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A comparison principle for minimizers

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Abstract. We give some conditions that ensure the validity of a *Comparison principle* for the minimizers of integral functionals, without assuming the validity of the *Euler–Lagrange* equation. We deduce a *weak maximum principle* for (possibly) degenerate elliptic equations and, together with a generalization of the *bounded slope condition*, the Lipschitz continuity of minimizers. To prove the main theorem we give a result on the existence of a representative of a given Sobolev function that is absolutely continuous along the trajectories of a suitable autonomous system. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un principe de comparaison pour les minima

Résumé. *Nous donnons des conditions qui assurent la validité d'un* principe de comparaison *pour les minimums d'une fonctionnelle intégrale qui ne satisfont pas nécessairement à l'*équation d'Euler–Lagrange. *Nous en déduisons un* principe de maximum faible *pour les équations elliptiques* (*éventuellement*) *dégénerées et, en généralisant la* condition de la pente bornée, *la Lipschitz continuité des minimums. La preuve du théorème principal se base sur l'éxistence d'un représentant d'une fonction de Sobolev donnée qui est absolument continu sur les trajectoires d'un système autonome convenable.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Nous fixons un ouvert borné Ω de $\mathbb{R}^n$. La fonction $L(x,z,p)$ est définie dans $\Omega\times\mathbb{R}\times\mathbb{R}^n$ et ℓ est une fonction dans $\mathrm{W}^{1,q}(\Omega)$, $q\geqslant 1$. La fonction $\bar u$ est dans $\mathrm{W}^{1,q}(\Omega)$ et on pose $\mathrm{W}^{1,q}_{\bar u}(\Omega)=\bar u+\mathrm{W}^{1,q}_0(\Omega)$. Dans cette partie nous nous référons aux hypothèses A, A′, B et D du texte anglais qui suit.

THÉORÈME PRINCIPAL 1 ([4]). – *On suppose que* (L,ℓ) *satisfait à l'hypothèse* B. *Soit* w *un minimum de*

$$I(u)=\int_\Omega L\big(x,u(x),\nabla u(x)\big)\,\mathrm{d}x$$

dans $\mathrm{W}^{1,q}_{\bar u}(\Omega)$. *Si* $w\leqslant\ell$ *dans* $\partial\Omega$, *alors* $w\leqslant\ell$ *presque partout dans* Ω.

Note présentée par Haïm BRÉZIS.

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Remarque 1. – Le théorème précédent ne se déduit pas du *principe de comparaison* classique pour les équations elliptiques du moment que nous n'introduisons pas d'hypothèses de croissance sur L (en particulier, nous ne savons pas a priori si w satisfait l'équation d'Euler–Lagrange) ni la condition classique d'ellipticité sur L.

La preuve du théorème 1 sous l'hypothèse B4 se base sur notre généralisation d'un théorème classique ([3], Th. 1.41) qui assure, pour toute fonction de Sobolev donnée, l'existence d'un représentant qui est absolument continu sur les droites.

THÉORÈME 2 ([5]). – *Soient v dans $\mathrm{W}_0^{1,q}(\Omega)$, E, F ouverts dans $\mathbb{R}^n$ et F un sous-ensemble de Ω. Soit $\Lambda : E \to F$ une application bilipschitzienne. Supposons que, pour presque tout ξ dans $\mathbb{R}^{n-1}$, $\Lambda(\partial E \cap \{(t,\xi) : t \in \mathbb{R}\}) \subset \partial\Omega$. Alors il existe un représentant v^* de v dans $\mathrm{W}_0^{1,q}(\Omega)$ qui s'annule sur $\partial\Omega$ et tel que l'application $t \mapsto (v^* \circ \Lambda)(t,\xi)$ est absolument continue pour presque tout ξ. La dérivée classique de $v^* \circ \Lambda$ et la dérivée faible de $v \circ \Lambda$ par rapport à t coïncident presque partout sur E.*

Remarque 2. – Si $E = F = \Omega$ on retrouve le théorème 1.41 de [3] ; dans le cas de notre interêt F est un sous-ensemble propre de Ω.

Nous appliquons le théorème principal aux *équations variationnelles* qui sont de la forme $\operatorname{div} L_p(x,u,\nabla u) - L_z(x,u,\nabla u) = 0$ et nous obtenons un *principe de maximum faible* pour l'équation elliptique (éventuellement dégénerée) $\operatorname{div}(A(x)\nabla u)) - c(x)u = 0$. Nous généralisons ensuite dans [4] la *condition de la pente bornée* introduite dans le contexte variationnel par Stampacchia dans [6].

DÉFINITION 1 (condition de la pente bornée généralisée $(\mathrm{CPBG})_K$). – Nous disons que le couple $(L,\bar{u})$ satisfait la $(\mathrm{CPBG})_K$ si $\bar{u} : \Omega \to \mathbb{R}$ est lipschitzienne de constante K et pour tout $\bar{x}$ dans $\partial\Omega$ il existe deux fonctions lipschitziennes $\ell_{\bar{x}}^1$, $\ell_{\bar{x}}^2 : \Omega \to \mathbb{R}$ de constante K telles que les couples $(L,\ell_{\bar{x}}^2)$ (resp. $(L,\ell_{\bar{x}}^1)$) vérifient l'Hypothèse A (resp. A$'$) et

$$\forall x \in \partial\Omega \quad \ell_{\bar{x}}^1(x) \leqslant \bar{u}(x) \leqslant \ell_{\bar{x}}^2(x) \quad \text{et} \quad \ell_{\bar{x}}^1(\bar{x}) = \bar{u}(\bar{x}) = \ell_{\bar{x}}^2(\bar{x}).$$

THÉORÈME 3 ([4]). – *Supposons que $L(x,z,p) = f(p) + g(x,z)$ vérifie l'Hypothèse* D *et que $(L,\bar{u})$ satisfait la* $(\mathrm{CPBG})_K$. *Soit w une solution du problème* :

$$\textit{minimiser} \quad I(u) = \int_\Omega L(x,u,\nabla u)\,\mathrm{d}x \quad \textit{dans } \mathrm{W}_{\bar{u}}^{1,q}(\Omega).$$

Alors w est lipschitzienne et satisfait l'équation d'Euler–Lagrange faible :

$$\int_\Omega f_p\big(\nabla w(x)\big) \cdot \nabla\varphi(x) + g_z\big(x,w(x)\big)\varphi(x)\,\mathrm{d}x = 0,$$

pour toute fonction φ dans $\mathcal{C}_c^\infty(\Omega)$.

Remarque 3. – Les théorèmes 1 et 3 généralisent des résultats récents obtenus par Cellina dans [1] pour des fonctions $L(x,z,p) = f(p)$ de la seule variable p avec ℓ affine.

We fix an open bounded subset Ω of $\mathbb{R}^n$. The function $L(x,z,p) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a normal integrand. Let $q \geqslant 1$, $1/q + 1/q' = 1$; for k in $\mathrm{L}^{q'}(\Omega;\mathbb{R}^n)$ and h in $\mathrm{L}^{q'}(\Omega)$ we write that $\operatorname{div} k - h \leqslant 0$ (resp. $\geqslant 0$) weakly if

$$\forall \eta \in \mathrm{W}_0^{1,q}(\Omega),\ \eta \geqslant 0, \quad \int_\Omega k \cdot \nabla\eta + h\eta\,\mathrm{d}x \geqslant 0 \quad (\text{resp. } \leqslant 0). \tag{1}$$

We also write that $\operatorname{div} k - h < 0$ (resp. $\operatorname{div} k - h > 0$) or that the inequality $\operatorname{div} k - h \leqslant 0$ (resp. $\operatorname{div} k - h \geqslant 0$) is *strict* if (1) is strict whenever $\eta \neq 0$. In what follows ℓ is a function in $\mathrm{W}^{1,q}(\Omega)$.

Assumption A. – The pair (L, ℓ) satisfies Assumption A if either A1 or A2 holds:

A1. The function $L(x,z,p)$ is convex in p, increasing in z and there exists k in $\mathrm{L}^{q'}(\Omega;\mathbb{R}^n)$ such that $k(x) \in \partial_p L(x, \ell(x), \nabla\ell(x))$ a.e. and $\operatorname{div} k \leqslant 0$ weakly.

A2. The function $L(x,z,p) = f(x,p) + g(x,z)$ is convex in (z,p) and there exist k in $\mathrm{L}^{q'}(\Omega;\mathbb{R}^n)$ and h in $\mathrm{L}^{q'}(\Omega)$ such that $k(x) \in \partial_p f(x, \nabla\ell(x))$ a.e., $h(x) \in \partial_z g(x, \ell(x))$ a.e. and $\operatorname{div} k - h \leqslant 0$ weakly.

Remark 1. – When L is differentiable the assumptions A1, A2 can be rewritten as:

A1. The function $L(x,z,p)$ is convex in p, increasing in z and ℓ is a *weak supersolution* of the equation $\operatorname{div} L_p(x,u,\nabla u) = 0$, i.e., $\operatorname{div} L_p(x,\ell(x),\nabla\ell(x)) \leqslant 0$ weakly.

A2. The function $L(x,z,p) = f(x,p) + g(x,z)$ is convex in (z,p) and ℓ is a *weak supersolution* of the equation $\operatorname{div} f_p(x,\nabla u) - g_z(x,u) = 0$, i.e., $\operatorname{div} f_p(x,\nabla\ell(x)) - g_z(x,\ell(x)) \leqslant 0$ weakly.

Remark 2 *(Assumption* A′*).* – We also use a dual set of hypothesis that we call Assumption A′ (A′1, A′2) where the inequalities in Assumption A are reversed and the term "increasing" in Assumption A1 is replaced by "decreasing" in Assumption A′1.

We introduce some notation and a definition before stating Assumption B. For every x in Ω let $L^x : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $L^x(p) = L(x,\ell(x),p)$, ($p \in \mathbb{R}^n$). Assume that (L,ℓ) satisfies Assumption A. Then, for every x in Ω, to the selection $k(x)$ of $\partial_p L(x,\ell(x),\nabla\ell(x))$ we associate the exposed face $F^L(x)$ of the epigraph of L^x containing $(\nabla\ell(x), L^x(\nabla\ell(x)))$ defined by:

$$F^L(x) = \operatorname{epi} L^x \cap \big\{(p,\zeta) \in \mathbb{R}^n \times \mathbb{R} : \zeta - L^x\big(\nabla\ell(x)\big) = k(x)\cdot\big(p - \nabla\ell(x)\big)\big\}.$$

We denote by $F^L_{\mathbb{R}^n}(x)$ the vector space associated to the projection of $F^L(x)$ onto the first n coordinates of $\mathbb{R}^n \times \mathbb{R}$.

Definition 1 (The cone property, [5]). – We say that the vector field $S : \Omega \to \mathbb{R}^n$ satisfies the *cone property* if S is Lipschitz continuous in Ω and there exist a vector u in $\mathbb{R}^n$ and $\alpha > 0$ such that $S(x)\cdot u \geqslant \alpha$ for every x in Ω.

Assumption B. – We say that the pair (L,ℓ) satisfies Assumption B if at least one of the following conditions B1, B2, B3 or B4 holds.

B1. The pair (L,ℓ) satisfies A1 and (just) one of the following items B1.1 or B1.2 holds:
- B1.1. The inequality $\operatorname{div} k \leqslant 0$ is strict;
- B1.2. The map $z \mapsto L(x,z,p)$ is strictly increasing for almost every x and every p.

B2. The pair (L,ℓ) satisfies A2 and (just) one of the following items B2.1, B2.2 or B2.3 holds:
- B2.1. The inequality $\operatorname{div} k - h \leqslant 0$ is strict;
- B2.2. The faces of $z \mapsto g(x,z)$ containing $(\ell(x), g(x,\ell(x)))$ are reduced to a point;
- B2.3. The function L does not depend on p, i.e., $L(x,z,p) = g(x,z)$, and the non trivial faces of $z \mapsto g(x,z)$ have strictly positive slope.

B3. The pair (L,ℓ) satisfies Assumption A (i.e., A1 or A2) and the faces of the map $p \mapsto L(x,\ell(x),p)$ containing $(\nabla\ell(x), L(x,\ell(x),\nabla\ell(x)))$ are reduced to a point.

B4. The pair (L,ℓ) satisfies Assumption A and there exists $S : \Omega \to \mathbb{R}^n$ with the cone property such that, for almost every x in Ω, $S(x)$ is orthogonal to $F^L_{\mathbb{R}^n}(x)$.

Remark 3. – The strict convexity of $z \mapsto g(x,z)$ implies B2.2 and B2.3 and the strict convexity of $p \mapsto L(x,z,p)$ implies B3.

We fix $\bar{u}$ in $\mathrm{W}^{1,q}(\Omega)$ and we set $\mathrm{W}^{1,q}_{\bar{u}}(\Omega) = \bar{u} + \mathrm{W}^{1,q}_0(\Omega)$.

MAIN THEOREM 2 ([4]). – *Let* (L,ℓ) *satisfy Assumption* B *and* w *be a minimizer of*

$$I(u) = \int_\Omega L\big(x, u(x), \nabla u(x)\big)\,\mathrm{d}x$$

in $\mathrm{W}^{1,q}_{\bar u}(\Omega)$*. Assume that* $w \leqslant \ell$ *on* $\partial\Omega$*. Then* $w \leqslant \ell$ *a.e. in* Ω*.*

Remark 4. – Theorem 2 generalizes Theorem 1 of [1] where the same conclusion is obtained in the case where ℓ is affine and $L(x,z,p) = f(p)$ is a function of the variable p.

The next result is used in the proof of the main theorem under B4.

THEOREM 3 ([5]). – *Let* v *be a positive function in* $\mathrm{W}^{1,q}_0(\Omega)$*. Assume that there exists a vector field* S *satisfying the* cone property *and* $S \cdot \nabla v \leqslant 0$ *a.e. on* Ω*. Then* $v = 0$ *a.e. on* Ω*.*

The proof of Theorem 3 is based on our generalization of a classical theorem on the existence of a representative of a given Sobolev function that is absolutely continuous on lines ([3], Th. 1.41).

THEOREM 4 ([5]). – *Let* v *be a function in* $\mathrm{W}^{1,q}_0(\Omega)$*,* E*,* F *be open in* $\mathbb{R}^n$ *and* F *be a subset of* Ω*. Let* $\Lambda : E \to F$ *be invertible and Lipschitz continuous with its inverse. We assume that, for almost every* ξ *in* $\mathbb{R}^{n-1}$*,* $\Lambda(\partial E \cap \{(t,\xi) : t \in \mathbb{R}\}) \subset \partial\Omega$*. Then there exists a representative* v^* *of* v *in* $\mathrm{W}^{1,q}_0(\Omega)$ *such that* v^* *vanishes on* $\partial\Omega$ *and the map* $t \mapsto (v^* \circ \Lambda)(t,\xi)$ *is absolutely continuous for almost every* ξ*. Moreover, the classical partial derivative of* $v^* \circ \Lambda$ *and the weak derivative of* $v \circ \Lambda$ *with respect to the first variable agree almost everywhere on* E*.*

Remark 5. – The assumption of Theorem 4 is fulfilled if, for instance, $E = F = \Omega$; when Λ is the identity our result gives Theorem 1.41 of [3]. In the case of our interest the set F is a proper subset of Ω.

THEOREM 5 ([4]). – *Let* ℓ *be such that* $\ell = \inf_{\lambda\in\Lambda} \ell_\lambda$ *for some family of functions* ℓ_λ *in* $\mathrm{W}^{1,q}(\Omega)$*. Assume that* (L,ℓ_λ) *satisfies Assumption* B *for every* λ *in* Λ *and that* w *is a minimizer of* I *on* $\mathrm{W}^{1,q}_{\bar u}(\Omega)$ *satisfying* $w \leqslant \ell_\lambda$ *on* $\partial\Omega$ *for every* λ *in* Λ*. Then* $w \leqslant \ell$ *a.e. on* Ω*. In particular, if* ℓ *belongs to* $\mathrm{W}^{1,q}(\Omega)$ *and* $w \leqslant \ell$ *on* $\partial\Omega$ *then* $w \leqslant \ell$ *a.e. on* Ω*.*

Remark 6. – There exists pairs (L,ℓ) satisfying the hypothesis of Theorem 5 but not Assumption B which seems then to be rather technical.

Assuming that L is smooth, we apply the main theorem to the *variational equation*:

$$\operatorname{div} L_p(x,u,\nabla u) - L_z(x,u,\nabla u) = 0 \quad \text{weakly.} \tag{2}$$

THEOREM 6 ([4]). – *Let* $L(x,z,p)$ *be convex in* (z,p) *for almost every* x *in* Ω*,* ℓ *be such that the pair* (L,ℓ) *satisfy Assumption* B *and let* w *in* $\mathrm{W}^{1,q}(\Omega)$ *be a solution to* (2) *such that the map* $x \mapsto L(x,w(x),\nabla w(x))$ *belongs to* $\mathrm{L}^1(\Omega)$ *and* $w \leqslant \ell$ *on* $\partial\Omega$*. Then* $w \leqslant \ell$ *on* Ω*.*

Remark 7. – We point out here what are the main differences between Theorem 6 and the classical result. First, we allow L, ℓ and w to be nonsmooth; moreover, we do not impose the ellipticity assumption, allowing thus the epigraph of the function $p \mapsto L(x,z,p)$ to have nontrivial faces. Instead we require that L is either monotonic in z (in Assumptions B1, B3, B4) or the sum of two functions (in Assumptions B2, B3, B4); in this situation Theorem 6 under Assumption B3 generalizes Theorem 10.7 of [2]. We don't cover however the case where L is elliptic but is neither monotonic in z nor the sum of two functions.

The application of Theorem 6 yields a comparison principle for elliptic equations in divergence form of the type $\operatorname{div}(A(x)\nabla u)) - c(x)u = 0$, where, for every x in Ω, $A(x)$ is a $n \times n$ symmetric matrix whose coefficients belong to $\mathrm{L}^\infty(\Omega)$, c is a function in $\mathrm{L}^\infty(\Omega)$ and (A,c,ℓ) satisfy Assumption C below.

Assumption C. – We say that (A,c,ℓ) satisfy Assumption C if the matrix $A(x)$ is positive semidefinite and $c\geqslant 0$ a.e. in Ω, ℓ in $\mathrm{W}^{1,2}(\Omega)$ is a weak supersolution of the equation $\operatorname{div}(A(x)\nabla u)-c(x)u=0$ and (just) one of the following condition hold:

C1. The function ℓ is a strict supersolution of $\operatorname{div}(A(x)\nabla u)-c(x)u=0$, i.e.:

$$\forall\varphi\in\mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \varphi\geqslant 0 \text{ in } \Omega,\ \varphi\neq 0,\quad \int_\Omega A(x)\nabla\ell(x)\cdot\nabla\varphi(x)+c(x)\ell(x)\varphi(x)\,\mathrm{d}x>0.$$

C2. The function c is strictly positive a.e. in Ω.

C3. The matrix $A(x)$ is positive definite for almost every x in Ω.

C4. There exists a function $S:\Omega\to\mathbb{R}^n$ satisfying the cone property such that, for almost every x in Ω, $S(x)$ is orthogonal to the kernel of $A(x)$.

We deduce a *weak maximum principle* for elliptic equations in divergence form.

THEOREM 7 ([4]). – *Let A be positive definite, $c\geqslant 0$ a.e. in Ω and assume that* (*just*) *one of the Assumptions* C2, C3 *or* C4 *hold. Let w in* $\mathrm{W}^{1,2}(\Omega)$ *be a weak solution to the equation* $\operatorname{div}(A(x)\nabla u)-c(x)u=0$. *Then* $w\leqslant\sup_{\partial\Omega}w^+$ *in* Ω.

Remark 8. – We point out that the classical weak maximum principle ([2], Th. 8.1), requires that the equation is strictly elliptic, i.e., that $p\cdot A(x)p\geqslant\lambda|p|^2$ ($\lambda>0$) for all $x,\ p$. This condition is contained in our Assumption C3, where we just require ellipticity. Further, our conditions C2 and C4 allow the equation to be degenerate.

Assumption D. – We say that L satisfies Assumption D if $L(x,z,p)=f(p)+g(x,z)$ is convex in (z,p), the function $g(x,z)$ is continuously differentiable in z and the derivative g_z is Lipschitz continuous. Moreover, one of the two following properties holds:

$$\inf\{g_{zz}(x,z):\,(x,z)\in\Omega\times\mathbb{R}\}>0;\tag{3}$$

$$f \text{ is superlinear and strictly convex, } g(x,z)=h(z).\tag{4}$$

We set $i=\inf\{g_{zz}(x,z):\,(x,z)\in\Omega\times\mathbb{R}\}$, $s=\sup\{|\nabla_x g_z(x,z)|:\,(x,z)\in\Omega\times\mathbb{R}\}$ and

$$M(g,K)=\begin{cases}\max\{s/i,\,K\} & \text{if (3) holds,}\\ K & \text{if (4) holds.}\end{cases}$$

We generalize in [4] the *bounded slope condition* $(\mathrm{BSC})_K$ introduced in the variational context by Stampacchia in [6] for functions L of the form $L(x,z,p)=f(p)$.

DEFINITION 8 (The generalized bounded slope condition $(\mathrm{GBSC})_K$). – We say that the pair $(L,\bar{u})$ satisfies the $(\mathrm{GBSC})_K$ if $\bar{u}:\Omega\to\mathbb{R}$ is Lipschitz continuous with Lipschitz constant K and for every $\bar{x}$ in $\partial\Omega$ there exist two Lipschitz continuous functions $\ell^1_{\bar{x}},\ \ell^2_{\bar{x}}:\Omega\to\mathbb{R}$ with Lipschitz constant K such that the pair $(L,\ell^2_{\bar{x}})$ (resp. $(L,\ell^1_{\bar{x}})$) satisfies Assumption A (resp. A′) and moreover,

$$\forall x\in\partial\Omega,\quad \ell^1_{\bar{x}}(x)\leqslant\bar{u}(x)\leqslant\ell^2_{\bar{x}}(x)\quad\text{and}\quad \ell^1_{\bar{x}}(\bar{x})=\bar{u}(\bar{x})=\ell^2_{\bar{x}}(\bar{x}).$$

The next results generalize Theorem 2 and Theorem 3 of [1] where the same conclusions are obtained for $L(x,z,p)=f(p)$ and for data satisfying the $(\mathrm{BSC})_K$.

THEOREM 9 ([4]). – *Let L satisfy Assumption* D *and assume that the pair $(L,\bar{u})$ fulfills the* $(\mathrm{GBSC})_K$. *Let w be a solution to the problem*

$$\text{(P)}\qquad \textit{minimize}\quad I(u)=\int_\Omega L(x,u,\nabla u)\,\mathrm{d}x\quad \textit{on}\ \mathrm{W}^{1,q}_{\bar{u}}(\Omega).$$

Then w is Lipschitz continuous, a Lipschitz constant being $M(g,K)$.

The proof of Theorem 9 is obtained via a recent result presented in [7].

COROLLARY 10 ([4]). – *Under the assumptions of Theorem* 9 *the problem* (P) *and the problem*

$$(\mathrm{P})' \qquad \textit{minimize} \quad I(u)=\int_\Omega L(x,u,\nabla u)\,\mathrm{d}x \quad \textit{on } \mathrm{W}^{1,q}_{\bar u}(\Omega),\ |\nabla u|\leqslant M(g,K) \textit{ a.e.}$$

have the same set of solutions.

THEOREM 11 ([4]). – *Let L satisfy Assumption* D, *f be continuously differentiable and assume that $(L,\bar u)$ fulfills the* $(\mathrm{GBSC})_K$. *Let w be a solution to Problem* (P). *Then w is Lipschitz continuous and it satisfies the Euler–Lagrange equation, i.e.,*

$$\int_\Omega f_p\big(\nabla w(x)\big)\cdot\nabla\varphi(x)+g_z\big(x,w(x)\big)\varphi(x)\,\mathrm{d}x=0$$

for every function φ in $\mathcal{C}^\infty_c(\Omega)$.

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