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Absolutely continuous representatives on curves for Sobolev functions

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Abstract

We consider a class of Lipschitz vector fields $S : \Omega \to \mathbb{R}^n$ whose values lie in a suitable cone and we show that the trajectories of the system $x' = S(x)$ admit a parametrization that is invertible and Lipschitz with its inverse. As a consequence, every v in $W^{1,1}(\Omega)$ admits a representative that is absolutely continuous on almost every trajectory of $x' = S(x)$. If S is an arbritrary Lipschitz field the same property does hold locally at every x such that $S(x) \neq 0$.

Keywords: Absolutely continuous; Representatives; Parametrization of the trajectories; Composition of Sobolev functions; Change of variable formula for Sobolev functions

1. Introduction

The results that we present here are motivated by the following problem. Let Ω be an open bounded subset of $\mathbb{R}^n$, $S : \Omega \to \mathbb{R}^n$ be a Lipschitz vector field such that the trajectories of $x' = S(x)$ leave Ω in a finite time and v in $\mathcal{C}^1(\Omega)$ be equal to 0 in the boundary $\partial\Omega$ of Ω and be such that its gradient $\nabla v(x)$ is orthogonal to $S(x)$ for every x in Ω. It then follows immediately that v vanishes on Ω. In fact let $x_0 \in \Omega$ and $x :]t_1, t_2[\to \Omega$ be the maximal solution of

$$x' = S(x), \qquad x(0) = x_0; \tag{A}$$

our assumption implies that $x(t_2) = \lim_{t \to t_2} x(t)$ belongs to $\partial\Omega$. We have

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$$v(x_0)=v\big(x(0)\big)-v\big(x(t_2)\big)=\int_{t_2}^{0}(v\circ x)'(t)\,dt$$

$$=\int_{t_2}^{0}\nabla v\big(x(t)\big)\cdot x'(t)\,dt=\int_{t_2}^{0}\nabla v\big(x(t)\big)\cdot S\big(x(t)\big)\,dt=0, \qquad (*)$$

proving the claim. We point out that the same conclusion does not hold if the trajectories of (A) do not leave Ω in a finite time; let for instance $v(x,y)=x^2+y^2-1$, $\Omega=\{(x,y): x^2+y^2<1\}$, $S(x,y)=(-y,x)$: in this case $\nabla v(x,y)$ is orthogonal to $S(x,y)$ and $v=0$ on $\partial\Omega$ but $v<0$ on Ω.

Here we extend the above vanishing property in the case where v is a function in $W_0^{1,1}(\Omega)$. This result turns out to be an important tool in establishing the validity of a comparison principle for the minimizers of integral functionals in our paper [7].

In this situation the previous reasoning cannot be directly applied. The main problem is that, if x is a trajectory of (A), the composed function $v\circ x$ may not be absolutely continuous. This can be regarded as a particular case of the problem of the composition of Sobolev functions. The known results on this topic [6] do not apply to our case. If S is a nonzero constant field the problem can be easily solved since a classical result shows that, once we fix a hyperplane Π that is transversal to S, there exists a representative of v that is equal to zero on $\partial\Omega$ and absolutely continuous on almost every trajectory through Π. In Section 3 we consider the class of vector fields whose values lie in a suitable cone (*cone property*, see Definition 2.6), allowing the trajectories of (A) to leave Ω in a finite time. The problem is reduced to the case where S is constant by showing that if we fix a hyperplane Π that is transversal to the cone containing the values of S then we can parametrize Ω by means of a function $\Psi(t,\xi)$, whose ξ-sections are the trajectories of (A), that is a bi-Lipschitz homeomorphism. This allows us to look at the composition of v with the trajectories of (A) as the ξ-sections of $v\circ\Psi(t,\xi)$. The change of variables formula for Sobolev functions shows that there exists a representative of $v\circ\Psi$ that is absolutely continuous on almost every section. However this is not enough since by $(*)$ we need to evaluate v at the endpoints of the trajectories. For this purpose we slightly modify the proof of the aforementioned classical result to show that actually there exists a representative v^* of v such that $v^*\circ\Psi$ is absolutely continuous on almost every section and, further, v^* vanishes on $\partial\Omega$. We point out that, in order to obtain the lipschitzianity of Ψ^{-1}, it is essential to assume the *cone property* as we show in Example 3.4.

Finally, in Section 4, we consider an arbitrary Lipschitz field $S:\Omega\to\mathbb{R}^n$ and we prove that if $S(\bar{x})$ is nonzero then the previous map Ψ constructed upon a hyperplane Π through $\bar{x}$ is orthogonal to $S(\bar{x})$ is injective and has a Lipschitz inverse in a neighbourhood of $(0,\bar{x})\in\mathbb{R}\times\Pi$. As a consequence, if v is in $W^{1,1}(\Omega)$ and τ is a positive real number, there exists a representative v^* of v such that for every ξ in a neighbourhood of $\bar{x}$ in Π, except a set of $(n-1)$-Hausdorff measure zero, the function $t\mapsto v^*(x_\xi(t))$ is absolutely continuous on $[0,\tau]$, where x_ξ is the solution of $x'=S(x)$, $x(0)=\xi$.

2. Notation and preliminary results

Notation. If A is a subset of $\mathbb{R}^n$ we denote $\operatorname{cl} A$ (respectively, ∂A) the closure (respectively, the boundary) of A. Given two vectors a and b in $\mathbb{R}^n$ and $r > 0$ we denote by $a \cdot b$ the usual scalar product in $\mathbb{R}^n$, by $|a|$ the euclidean norm of a and by $B(a, r)$ the ball centered in a of radius r; by $\operatorname{diam}(A)$ we denote the diameter of A, i.e., $\operatorname{diam}(A) = \sup\{|x^2 - x^1|\colon x^1, x^2 \in A\}$ and by $\operatorname{dist}(a, A)$ the distance of a to A. The standard orthonormal basis of $\mathbb{R}^n$ is $e_1, \dots, e_n$. If $f : X \to Y$ is a function and W is a subset of X (respectively, Z is a subset of Y) the image of W (respectively, the inverse image of Z) through f is denoted by $f(W)$ (respectively, $f^{-1}(Z)$).

By $\mathcal{C}^\infty(\Omega)$ (respectively, $\mathcal{C}_c^\infty(\Omega)$) we denote the space of infinitely differentiable functions in Ω (respectively, infinitely differentiable functions with compact support in Ω) and, for $q \geqslant 1$ and A open in $\mathbb{R}^n$, $L^q(A)$ (respectively, $W^{1,q}(A)$, $W_0^{1,q}(A)$) is the usual space of the Lebesgue functions (respectively, the first order Sobolev spaces) of exponent q; for f in $W^{1,q}(\Omega)$ (respectively, differentiable in Ω) the weak derivative (respectively, classical derivative) of f with respect to x_i is denoted by f_{x_i} (respectively, $D_{x_i} f$) and its gradient by ∇f. For a function f of one variable the classical derivative is often denoted by f'. The composition of the functions f and g (whenever it is defined) is denoted by $f \circ g$, the inverse of f is denoted by f^{-1}.

Let Ω be an open bounded subset of $\mathbb{R}^n$. A classical result [8, Theorem 1.41] states that a function in $W_0^{1,q}(\Omega)$ admits a representative that is absolutely continuous on almost all line segments that are parallel to the coordinate axes. A slight modification of its proof yields the following generalization to the composition of Sobolev functions with invertible, Lipschitz mappings.

Proposition 2.1. *Let v be a function in $W_0^{1,q}(\Omega)$, E, F be open in $\mathbb{R}^n$ and F be a subset of Ω. Let $\Lambda : \operatorname{cl} E \to \operatorname{cl} F$ be Lipschitz, invertible and its inverse be Lipschitz. Assume that, for almost every ξ in $\mathbb{R}^{n-1}$,*

$$\Lambda\big(\partial E \cap \big\{(t, \xi)\colon t \in \mathbb{R}\big\}\big) \subset \partial\Omega. \tag{2.1}$$

There exists a representative v^* of v in $W_0^{1,q}(\Omega)$ such that v^* vanishes on $\partial\Omega$ and the map $t \mapsto (v^* \circ \Lambda)(t, \xi)$ is absolutely continuous for almost every ξ. Moreover, the classical partial derivative of $v^* \circ \Lambda$ and the weak derivative of $v \circ \Lambda$, with respect to the first variable t, agree almost everywhere on E.

Remark 2.2. It is well known [9, Theorem 2.2.2] that the composed map $v \circ \Lambda$ belongs to $W^{1,q}(E)$, even without assuming (2.1). Assumption (2.1) is fulfilled if, for instance, E and F coincide with Ω; in this case, however, the claim follows easily from Theorem 1.41 of [8] itself. We will apply Proposition 2.1 in the proof of Theorem 3.5 where the set F is a proper subset of Ω.

If E is an open subset of $\mathbb{R}^n$ for ξ in the projection of E onto the last $n-1$ coordinates we denote by $E(\xi)$ the ξ-section of E, i.e.,

$$E(\xi) = \big\{t \in \mathbb{R}\colon\ (t,\xi) \in E\big\}.$$

Remark 2.3. Proposition 2.1, in the case where Λ is the identity map, yields [8, Theorem 1.41].

Definition 2.4 (the flux Φ). Let $S : \Omega \to \mathbb{R}^n$ be Lipschitz. Let $\bar{x}$ in Ω, $I_{\bar{x}}$ be the maximal interval of definition of the maximal solution of

$$x' = S(x), \qquad x(0) = \bar{x}, \tag{2.2}$$

and $\Phi(t,\bar{x})$ be its value at t. The *domain* D of Φ is defined by

$$D = \big\{(t,\bar{x}) \in \mathbb{R} \times \Omega\colon\ t \in I_{\bar{x}}\big\}.$$

We recall that for every s, t, x such that the pairs (s,x) and $(t, \Phi(s,x))$ are in D then $\Phi(t,\Phi(s,x)) = \Phi(t+s,x)$. We will often denote by $x_\xi(t)$ the point $\Phi(t,\xi)$.

Remark 2.5. It is well known, see for instance [5, Theorem 3.1.1], that D is open and Φ is locally Lipschitz; actually Φ is Lipschitz if the solutions to (2.2) leave Ω in a uniformly bounded time.

Definition 2.6 (the cone property). We say that the vector field $S : \Omega \to \mathbb{R}^n$ satisfies the *cone property* if S is Lipschitz in Ω and there exist a vector u in $\mathbb{R}^n$ and $\alpha > 0$ such that $S(x) \cdot u \geqslant \alpha$ for every x in Ω.

Remark 2.7. When Ω is bounded, S being Lipschitz, this condition implies that there exists a positive β such that the vector $S(x)$ belongs to the cone $\{y \in \mathbb{R}^n\colon\ y \cdot u \geqslant \beta|y|\}$ for every x.

The following result is straightforward.

Proposition 2.8. *Let $S : \Omega \to \mathbb{R}^n$ be a vector field satisfying the cone property, $\bar{x}$ in Ω and $x : I_{\bar{x}} =]t_1(\bar{x}), t_2(\bar{x})[\to \Omega$ $(-\infty \leqslant t_1(\bar{x}) \leqslant t_2(\bar{x}) \leqslant +\infty)$ be the maximal solution of the autonomous system* (2.2). *Then for every t in $I_{\bar{x}}$ the following inequalities hold*:

$$\begin{cases} (x(t) - \bar{x}) \cdot u \geqslant \alpha t & \text{if } t \geqslant 0, \\ (x(t) - \bar{x}) \cdot u \leqslant \alpha t & \text{if } t \leqslant 0. \end{cases}$$

Moreover, if Ω is bounded there exist $T > 0$ depending only on Ω such that $|t_1(\bar{x})| \leqslant T$ and $|t_2(\bar{x})| \leqslant T$; moreover, the limits $\lim_{t \to t_2(\bar{x})} x(t)$, $\lim_{t \to t_1(\bar{x})} x(t)$ exist and belong to the boundary of Ω.

3. Lipschitz fields that satisfy the cone property

In this section Ω is an open bounded subset of $\mathbb{R}^n$, the function $S : \Omega \to \mathbb{R}^n$ is assumed to satisfy the cone property and the vector u is as in Definition 2.6. We fix a hyperplane Π that is orthogonal to u and such that $\Omega \cap \Pi \neq \emptyset$. We set

$$A_\Pi = \{(t, \xi) \in \mathbb{R} \times \Pi\colon\ (t, \xi) \in D\}$$

and we introduce the parametrization Ψ_Π of the trajectories of the system $x' = S(x)$ that intersect Π by

$$\Psi_\Pi : A_\Pi \subset \mathbb{R} \times \Pi \to \Omega, \quad (t, \xi) \mapsto \Psi_\Pi(t, \xi) = \Phi(t, \xi);$$

Ψ is the restriction of Φ to $\mathbb{R} \times \Pi$. When no ambiguity may occur we will often write Ψ instead of Ψ_Π and A instead of A_Π.

Lemma 3.1. *For every $\bar{x}$ in Ω the function $t \mapsto \Phi(t, \bar{x}) \cdot u$ is strictly increasing on $I_{\bar{x}}$.*

Proof. The derivative with respect to t of the map $t \mapsto \Phi(t, \bar{x}) \cdot u$ is given by $S(\Phi(t, \bar{x})) \cdot u$ and is thus greater than the strictly positive constant α. $\square$

Theorem 3.2. *The domain A_Π of Ψ_Π is open in $\mathbb{R} \times \Pi$ and the image $B = \Psi_\Pi(A_\Pi)$ of A_Π through Ψ_Π is open in $\mathbb{R}^n$. The map Ψ_Π is Lipschitz, invertible and its inverse is Lipschitz.*

Remark 3.3. In the proof of this result the continuity of Ψ_Π and its invertibility follow easily from the properties of the flux of S. However, the fact that the inverse Ψ_Π^{-1} is Lipschitz is not trivial at all; moreover, as we show in the following example this conclusion cannot be obtained if the field S, instead of satisfying the cone property, is just such that the trajectories of the associated dynamical system leave Ω in a finite time.

Example 3.4. Let $\Omega = \{(x, y) \in \mathbb{R}^2\colon\ x > 0,\ x^2 + y^2 < 1\}$, $S(x, y) = (-y, x)$ and Π be the x-axis. For every $\bar{x} \in]0, 1[$ we have

$$\Psi(t, \bar{x}) = (\bar{x} \cos t, \bar{x} \sin t),$$

and thus the inverse map is given by

$$\Psi^{-1}(x, y) = \big(t(x, y), \bar{x}(x, y)\big),$$

where

$$t(x, y) = \arctan(y/x), \qquad \bar{x}(x, y) = \sqrt{x^2 + y^2}.$$

Clearly $\bar{x}(x, y)$ is Lipschitz whereas $t(x, y)$ is not. Here the solutions to $(x', y') = S(x, y)$ leave the domain in a finite time (namely in a time less than π), however S does not satisfy the cone property on Ω.

Proof of Theorem 3.2. We subdivide the proof into several steps.

(a) *The map Ψ is injective and A, B are open.*

It is not restrictive to assume that $u = e_n$ and that the hyperplane Π is defined by

$$\Pi = \{(x_1, \dots, x_n) \in \mathbb{R}^n\colon\ x_n = 0\}.$$

The set A is open since D is open. Let (s, ν), (t, ξ) in A be such that $\Psi(s, \nu) = \Psi(t, \xi)$. Then

$$\Phi(s, \nu) = \Phi(t, \xi), \tag{3.1}$$

and therefore by applying the map $\Phi(-s, \cdot)$ on both sides of the latter equality we obtain $\nu = \Phi(0, \nu) = \Phi(t - s, \nu)$. Taking the scalar product with e_n on each side of the equalities we get $\Phi(t - s, \xi) \cdot e_n = 0 = \Phi(0, \xi)$; the strict monotonicity of the map $\tau \mapsto \Phi(\tau, \xi)$ then implies that $t = s$. It follows from (3.1) that $\nu = \xi$ and Ψ is injective. Clearly, by Remark 2.5 and Proposition 2.8, the map Ψ is Lipschitz: the *invariance of domain theorem* [4, Theorem 3.30] then implies that the set $B = \Psi(A)$ is open.

We are now concerned with the proof of the Lipschitz continuity of the inverse of Ψ: in what follows (s, ν), (t, ξ) are in A and we set

$$y = \Psi(s, \nu), \qquad z = \Psi(t, \xi).$$

Our goal is to prove the existence of a constant β depending only on Ω and S satisfying

$$\max\{|t - s|, |\xi - \nu|\} \leqslant \beta |z - y|. \tag{3.2}$$

In the next steps (b) and (c) we state some intermediate results towards this scope.

(b) *If s and t have opposite signs then*

$$\alpha |t - s| \leqslant |z - y|, \tag{3.3}$$

otherwise there exists a positive constant K depending only on Ω and S such that

$$\alpha |t - s| \leqslant K \min\{|s|, |t|\} |\xi - \nu| + |z - y|. \tag{3.4}$$

Moreover, denoting by L_Φ the Lipschitz constant of Φ, we have

$$|\xi - \nu| \leqslant L_\Phi (|t - s| + |z - y|). \tag{3.5}$$

Since the derivative of the map $t \mapsto \Phi(t, x)$ is $S(\Phi(t, x))$ then we have

$$y = \nu + \int_0^s S(\Psi(r, \nu))\, dr, \qquad z = \xi + \int_0^t S(\Psi(r, \xi))\, dr.$$

Assume that s and t have opposite signs, for instance that $s \leqslant 0 \leqslant t$. Then

$$z - y = (\xi - \nu, 0) + \int_0^t S(\Psi(r, \xi))\, dr - \int_0^s S(\Psi(r, \nu))\, dr,$$

whence, taking the scalar product with e_n on both sides of the equality,

$$\int_0^t S\big(\Psi(r,\xi)\big)\cdot e_n\,dr - \int_0^s S\big(\Psi(r,\nu)\big)\cdot e_n\,dr = (z-y)\cdot e_n. \tag{3.6}$$

Since

$$\int_0^t S\big(\Psi(r,\xi)\big)\cdot e_n\,dr \geqslant \int_0^t \alpha\,dr = \alpha t,$$
$$\int_s^0 S\big(\Psi(r,\nu)\big)\cdot e_n\,dr \geqslant \int_s^0 \alpha\,dr = -\alpha s,$$

then (3.6) yields

$$|z-y| \geqslant (z-y)\cdot e_n \geqslant \alpha(t-s) = \alpha|t-s|,$$

which proves (3.3). Assume that s and t have the same sign and without restriction that $0 \leqslant s \leqslant t$. Then (s,ν) and (s,ξ) belong to A for every s in $[0,t]$ and thus we can write

$$z-y = (\xi-\nu,0) + \int_0^s S\big(\Psi(r,\xi)\big) - S\big(\Psi(r,\nu)\big)\,dr + \int_s^t S\big(\Psi(r,\xi)\big)\,dr;$$

taking again the scalar product with e_n on both sides of the latter equality, we obtain

$$\int_s^t S\big(\Psi(r,\xi)\big)\cdot e_n\,dr = -\int_0^s \big[S\big(\Psi(r,\xi)\big) - S\big(\Psi(r,\nu)\big)\big]\cdot e_n\,dr + (z-y)\cdot e_n.$$

We recall that $S(y)\cdot e_n \geqslant \alpha$ for every y in Ω; thus denoting by L_S (respectively, L_Ψ) the Lipschitz constant of S (respectively, Ψ), we obtain

$$\alpha|t-s| \leqslant L_S L_\Psi |s||\xi-\nu| + |z-y|,$$

proving (3.4). By definition we have

$$\nu = \Phi(-s,y), \qquad \xi = \Phi(-t,z);$$

the Lipschitz continuity of Φ yields

$$|\xi-\nu| \leqslant L_\Phi\big(|t-s| + |z-y|\big),$$

proving (3.5).

(c) *There exist two constants T_1 and C depending only on Ω and S such that*

$$\max\big\{|t-s|,\ |\xi-\nu|\big\} \leqslant C|z-y|,$$

for every s,t satisfying

$$\min\big\{|s|,|t|\big\} \leqslant T_1.$$

Set $\tau = \min\{|s|, |t|\}$; the inequalities (3.3) and (3.4) together with (3.5) yield

$$\begin{aligned}|t-s| &\leqslant \frac{K\tau L_\Phi}{\alpha}\big(|t-s|+|z-y|\big)+\frac{1}{\alpha}|z-y| \\ &= \frac{K\tau L_\Phi}{\alpha}|t-s|+\frac{K\tau L_\Phi+1}{\alpha}|z-y|,\end{aligned}$$

so that

$$|t-s|\left(1-\frac{KL_\Phi\tau}{\alpha}\right)\leqslant \frac{K\tau L_\Phi+1}{\alpha}|z-y|.$$

Choosing $0<T_1<\alpha/(KL_\Phi)$ and $\tau \leqslant T_1$ we obtain $|t-s|\leqslant C_1|z-y|$ for some constant C_1 depending only on Ω and S. Now (3.5) gives $|\xi-\nu|\leqslant L_\Phi(C_1+1)|z-y|$, proving the claim.

(d) *The inverse of Ψ is Lipschitz.*

If s and t have opposite signs the application of (3.3) together with (3.5) yields

$$|\xi-\nu|\leqslant L_\Phi\left(\frac{1}{\alpha}+1\right)|z-y|,$$

proving the validity of (3.2). Assume now, without restriction, that s and t are both positive. Remark first that by Proposition 2.8 for every (t,ξ) in A we have

$$\big|\Psi(t,\xi)\cdot e_n\big| = \big|\big(\Phi(t,\xi)-\Phi(0,\xi)\big)\cdot e_n\big| \geqslant \alpha|t|. \tag{3.7}$$

Therefore, if we set $\Delta=\alpha T_1$ the inequality $|\Psi(t,\xi)\cdot e_n|\leqslant\Delta$ implies that $|t|\leqslant T_1$. For k in $\mathbb{Z}$ we set

$$h_k = k\Delta, \qquad \Pi_k=\big\{(x_1,\ldots,x_n)\in\mathbb{R}^n:\ x_n=h_k\big\}.$$

We define Ψ_k by $\Psi_k(t,\xi)=\Phi(t,(\xi,h_k))$ for every ξ in Π; notice that $\Psi_0=\Psi$. As in (3.7) Proposition 2.8 implies that

$$\big|\Psi_k(t,\xi)\cdot e_n - h_k\big|\geqslant\alpha|t|. \tag{3.8}$$

We may assume that $y\cdot e_n\leqslant z\cdot e_n$: let m in $\mathbb{N}$ be such that $h_m\leqslant y\cdot e_n<h_{m+1}$. Remark that m is bounded above by a constant depending only on Ω and S: in fact, from the latter inequality we deduce that $m\Delta\leqslant y\cdot e_n = y\cdot e_n-\bar y\cdot e_n$ for every $\bar y$ in $\Pi\cap\Omega$ and therefore m is bounded above by the constant $(2\operatorname{diam}\Omega)/\Delta$. By the continuity and the monotonicity of the maps $t\mapsto\Psi(t,\xi)\cdot e_n$ and $t\mapsto\Psi(t,\nu)\cdot e_n$, there exist

$$\begin{aligned}&s_0=0<s_1<\cdots<s_m\leqslant s=s_{m+1},\\ &t_0=0<t_1<\cdots<t_m<t=t_{m+1},\end{aligned}$$

such that

$$\Psi(s_k,\nu)\cdot e_n=h_k,\quad \Psi(t_k,\xi)\cdot e_n=h_k\quad (k=0,\ldots,m),$$

so that $\Psi(s_k,\nu)=(p^k,h_k)$ and $\Psi(t_k,\xi)=(q^k,h_k)$ for some p^k,q^k in $\mathbb{R}^{n-1}$; notice that $0\leqslant\Psi(s,\nu)\cdot e_n-h_m\leqslant\Delta$, $p^0=\nu$ and $q^0=\xi$. The properties of the flux Φ allow us to write that, for k in $\{0,\dots,m\}$,

$$\begin{aligned}\Psi(s_{k+1},\nu)&=\Phi(s_{k+1},\nu)=\Phi\big(s_{k+1}-s_k,\Phi(s_k,\nu)\big)=\Phi\big(s_{k+1}-s_k,\Psi(s_k,\nu)\big)\\&=\Phi\big(s_{k+1}-s_k,(p_k,h_k)\big)=\Psi_k(s_{k+1}-s_k,p_k).\end{aligned}$$

Since $|\Psi(s_{k+1},\nu)\cdot e_n-h_k|\leqslant\Delta$ then $|\Psi_k(s_{k+1}-s_k,p_k)\cdot e_n-h_k|\leqslant\Delta$, hence (3.8) implies that $s_{k+1}-s_k\leqslant T_1$; analogously, for k in $\{0,\dots,m-1\}$, we obtain that $t_{k+1}-t_k\leqslant T_1$ (the case $k=m$ is excluded here since $t_{m+1}-t_m=t-t_m$ may be greater than T_1). Starting with $k=m$ we are therefore led to the following situation:

$$\begin{aligned}&y=\Psi_0(s_{m+1},p^0)=\Psi_m(s_{m+1}-s_m,p^m),\quad 0\leqslant s_{m+1}-s_m\leqslant T_1,\\&z=\Psi_0(t_{m+1},q^0)=\Psi_m(t_{m+1}-t_m,p^m),\quad 0\leqslant t_{m+1}-t_m.\end{aligned}$$

We point out here that the previous steps (b) and (c) do obviously apply when $y=\Psi_k(s,\nu)$ and $z=\Psi_k(t,\xi)$ (k in $\{0,\dots,m\}$). It follows from step (c) that

$$|q^m-p^m|\leqslant C|z-y|. \tag{3.9}$$

If $m=0$ we obtain $|\xi-\nu|\leqslant C|z-y|$. Otherwise we write that

$$\begin{aligned}&p^m=\Psi_{m-1}(s_m-s_{m-1},p^{m-1}),\quad 0\leqslant s_m-s_{m-1}\leqslant T_1,\\&q^m=\Psi_{m-1}(t_m-t_{m-1},q^{m-1}),\quad 0\leqslant t_m-t_{m-1}\leqslant T_1,\end{aligned}$$

so that again step (c) yields $|q^{m-1}-p^{m-1}|\leqslant C|q^m-p^m|$ and (3.9) gives

$$|q^{m-1}-p^{m-1}|\leqslant C^2|z-y|.$$

Hence, after $m+1$ steps we obtain

$$|\xi-\nu|\leqslant C^{m+1}|z-y|.$$

Finally we use (3.4) to deduce that

$$|t-s|\leqslant\left(\frac{K}{\alpha}\min\big\{|s|,|t|\big\}C^{m+1}+1\right)|z-y|.$$

Now m,s,t are bounded above by constants depending only on Ω and S: the conclusion follows. □

We apply now Proposition 2.1 with $\Lambda=\Psi_\Pi$. We recall that, for ξ in Π, we denote by $]t_1(\xi),t_2(\xi)[$ the maximal interval of the solution x_ξ to $x'=S(x)$, $x(0)=\xi$.

Theorem 3.5. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and v be a function in $W_0^{1,q}(\Omega)$. There exists a representative v^* of v in $W_0^{1,q}(\Omega)$ such that the map*

$$t\mapsto v^*\big(x_\xi(t)\big) \tag{3.10}$$

is absolutely continuous for almost every ξ in Π and

$$v^*\big(x_\xi\big(t_1(\xi)\big)\big) = v^*\big(x_\xi\big(t_2(\xi)\big)\big) = 0. \tag{3.11}$$

Moreover, the classical partial derivative with respect to t of $v^* \circ x_\xi$ is

$$D_t(v^* \circ x_\xi)(t) = \nabla v\big(x_\xi(t)\big) \cdot S\big(x_\xi(t)\big), \tag{3.12}$$

for almost every (t, ξ) in A.

Proof. We set $\Lambda = \Psi_\Pi$, $E = A$, $F = \Psi_\Pi(A)$. Remark first that for every ξ in Π we have $E(\xi) =]t_1(\xi), t_2(\xi)[$ and that $\Psi(t_1(\xi), \xi)$, $\Psi(t_2(\xi), \xi)$ belong to $\partial\Omega$. We can thus apply Proposition 2.1: there exists a representative v^* of v in $W_0^{1,q}(\Omega)$ vanishing on $\partial\Omega$ (so that $v^*(\Psi(t_1(\xi), \xi)) = v^*(\Psi(t_2(\xi), \xi)) = 0$) and satisfying (3.10) and (3.12). $\square$

We are now in the position to extend to Sobolev functions the result that we proved for smooth functions in the introduction.

Theorem 3.6. *Let Ω be an open bounded subset of $\mathbb{R}^n$ and v be a positive function in $W_0^{1,q}(\Omega)$. Assume that there exists a vector field $S : \Omega \to \mathbb{R}^n$ satisfying the cone property such that $S \cdot \nabla v \leqslant 0$ a.e. on Ω. Then $v = 0$ a.e. on Ω.*

Proof. Let $N = \{x \in \Omega\colon v(x) > 0\}$ and assume that N is non negligible. Let the vector u be as in Definition 2.6 and Π be a hyperplane that is orthogonal to u and whose intersection with N is non negligible (for the $(n-1)$-Hausdorff measure in Π); let $\Psi = \Psi_\Pi$ be the map defined above. By Theorem 3.5 there exists a representative v^* of v such that the map $t \mapsto v^* \circ x_\xi(t)$ is absolutely continuous for almost every ξ, its derivative being

$$\nabla v\big(x_\xi(t)\big) \cdot S\big(x_\xi(t)\big) = \nabla v\big(\Psi(t, \xi)\big) \cdot S\big(\Psi(t, \xi)\big) \quad \text{a.e. in }]t_1(\xi), t_2(\xi)[,$$

and $v^*(x_\xi(t_1(\xi))) = 0$. Therefore, for almost every ξ in $\Pi \cap N$ we have

$$0 < v^*(\xi) = v^*\big(x_\xi(0)\big) - v^*\big(x_\xi\big(t_1(\xi)\big)\big) = \int_{t_1(\xi)}^{0} \nabla v\big(\Psi(t, \xi)\big) \cdot S\big(\Psi(t, \xi)\big)\, dt,$$

and therefore the $(n-1)$-dimensional integral of v^* on $\Pi \cap N$ is given by

$$\begin{aligned} \int_{\Pi\cap N} v^*(\xi)\, d\xi &= \int_{\Pi\cap N} \left\{ \int_{t_1(\xi)}^{0} \nabla v\big(\Psi(t, \xi)\big) \cdot S\big(\Psi(t, \xi)\big)\, dt \right\} d\xi \\ &= \int_{A_1} \nabla v\big(\Psi(t, \xi)\big) \cdot S\big(\Psi(t, \xi)\big)\, dt\, d\xi, \end{aligned}$$

where we set $A_1 = \{(t, \xi) \in \mathbb{R} \times \Pi\colon \xi \in N,\ t \leqslant 0\}$ and $v = 0$ out of Ω. Theorem 3.2 allows us to apply the change of variables formula for Sobolev functions [9, Theorem 2.2.2]: denoting by $J_{\Psi^{-1}}$ the Jacobian of the inverse of Ψ we are thus led to the inequalities

$$0 < \int\limits_{\Pi \cap N} v^*(\xi)\, d\xi = \int\limits_{\Psi(A_1)} \nabla v(x) \cdot S(x) \left| J_{\Psi^{-1}}(x) \right| dx \leqslant 0,$$

a contradiction. It follows that N is negligible. □

Remark 3.7. This result plays an important role in the proof of our *comparison principle* for the minimizers of integral functionals in [7, Theorem 3.14].

4. The general case

We give now a local version of Theorem 3.2 for a wider class of vector fields. Let Ω be an open subset of $\mathbb{R}^n$, $S : \Omega \to \mathbb{R}^n$ be Lipschitz. For every x in Ω such that $S(x) \neq 0$ we set Π_x to be the hyperplane through x that is orthogonal to $S(x)$. If D denotes the domain of the flux of the autonomous system associated to S, we set $A_x = D \cap (\mathbb{R} \times \Pi_x)$ and Ψ_x to be the restriction of the flux Φ to A_x, i.e.,

$$\Psi_x : A_x \to \mathbb{R}^n, \quad (t, \xi) \mapsto \Psi_x(t, \xi) = \Phi(t, \xi).$$

We first show that S does satisfy the cone property in a neighbourhood of a point where it does not vanish.

Lemma 4.1. *Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz and $\bar{x} \in \mathbb{R}^n$ be such that $S(\bar{x}) \neq 0$. There exist $\rho > 0$, $\alpha > 0$ such that*

$$\forall x, y \in B(\bar{x}, \rho) \quad S(x) \cdot S(y) \geqslant \alpha. \tag{4.1}$$

Moreover, when $|S|$ is bounded below by a strictly positive constant, the constants ρ and α depend only on S.

Proof. We have $S(x) \cdot S(y) = (S(x) - S(y)) \cdot S(y) + |S(y)|^2$; now $|S(x) - S(y)| \leqslant L|x - y|$, where L is the Lipschitz constant of S. Let $r > 0$ and $m > 0$ be a lower bound of $|S|$ on $B(\bar{x}, r)$. For x, y in $B(\bar{x}, r)$ we have $S(x) \cdot S(y) \geqslant |S(y)|^2 - L|x - y|$: it is enough to set $\alpha = m^2/2$ and $\rho = \min\{r, m^2/(4L)\}$. □

Lemma 4.2. *Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz, $\bar{x} \in \mathbb{R}^n$ be such that $S(\bar{x}) \neq 0$ and let $\rho > 0$. There exist $T > 0$, $R > 0$ such that*

$$\forall (t, \xi) \in B(\bar{x}, R) \times \Pi_{\bar{x}} \quad \left| \Psi_{\bar{x}}(t, \xi) - \bar{x} \right| \leqslant \rho/2. \tag{4.2}$$

Moreover, when $|S|$ is bounded, the two constants T and R depend only on ρ.

Proof. We set $\Psi = \Psi_{\bar{x}}$ and write that $\Psi(t, \xi) - \bar{x} = \Psi(t, \xi) - \Psi(t, \bar{x}) + \Psi(t, \bar{x}) - \Psi(0, \bar{x})$. Let $\omega(t) = \Psi(t, \xi) - \Psi(t, \bar{x})$. We have

$$\left| \omega'(t) \right| = \left| S\big(\Psi(t, \xi)\big) - S\big(\Psi(t, \bar{x})\big) \right| \leqslant L \left| \Psi(t, \xi) - \Psi(t, \bar{x}) \right| = L \left| \omega(t) \right|,$$

and from Gronwall's lemma we obtain $|\omega(t)| \leqslant |\omega(0)| e^{L|t|} = |\xi - \bar{x}| e^{L|t|}$; moreover,

$$\left|\Psi(t,\bar{x})-\Psi(0,\bar{x})\right|=\left|\Phi(t,\bar{x})-\bar{x}\right|=\left|\int_0^t S\big(\Phi(s,\bar{x})\big)\,ds\right|.$$

Fix $\tau>0$ and let $M=\sup\{|S(\Phi(s,\bar{x}))|;\ |s|\leqslant\tau\}$: then $|\Psi(t,\xi)-\bar{x}|\leqslant|\xi-\bar{x}|e^{L|t|}+M|t|$ for every $|t|\leqslant\tau$. By continuity the right-hand side of the above inequality is smaller than $\rho/2$ if (t,ξ) belongs to a suitable neighbourhood $]-T,T[\times B(\bar{x},R)$ $(T<\tau)$ of $(0,\bar{x})$; if S is bounded above we choose $M=\sup\{|S(x)|;\ x\in\mathbb{R}^n\}$ and then T depends only on ρ. $\square$

We state now the local version of Theorems 3.2 and 3.5 for arbitrary Lipschitz fields.

Theorem 4.3. *Let $S:\mathbb{R}^n\to\mathbb{R}^n$ be Lipschitz and $\bar{x}$ in $\mathbb{R}^n$ be such that $S(\bar{x})\neq 0$. Then there exist two strictly positive real numbers R and T such that the map $\Psi_{\bar{x}}$ restricted to $]-T,T[\times B(\bar{x},R)$ is Lipschitz, injective and its left inverse is Lipschitz. Moreover, when $|S|$ is bounded and bounded below by a strictly positive constant, then R and T depend only on S.*

Proof. Let ρ,α,T,R be as in Lemmas 4.1 and 4.2. Let $u=S(\bar{x})$; by Lemma 4.1 we have $S(x)\cdot u\geqslant\alpha$ for every x in $B(\bar{x},\rho)$. We consider a Lipschitz vector field $\overline{S}$ that coincides with S on the ball $B(\bar{x},\rho/2)$ and that satisfies $\overline{S}(x)\cdot u\geqslant\min\{\alpha,1\}$ for every x in $\mathbb{R}^n$; it is enough to set $\overline{S}(x)=\lambda(x)S(x)+(1-\lambda(x))u$, where λ is a smooth function with compact support in $B(\bar{x},\rho)$, values in $[0,1]$ and equal to 1 in $B(\bar{x},\rho/2)$. Let $\overline{\Phi}$ be the flux associated to $\overline{S}$ and $\overline{\Psi}_{\bar{x}}$ be its restriction to $\mathbb{R}\times\Pi_{\bar{x}}$. By Theorem 3.2 the map $\overline{\Psi}_{\bar{x}}$ is Lipschitz, injective and its inverse is Lipschitz. Let $|t|<T$ and $|\xi-\bar{x}|<R$, by (4.2) we deduce that $S(\Phi(t,\xi))=\overline{S}(\Phi(t,\xi))$; by uniqueness we have $\overline{\Phi}(t,\xi)=\Phi(t,\xi)$ which proves that $\overline{\Psi}_{\bar{x}}(t,\xi)=\Psi_{\bar{x}}(t,\xi)$. It follows that the restriction of Ψ to $]-T,T[\times B(\bar{x},R)$ is Lipschitz, injective and its inverse is Lipschitz. $\square$

We recall that if $\bar{x}$ is in Ω we denote by $I_{\bar{x}}$ the maximal interval of definition of the solution to $x'=S(x)$, $x(0)=\bar{x}$.

Theorem 4.4. *Let $v\in W^{1,1}(\Omega)$, $S:\Omega\to\mathbb{R}^n$ be Lipschitz and x_0 in Ω be such that $S(x_0)\neq 0$. For every $0<\tau<\sup I_{x_0}$ there exists a representative v^* of v and $R_0>0$ such that for almost every ξ in $B(x_0,R_0)\cap\Pi_{x_0}$ the map $t\mapsto v^*(x_\xi(t))$ is absolutely continuous on $[0,\tau]$.*

Proof. It is not restrictive to assume that there exist two real numbers $0<m\leqslant M$ such that $m\leqslant|S(x)|\leqslant M$ for every x in Ω. In fact, let Γ be the curve $\Gamma=\{\Phi(t,x_0),\ 0\leqslant t\leqslant\tau\}$. Since $S(x_0)\neq 0$ and S is Lipschitz then S does not vanishes on Γ. By continuity there exist $\varepsilon>0$ and $m>0$ such that $|S(x)|\geqslant m$ for every x in W, where $W=\{x\in\Omega:\ \operatorname{dist}(x,\Gamma)<\varepsilon\}$. Moreover, there exists $r>0$ such that, for every x in $B(x_0,r)$, τ belongs to I_x and $\Phi(t,x)\in W$ for every $t\in[0,\tau]$. By [3, Section 3.1.1] there exists a Lipschitz field $\overline{S}:\mathbb{R}^n\to\mathbb{R}^n$ such that $\overline{S}=S$ on W, $|\overline{S}|$ is bounded and $\inf_{\mathbb{R}^n}|\overline{S}|=\inf_W|S|$. We fix T,R,ρ and α as in Lemmas 4.1 and 4.2; our assumption implies that these constants do not depend on the choice of x_0. In many parts of the proof we write Ψ instead of Ψ_{x_0}. We subdivide the proof into several steps.

(a) *There exist a representative* v^* *of* v *and a subset* N_0 *of* Π_{x_0} *of* $(n-1)$*-Hausdorff measure* 0 *such that for every* ξ *in* $B(x_0, R) \cap \Pi_{x_0} \setminus N_0$ *the map* $t \mapsto v^*(\Psi(t, \xi))$ *is absolutely continuous in* $]-T, T[$.

By Theorem 4.3, $w = v \circ \Psi$ belongs to $W^{1,1}(E)$ where $E =]-T, T[\times (B(x_0, R) \cap \Pi_{x_0})$. By Theorem 1.41 in [9] there exists a representative $w^* \in W^{1,1}(E)$ of w and a negligible subset N_0 of Π_{x_0} such that, for every $\xi \in B(x_0, R) \cap \Pi_{x_0} \setminus N_0$, the function $t \mapsto w^*(t, \xi)$ is absolutely continuous in $]-T, T[$. The function v^* defined by $v^* = w^* \circ \Psi^{-1}$ on $\Psi(E)$, $v^* = v$ otherwise, fulfills (a).

(b) *There exists* $R_0 \leqslant R$ *such that*

$$\forall \xi \in B(x_0, R_0) \cap \Pi_{x_0}\ \exists t_0(\xi) \in]0, T[: \quad \Psi\big(t_0(\xi), \xi\big) \in \Pi_{\Phi(T/2, x_0)};$$

moreover, $t_0(\xi)$ *is unique.*

Let $x_1 = \Phi(T/2, x_0)$ and $\Pi^-_{x_1}$, $\Pi^+_{x_1}$ be the open half-planes defined by

$$\Pi^-_{x_1} = \big\{x \in \mathbb{R}^n\colon\ (x - x_1) \cdot S(x_1) < 0\big\},$$
$$\Pi^+_{x_1} = \big\{x \in \mathbb{R}^n\colon\ (x - x_1) \cdot S(x_1) > 0\big\}.$$

By Lemma 4.1 we have $S(x) \cdot S(x_1) \geqslant \alpha$ on $B(x_0, \rho)$. Proposition 2.8 implies that $x_0 = \Phi(-T/2, x_1) \in \Pi^-_{x_1}$ and that $x_2 = \Phi(T, x_0) = \Phi(T/2, x_1) \in \Pi^+_{x_1}$. Let U be a neighbourhood of x_2 contained in $\Pi^+_{x_1}$. By continuity there exists R_0 such that $\Phi(T, \xi) \in U$ for every ξ in $B(x_0, R_0) \cap \Pi_{x_0}$. For every such ξ the path $t \mapsto \Phi(t, \xi)$ joins $\xi \in \Pi^-_{x_1}$ to $\Phi(T, \xi) \in \mathbb{R}^n \setminus \operatorname{cl} \Pi^-_{x_1}$; a connection argument yields the existence of $t_0(\xi)$ such that $\Phi(t_0(\xi), \xi)$ belongs to the boundary of $\Pi^-_{x_1}$, i.e., to Π_{x_1}. The uniqueness of $t_0(\xi)$ follows immediately from Proposition 2.8.

(c) *The map*

$$B(x_0, R_0) \cap \Pi_{x_0} \to \Pi_{x_1}, \quad \xi \mapsto \Psi\big(t_0(\xi), \xi\big)$$

is Lipschitz, injective and its inverse is Lipschitz.

The injectivity follows by the uniqueness of the solutions of the Cauchy problem. Since the inverse map can be represented in a similar way by considering the field $(-S)$ instead of S it is enough to prove the map t_0 is Lipschitz. Let ξ_1, ξ_2 in $B(x_0, R_0) \cap \Pi_{x_0}$ and assume that $t_0(\xi_2) \geqslant t_0(\xi_1)$. Since $(\Phi(t_0(\xi_2), \xi_2) - \Phi(t_0(\xi_1), \xi_1)) \cdot S(x_1) = 0$ then

$$\int_{t_0(\xi_1)}^{t_0(\xi_2)} S\big(\Phi(s, \xi_2)\big) \cdot S(x_1)\, ds$$
$$= (\xi_2 - \xi_1) \cdot S(x_1) + \int_0^{t_0(\xi_1)} \big(S\big(\Phi(s, \xi_1)\big) - S\big(\Phi(s, \xi_2)\big)\big) \cdot S(x_1)\, ds;$$

since Φ is Lipschitz, S is bounded and $S(x) \cdot S(x_1) \geqslant \alpha > 0$ there exists $C > 0$ such that $\alpha|t_0(\xi_2) - t_0(\xi_1)| \leqslant C|\xi_2 - \xi_1|$, which proves the claim.

(d) *End of the proof.*

Let m in $\mathbb{N}$ be such that $(m+1)T/2 \geqslant \tau$. For every k in $\{0, \dots, m\}$ we set $x_k = \Phi(kT/2, x_0)$ and

$$\Psi_k : \mathbb{R} \times \Pi_{x_k} \to \Omega, \quad (t, \xi^k) \mapsto \Phi(t, \xi^k).$$

We remark that $\Psi_0 = \Psi$. The claims (b) and (c) applied to the point x_k instead of x_0 imply that there exists R_k such that, for every $\xi^k \in B(x_k, R_k) \cap \Pi_k$, $\Psi_k(t_k(\xi^k), \xi^k)$ belongs to $\Pi_{x_{k+1}}$ for some $t_k(\xi^k) \in [0, T]$ and the map $\xi^k \mapsto \Psi_k(t_k(\xi^k), \xi^k)$ is Lipschitz, injective and its inverse is Lipschitz. By choosing R_k small enough we may assume that for every k we have

$$\forall \xi^k \in B(x_k, R_k) \cap \Pi_{x_k} \quad \left|\Psi_k\big(t_k(\xi^k), \xi^k\big) - x_{k+1}\right| \leqslant R_{k+1}.$$

We set $T_k(\xi) = t_1(\xi) + \dots + t_k(\xi)$; it follows that for every ξ in $B(x_0, R_0) \cap \Pi_{x_0}$ and t in $]T_k(\xi) - T, T_k(\xi) + T[$ we have

$$\Phi(t, \xi) = \Psi_k\big(t - T_k(\xi), \Phi(T_k(\xi), \xi)\big). \tag{4.3}$$

By (c) there exist $N_k \subset \Pi_{x_k}$ and $N_{k+1} \subset \Pi_{x_{k+1}}$ of $(n-1)$-Hausdorff measure zero and v_k^*, v_{k+1}^* equal to v a.e. such that the map $t \mapsto v_k^*(\Psi_k(t, \xi^k))$ is absolutely continuous on $]-T, T[$ for every ξ^k in $B(x_k, R_k) \cap \Pi_{x_k} \setminus N_k$. Since the inverse of $\xi^k \mapsto \Psi_k(t_k(\xi^k), \xi^k)$ is Lipschitz the inverse image of N_{k+1} has $(n-1)$-Hausdorff measure zero and therefore it is not restrictive to assume that

$$\forall \xi^k \in B(x_k, R_k) \cap \Pi_{x_k} \setminus N_k \quad \Psi_k\big(t_k(\xi^k), \xi^k\big) \notin N_{k+1}.$$

It follows from (4.3) that for every ξ in $B(x_0, R_0) \cap \Pi_{x_0} \setminus N_0$ the map $t \mapsto v_k^*(\Phi(t, \xi))$ is absolutely continuous on $]T_k(\xi) - T, T_k(\xi) + T[$. Let $E_k = \Psi_k(]-T, T[\times B(x_k, R_k))$, $(\vartheta_k)_k$ be a partition of the unity of $(E_k)_k$ and set $v^* = \sum_{k=0}^m \vartheta_k v_k^*$. For every k the function $t \mapsto \vartheta_k(\Phi(t, \xi))$ is the composition of two Lipschitz functions and its support is contained in $]T_k(\xi) - T, T_k(\xi) + T[$; moreover, the map $t \mapsto v_k^*(\Phi(t, \xi))$ is absolutely continuous on the same interval. It follows that v^* fulfills the requirements of the claim. □

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References

[1] R.A. Adams, Sobolev Spaces, in: Pure Appl. Math., Vol. 65, Academic Press, New York, 1975.

[2] A. Cellina, On the bounded slope condition and the validity of the Euler–Lagrange equation, SIAM J. Control Optim. 40 (2001/02) 1270–1279.

[3] L. Evans, R. Gariepy, Measure Theory and Fine Properties of Functions, in: Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.
[4] I. Fonseca, W. Gangbo, Degree Theory in Analysis and Applications, in: Oxford Lecture Ser. Math. Appl., Vol. 2, Oxford Sci. Publ./Clarendon, Oxford, 1995.
[5] E. Hille, Lectures on Ordinary Differential Equations, in: Addison–Wesley Ser. Math., Addison–Wesley, Reading, MA, 1969.
[6] M. Marcus, V.J. Mizel, Absolute continuity on tracks and mappings of Sobolev spaces, Arch. Rational Mech. Anal. 45 (1972) 294–320.
[7] C. Mariconda, G. Treu, A comparison principle for minimizers, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 681–686.
[8] J. Malý, W.P. Ziemer, Fine Regularity of Solutions of Elliptic Partial Differential Equations, in: Math. Surveys Monographs, Vol. 51, American Mathematical Society, Providence, RI, 1997.
[9] W.P. Ziemer, Weakly Differentiable Functions, Springer-Verlag, New York, 1989.