdot}\,)$ is real analytic in $\mathbb{R}^+$. This fact also ensures that for $d \geq 2$ they are linearly independent on the unique connected component of $\Omega \setminus \{\xi_1, \dots, \xi_m\}$. Indeed, if they were dependent there, by continuity they would be dependent also on the whole Ω. But this not possible (for any $d \geq 1$ in fact), since if they were dependent, one of them would be a linear combination of the others, and would then result analytic at its own center. Again, Theorem 1.1 applies.

In the case of univariate TPS, $f_j(x) = |x - \xi_j|^k \log(|x - \xi_j|)$, $\Omega = (a, b)$ and $a < \xi_1 < \xi_2 < \cdots < \xi_m < b$. Assume that a linear combination $f(x) = \alpha_1 f_1(x) + \cdots + \alpha_m f_m(x) \equiv 0$ in $(\xi_\ell, \xi_{\ell+1})$, where we set $\xi_{-1} = a$ and $\xi_{m+1} = b$. Now, since the k-th derivatives $f_\ell^k(x)$ and $f_{\ell+1}^k(x)$ tend to ∞ as $x \to \xi_\ell^+$ and $x \to \xi_{\ell+1}^-$ respectively, due to the presence of the logarithmic factor, necessarily $\alpha_\ell = 0 = \alpha_{\ell+1}$, otherwise $f^k(x)$ would tend to ∞ at ξ_ℓ and $\xi_{\ell+1}$. Then, $f(x)$ being analytic in $(\xi_{\ell-1}, \xi_{\ell+2})$ and identically zero in the subinterval $(\xi_\ell, \xi_{\ell+1})$, it is $f(x) \equiv 0$ in the whole interval $(\xi_{\ell-1}, \xi_{\ell+2})$ (where we set for convenience $\xi_j = a$ for $j < 1$ and $\xi_j = b$ for $j > m$). Repeating the reasoning above on all the progressively enlarging subintervals, we then obtain that $\alpha_j = 0$ for all j, and thus $f_1, \dots, f_m$ are linearly independent on every open subinterval corresponding to consecutive centers, that is on every connected component of $(a, b) \setminus \{\xi_1, \dots, \xi_m\}$, and Theorem 1.1 then applies. $\square$

Remark 2.1. In the univariate case, for $m > k + 1$ univariate RP are certainly linearly dependent in every subinterval determined by consecutive centers, being polynomials of degree k there, and consequently Theorem 1.1 does not apply.

Remark 2.2. The fact that pairwise distinct interpolation points in general do not guarantee unisolvency, is well-known for total-degree polynomial spaces in dimension $d > 1$. This happens, in particular, when the points all lie on the zero set of a polynomial belonging to the same space. Consider, for example, interpolation of total-degree $k > 1$: if the interpolation points lie on the unit sphere $\{x \in \mathbb{R}^d : \|x\|_2 = 1\}$, they cannot be unisolvent, because if $p_k(x)$ is an interpolation polynomial at those points, such is also $p_k(x) + c(\|x\|_2^2 - 1)$ for any constant $c \in \mathbb{R}$.

Dealing with fixed center RBF, again the mere fact that the interpolation points are distinct does not guarantee unisolvency. To give a counterexample, in dimension $d = 2$ consider two basis functions, say $\phi(\|x-\xi\|_2)$ and $\phi(\|x-\eta\|_2)$. Clearly, the functions coincide on the axis of the segment connecting the centers ξ and η. Then, taking all the pairwise distinct interpolations points on such an axis the interpolation matrix is certainly singular since two columns coincide. The counterexample can be extended to any $d>1$ taking the locus of equidistant points from two centers, that is an hyperplane.

Another counterexample can be easily constructed in dimension $d \geq 3$ for an arbitrary finite number of RBF. Let be $2 \leq r < d$, $H = \{(x_1,\dots,x_r,0,\dots,0) \in \mathbb{R}^d\}$, $K = \{(0,\dots,0,x_{r+1},\dots,x_d) \in \mathbb{R}^d\}$. Then H and K are orthogonal and $\mathbb{R}^d = H \oplus K$. Let $S \subset H$ be any sphere centered at the origin and $R>0$ its radius. Then, for any $\xi \in S$ and any $x \in K$ by the Pythagorean Theorem, we get

$$\|x-\xi\|_2^2 = \|x+(-\xi)\|_2^2 = \|x\|_2^2 + \|\xi\|_2^2 = \|x\|_2^2 + R^2 \ .$$

Consequently, if we take at least two interpolation centers on S and all the distinct interpolation points on K, then the interpolation matrix is singular, having at least two equal columns.

Remark 2.3. The fact that least-squares approximation by Thin-Plate Splines (without any polynomial augmentation) can be desiderable in certain applications, has been recognized in the literature (cf. e.g. [17]). In this framework, it is important to have a nonsingular Gram matrix, see Remark 1.2. For the deterministic theory of least-squares RBF approximation we may quote e.g. [10, Ch.20], [12, §3.10] and the classical papers [18, 20].

On the other hand, Corollary 2.1 suggests the following operative procedure for *interpolation by TPS and RP without any polynomial augmentation*. Given a sample at random points $x_1,\dots,x_m$ distributed with respect to any given probability density, we can draw another random point distribution of the same length, say $\xi_1,\dots,\xi_m$ (for example using a uniform distribution or even the same density, the probability that this distributions have common points being null). Then, we can interpolate at $x_1,\dots,x_m$ with the RBF basis $\{\phi(\|x-\xi_j\|_2)\}$, $1 \leq j \leq m$, that is with the interpolation matrix $[\phi(\|x_i-\xi_j\|_2)]$, $1 \leq i,j \leq m$, which will be *a.s.* nonsingular.

In alternative, we can fix a priori a set of distinct centers $\xi_1,\dots,\xi_m$ with any general or application driven criterion, and then proceed by continuous random sampling of length m. Again, the interpolation matrix will be *a.s.* nonsingular.

As a last example, we give the following corollary concerning in particular polynomial interpolation on surfaces.

Corollary 2.2. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a surface that admits an analytic parametrization $x=\psi(u,v)$ from a connected open set $D \subset \mathbb{R}^2$, i.e. $\psi=(\psi_1,\psi_2,\psi_3)$ where $\psi_i : D \to \mathbb{R}^3$ are analytic and $\psi(D)=\mathcal{S}$. Moreover, let $\{p_j\}_{j\geq 1}$ be a set of trivariate polynomials that are linearly independent on $\mathcal{S}$ and $\{(u_i,v_i)\}_{i\geq 1}$ a randomly distributed sequence on D with respect to any given probability density. Then the points $\{x_i = \psi(u_i,v_i)\}_{1\leq i\leq m}$ are* a.s. *unisolvent for polynomial interpolation in* $\mathrm{span}(p_1,\dots,p_m)$.

Proof. The proof is immediate in view of Theorem 1.1, by observing that the functions $f_j(u,v) = p_j(\psi(u,v))$ are linearly independent and real analytic on $\Omega = D$. □

For the purpose of illustration, relevant examples are regions of sphere, torus and cylinder, where there are natural entire parametrizations of trigonometric or algebraic/trigonometric type (spherical, toroidal and cylindrical coordinates), as well as Cartesian graphs of bivariate analytic functions. Some applications of Corollary 2.2 can be found, e.g., in the paper [9] that extends the method in [8] to (Quasi)Monte-Carlo integration on surfaces with a regular analytic parametrization, where it is required that the points be distributed with respect to the surface measure density, that is $\sigma(u,v) = \|\partial_u\psi \times \partial_v\psi\|_2/area(S)$. Another application arises in the context of multinode Shepard interpolation on the sphere with enhanced polynomial reproduction, cf. [5].

Conclusions and perspectives. We have proposed a common framework for multivariate interpolation unisolvency by random sampling, namely spaces of a.e. analytic functions. This framework embodies apparently quite different approaches, such as polynomial interpolation and RBF interpolation.

The scope of the present result could go beyond the discussed examples. For example, one might ask whether it could be somehow applied to prove nonsingularity of unsymmetric collocation matrices within Kansa-like methods for BVPs (which is still an open problem [2, 11, 19]), confining the problem to the case of fixed centers and random collocation nodes. The main difficulty here is not only the fact that one has to deal with both, RBF and their partial derivatives, but above all that one should work with random sampling in different dimensions, say in the interior of the domain and on its boundary (concerning the latter, Corollary 2.2 might give a guideline). This challenging problem will be object of future investigations.

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