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A Comparison Principle and the Lipschitz Continuity for Minimizers

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We give some conditions that ensure the validity of a Comparison Principle for the minimizers of integral functionals, without assuming the validity of the Euler–Lagrange equation. We deduce a weak Maximum Principle for (possibly) degenerate elliptic equations and, together with a generalization of the Bounded Slope Condition, a result on the Lipschitz continuity of minimizers.

Keywords: Comparison principle, maximum principle, variational equation, Euler Lagrange equation, elliptic equation, Bounded Slope condition, regularity of minimizers

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1. Introduction

The classical *Comparison Principle* for *variational equations* [5, Thm. 10.7] concerns the weak solutions of the Euler–Lagrange equation

$$\text{div } L_p(x,u,\nabla u) - L_z(x,u,\nabla u) = 0 \tag{E}$$

where $L(x,z,p) : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is differentiable and convex in (z,p) (Ω being open and bounded in $\mathbf{R}^n$). It assumes that L is *elliptic*, i.e., L is twice differentiable in p and $\sum_{i,j} L_{p_ip_j}(x,z,p)\xi_i\xi_j > 0$ for every $(\xi_1,\dots,\xi_n) \neq 0$ and states that if w (resp. ℓ) in $\mathcal{C}^1(\Omega)$ is a weak solution (resp. weak supersolution) of (E) satisfying $w \leq \ell$ on the boundary of Ω then $w \leq \ell$ on Ω.

In this paper we study the problem of the validity of a *Comparison Principle* for functions w that are the solutions of the minimum problem

$$\text{minimize} \quad \int_\Omega L(x,u,\nabla u)\,dx \qquad \text{on } \bar{u} + W_0^{1,q}(\Omega), \tag{P}$$

$\bar{u}$ being a prescribed boundary datum in $W^{1,q}(\Omega)$, ($q \geq 1$). As it is well known, some polynomial growth conditions on L and its dervatives ensure that a minimizer of (P) is a

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weak solution to (E) [3]. The point here is that we do not assume any growth condition and thus a comparison principle for minimizers can not be deduced from the classical one. Our work was inspired by a recent result obtained by Cellina in [1] stating that if L is a function of the variable p, i.e. $L(x,z,p) = f(p)$, whose epigraph f has no n–dimensional faces, then a minimizer w of $I(u) = \int_\Omega f(\nabla u)\,dx$ that lies below an affine fuction $\ell(x) = a \cdot x + b$ on $\partial\Omega$ satisfies the inequality $w \le \ell$ on Ω.
In our Main Theorem we give a list of *seven* alternative assumptions (Assumption B) on the pair (L,ℓ) ensuring that if w is a solution of (P) satisfying $w \le \ell$ on $\partial\Omega$ then $w \le \ell$ on Ω (*Comparison Principle* for minimizers). We underline that we do not require any differentiability assumption on L and that in Assumptions B1.1, B1.2, B3, B4 we do not impose that L is the sum of two functions. Assumption B3 contains the case where L is elliptic and the other cases allow the epigraph of the convex function $p \mapsto L(x,z,p)$ to have some non trivial faces. As a particular case of Assumption B4 we obtain the hypothesis of Theorem 2 in [1], i.e. that $L(x,z,p) = f(p)$, ℓ is affine and the epigraph of f does not contain n–dimensional faces.
Section 4 is devoted to the applications of Theorem 3.14 to variational equations. The main difference with respect to the classical *Comparison Principle* is that we do not impose the ellipticity on L, although we require that the function L is either monotonic in z or the sum of two functions (i.e. $L(x,z,p) = f(x,p) + g(x,z)$). In particular we obtain a *weak Maximum Principle* for (possibly degenerate) elliptic equations.

In the last section we study the existence of a Lipschitz continuous solution to (P). In [1] Cellina studies the same problem for functions $L(x,z,p) = f(p)$ of the variable p and shows that, if the boundary datum $\bar u$ satisfies the *Bounded Slope Condition* (*B.S.C.*), introduced by Stampacchia in a variational context – i.e. at every point $\bar x$ of $\partial\Omega$ the function $\bar u$ is bounded from above and below by two affine functions, with uniformly bounded slopes, that coincide with $\bar u$ at $\bar x$ – and f is strictly convex, then the minimizer of $I(u) = \int_\Omega f(\nabla u)\,dx$ on $\bar u + W_0^{1,1}(\Omega)$ is Lipschitz continuous. We generalize the (*B.S.C.*) and say that the pair $(L,\bar u)$ satisfies the *Generalized Bounded Slope Condition* (*G.B.S.C.*) if at every point $\bar x$ of $\partial\Omega$ there exist a subsolution $\ell^1_{\bar x}$ and a supersolution $\ell^2_{\bar x}$ of (E), coinciding with $\bar u$ at $\bar x$ that are Lipschitz continuous with uniformly bounded Lipschitz constant and such that $\ell^1_{\bar x} \le \bar u \le \ell^2_{\bar x}$ on $\partial\Omega$. A slight modification of the results obtained by Treu and Vornicescu in [9] allows us to give in Theorem 5.1 some conditions on the function L (Assumption D) under which a solution to (P) that lies between two Lipschitz continuous functions in $\bar u + W_0^{1,q}(\Omega)$ is Lipschitz continuous too. Surprisingly it turns out that if the pair $(L,\bar u)$ fulfills the (*G.B.S.C.*) and L satisfies Assumption D then $(L,\ell^1_{\bar x})$ and $(L,\ell^2_{\bar x})$ fulfill the requirements of our *Comparison Principle* (Thm. 3.14): its application yields the Lipschitz continuity of the minimizers of I (Thm. 5.4).
In a particular case we obtain the conclusion of the aforementioned result of Cellina for functionals of the gradient without assuming the continuity of the minimizer. Just to mention an application we may consider a nonsmooth convex functional of the gradient with superlinear anisotropic growth. In this case the classical theory yields the existence of a minimizer in a Sobolev space in $W^{1,p}(\Omega)$, with no information on its continuity if $p < n$. Our result then implies that, if the boundary datum satisfies the (*B.S.C.*), the minima are necessarily Lipschitz. As in [1, Thm. 3], when L is smooth, this result implies that the minimizers of I satisfy the weak Euler–Lagrange equation.

2. Notation

In this paper we consider an open bounded set Ω of $\mathbf{R}^n$ ($n \ge 1$). For a subset A of $\mathbf{R}^n$ we denote by $\overline{A}$ (resp. ∂A) the closure (resp. the boundary) of A. Given two vectors a and b in $\mathbf{R}^n$ we denote by $a \cdot b$ their usual scalar product in $\mathbf{R}^n$ and by $|a|$ the euclidean norm of a. We set $B(0,R]$ to be the closed ball of radius R centered in 0.
By $C^\infty(\Omega)$ (resp. $C_c^\infty(\Omega)$) we denote the space of infinitely differentiable functions in Ω (resp. infinitely differentiable functions with compact support in Ω) and, for $r \ge 1$ and A open in $\mathbf{R}^n$, $L^r(A)$ (resp. $W^{1,r}(A)$, $W_0^{1,r}(A)$) is the usual space of the Lebesgue functions (resp. the usual first order Sobolev spaces) of exponent r; for u in $W^{1,r}(\Omega)$ the weak derivative of u with respect to x_i is denoted by u_{x_i} and for $u = (u_1,\dots,u_n)$ in $W^{1,r}(\Omega;\mathbf{R}^n)$ the divergence of u is div $u = \sum_i (u_i)_{x_i}$. We fix a real number $q \ge 1$ and we set q' to be its conjugate, i.e. $q' = \frac{q}{q-1}$. For u in $W^{1,q}(\Omega)$ by $u \ge 0$ on $\partial\Omega$ we mean that there exists a sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $C^\infty(\overline{\Omega})$, $\varphi_n \ge 0$ on $\partial\Omega$, converging to u in $W^{1,q}(\Omega)$. For u, v in $W^{1,q}(\Omega)$ by $u \le v$ in $\partial\Omega$ we mean that $v - u \ge 0$ in $\partial\Omega$. We set $u \wedge v = \min\{u,v\}$, $u^+ = \max\{u,0\}$; we recall that these functions still belong to $W^{1,q}(\Omega)$ and that if $u \le 0$ on $\partial\Omega$ then u^+ belongs to $W_0^{1,q}(\Omega)$.
For $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ being a convex function we denote by Dom(f) the *effective domain* of f i.e. Dom$(f) = \{p \in \mathbf{R}^n : f(p) < +\infty\}$; by epi$(f)$ its *epigraph*. A face of epi(f) is a convex extremal subset of epi(f); it is said to be exposed if it coincides with the intersection of epi(f) with a supporting hyperplane; it is said to be trivial if it is reduced to a point. The subdifferential of f at p (in the usual sense of convex analysis) is denoted by $\partial f(p)$. We will use the fact that if $k \in \partial f(p_1)$ and there exists $p_2 \ne p_1$ such that $f(p_2) = f(p_1) + k \cdot (p_2 - p_1)$ then the set

$$\{(p,\zeta) \in \mathbf{R}^n \times \mathbf{R} : \zeta = f(p_1) + k \cdot (p - p_1)\} \cap \text{epi}\,(f)$$

is a non trivial (exposed) face of epi(f).
If $L : \Omega \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R} \cup \{+\infty\}$, $(x,z,p) \longmapsto L(x,z,p)$, is differentiable with respect to z (resp. to $p = (p_1,\dots,p_n)$) we denote by L_z (resp. L_{p_i}, $i = 1,\dots,n$) the partial derivative of L with respect to z (resp. p_i) and by L_p or $\nabla_p L$ (simply ∇L if $L(x,z,p) = L(p)$) the gradient of L with respect to p. By $L_{x_ix_j}$, L_{zz}, $L_{p_ip_j}$ we denote the usual second derivatives of L. In the case where $L(x,z,p)$ is convex in z (resp. p) $\partial_z L$ (resp. $\partial_p L$) is the subdifferential of the map $z \longmapsto L(x,z,p)$ (resp. $p \longmapsto L(x,z,p)$). For $u : \Omega \to \mathbf{R}$, $v : \Omega \to \mathbf{R}^n$ and x in Ω we often write $L(x,u,v)$ instead of $L(x,u(x),v(x))$.

3. Main results

In what follows the function

$$L : \Omega \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R} \cup \{+\infty\}$$
$$(x,z,p) \longmapsto L(x,z,p)$$

is such that $x \longmapsto L(x,z(x),p(x))$ is measurable for every measurable $z : \Omega \to \mathbf{R}$ and $p : \Omega \to \mathbf{R}^n$ (this condition is fulfilled if, for instance, L is a normal integrand, see [4]); ℓ is a function in $W^{1,q}(\Omega)$.

For k in $L^{q'}(\Omega;\mathbf{R}^n)$ and h in $L^{q'}(\Omega)$ by div $k - h \le 0$ (resp. ≥ 0) we mean that the inequality holds in a weak sense, i.e.

$$\forall \eta \in W_0^{1,q}(\Omega),\ \eta \ge 0, \qquad \int_\Omega k \cdot \nabla\eta + h\eta\,dx \ge 0 \quad (\text{resp. } \le 0). \tag{1}$$

We also write that div $k-h<0$ (resp. div $k-h>0$) or that the inequality div $k-h\le 0$ (resp. div $k-h\ge 0$) is *strict* if (1) is strict whenever $\eta\neq 0$. It is obvious that if k is in $W^{1,q'}(\Omega;\mathbf{R}^n)$ and div $k(x)-h(x)\le 0$ (resp. <0) a.e. then div $k-h\le 0$ (resp. <0) weakly since, for every η in $W_0^{1,q}(\Omega)$, we have $\int_\Omega(\text{div } k-h)\eta\,dx=-\int_\Omega k\cdot\nabla\eta+h\eta\,dx$.

Assumption A. We say that the pair (L,ℓ) satisfies Assumption A if either $A1$ or A2 holds:

A1. The function $L(x,z,p)$ is convex in p, nondecreasing in z and there exists k in $L^{q'}(\Omega;\mathbf{R}^n)$ such that $k(x)\in\partial_pL(x,\ell(x),\nabla\ell(x))$ a.e. and

$$\text{div } k\le 0. \tag{2}$$

A2. The function $L(x,z,p)=f(x,p)+g(x,z)$ is convex in (z,p); there exist k in $L^{q'}(\Omega;\mathbf{R}^n)$ and h in $L^{q'}(\Omega)$ such that $k(x)\in\partial_pf(x,\nabla\ell(x))$ a.e., $h(x)\in\partial_zg(x,\ell(x))$ a.e. and

$$\text{div } k-h\le 0. \tag{3}$$

Remark 3.1. When $L(x,z,p)=f(x,p)$ is convex in p the Assumptions A1 and A2 are equivalent.

Remark 3.2. When $L(x,z,p)$ is differentiable with respect to z and p the Assumptions $A1$, $A2$ can be rewritten as

A1diff. The function $L(x,z,p)$ is convex in p, nondecreasing in z and

$$\text{div } L_p(x,\ell(x),\nabla\ell(x))\le 0$$

i.e. ℓ is a *supersolution* of the equation div $L_p(x,u,\nabla u)=0$.

A2diff. The function $L(x,z,p)=f(x,p)+g(x,z)$ is convex in (z,p) and

$$\text{div } f_p(x,\nabla\ell(x))-g_z(x,\ell(x))\le 0$$

i.e. ℓ is a *supersolution* of the equation div $f_p(x,\nabla u)-g_z(x,u)=0$.

Remark 3.3 (Assumption A′). In the sequel we will also use a dual set of hypothesis that we will call Assumption A′ (A′1, A′2) where the inequalities that appear in Assumption A are reversed and the term "nondecreasing" in Assumption A1 is replaced by "nonincreasing" in Assumption A′1.

Remark 3.4. We notice that if $L(x,z,p)=f(p)$ and f, ℓ are smooth then the divergence of $f_p(\nabla\ell(x))$ is the trace of the product of the Hessian of f at $\nabla\ell(x)$ and the Hessian of ℓ at x; in fact div $f_p(\nabla\ell(x))=\sum_i\partial_{x_i}(f_{p_i}(\nabla\ell(x)))=\sum_{i,j}f_{p_ip_j}(\nabla\ell(x))\ell_{x_ix_j}(x)$. It follows that Assumption A1diff is fulfilled if, for instance, f is convex and ℓ is concave (this is a consequence of the fact that if $(a_{ij})_{i,j}$ and $(b_{ij})_{i,j}$ are semidefinite then the quadratic form $\sum_{i,j}a_{ij}b_{ij}\xi_i\xi_j$ is semidefinite [6, Ex. VII.35]). However there exist $L(x,z,p)=f(p)$ convex and $\ell(x)$ non concave fulfilling A1, as for instance $f(p_1,p_2)=p_1^2+p_2^2$, $\ell(x_1,x_2)=x_1^2-2x_2^2$.

In what follows we fix $\bar u$ in $W^{1,q}(\Omega)$; we set $W_{\bar u}^{1,q}(\Omega)=\bar u+W_0^{1,q}(\Omega)$ and we define the functional I by

$$\forall u\in W_{\bar u}^{1,q}(\Omega)\qquad I(u)=\int_\Omega L(x,u(x),\nabla u(x))\,dx.$$

Proposition 3.5. *Let (L,ℓ) satisfy* Assumption A *and w be a minimizer of I in $W_{\bar u}^{1,q}(\Omega)$, i.e.*

$$I(w)=\min\{I(u):\ u\in W_{\bar u}^{1,q}(\Omega)\}.$$

Assume that $w\le l$ on $\partial\Omega$ and set $E^+=\{x\in\Omega:\ w(x)>\ell(x)\}$. Then

$$L(x,\ell(x),\nabla w(x))-L(x,\ell(x),\nabla\ell(x))=k(x)\cdot(\nabla w(x)-\nabla\ell(x))\quad\text{a.e. on } E^+ \tag{4}$$

If (L,ℓ) satisfies A1*, then*

$$L(x,w(x),\nabla\ell(x))=L(x,\ell(x),\nabla\ell(x))\quad\text{a.e. on } E^+; \tag{5}$$

If $L(x,z,p)=f(x,p)+g(x,z)$ and (L,ℓ) satisfies A2*, then*

$$g(x,w(x))-g(x,\ell(x))=h(x)(w(x)-\ell(x))\quad\text{a.e. on } E^+ \tag{6}$$

Moreover, if E^+ is non negligible, then the inequalities (2) *- if* A1 *holds - and* (3) *- if* A2 *holds - are not strict.*

Proof. We first remark that $w\wedge\ell$ belongs to $W_{\bar u}^{1,q}(\Omega)$ and therefore $I(w\wedge\ell)\ge I(w)$. Since the gradient of $w\wedge\ell$ is given by

$$\nabla(w\wedge\ell)=\begin{cases}\nabla\ell & \text{on } E^+\\ \nabla w & \text{on } \Omega\setminus E^+\end{cases}$$

(see [5]) then we obtain

$$0\ge I(w)-I(w\wedge\ell)=\int_{E^+}L(x,w,\nabla w)-L(x,\ell,\nabla\ell)\,dx. \tag{7}$$

Assume that Assumption A1 holds. Since $w>\ell$ on E^+, the monotonicity assumption yields $L(x,w(x),\nabla w(x))\ge L(x,\ell(x),\nabla w(x))$ a.e. on E^+, whereas the convexity assumption implies $L(x,\ell(x),\nabla w(x))-L(x,\ell(x),\nabla\ell(x))\ge k(x)\cdot(\nabla w(x)-\nabla\ell(x))$ a.e. on E^+. Hence, writing that

$$L(x,w,\nabla w)-L(x,\ell,\nabla\ell)=L(x,w,\nabla w)-L(x,\ell,\nabla w)+L(x,\ell,\nabla w)-L(x,\ell,\nabla\ell),$$

by (7) we have

$$\begin{aligned}0&\ge\int_{E^+}L(x,w,\nabla w)-L(x,\ell,\nabla w)\,dx+\int_{E^+}L(x,\ell,\nabla w)-L(x,\ell,\nabla\ell)\,dx\\&\ge\int_{E^+}k\cdot(\nabla w-\nabla\ell)\,dx.\end{aligned}$$

Now we remark that $\nabla w-\nabla\ell=\nabla(w-\ell)^+$ a.e. on E^+, $(w-\ell)^+\in W_0^{1,q}(\Omega)$ and $(w-\ell)^+=0$ on $\Omega\setminus E^+$. Therefore

$$\int_{E^+}k\cdot(\nabla w-\nabla\ell)\,dx=\int_\Omega k\cdot\nabla(w-\ell)^+\,dx\ge 0$$

by the assumption on the weak divergence of k. It follows that the above inequalities are indeed equalities. Then $\int_{E^+} k \cdot \nabla(w-\ell)^+ \, dx = 0$ so that, if E^+ is non negligible, the inequality div $k \le 0$ (in the weak sense) is not *strict*. It follows that

$$0 = \int_{E^+} L(x,w,\nabla w) - L(x,\ell,\nabla w)\, dx + \int_{E^+} L(x,\ell,\nabla w) - L(x,\ell,\nabla \ell) - k\cdot(\nabla w - \nabla \ell)\, dx$$

and the two integrands are non negative: we deduce that

$$\begin{cases} L(x,w(x),\nabla w(x)) = L(x,\ell(x),\nabla w(x)) \quad \text{and} \\ L(x,\ell(x),\nabla w(x)) - L(x,\ell(x),\nabla\ell(x)) = k(x)\cdot(\nabla w(x) - \nabla \ell(x)) \end{cases} \quad \text{a.e. on } E^+,$$

proving the claim under Assumption A1.
If Assumption A2 holds then (7) can be rewritten as

$$0 \ge \int_{E^+} f(x,\nabla w) - f(x,\nabla \ell)\, dx + \int_{E^+} g(x,w) - g(x,\ell)\, dx. \tag{8}$$

The convexity of $z \mapsto g(x,z)$ and $p \mapsto f(x,p)$ yields

$$\begin{cases} f(x,\nabla w(x)) - f(x,\nabla\ell(x)) \ge k(x)\cdot(\nabla w(x) - \nabla\ell(x)) \\ g(x,w(x)) - g(x,\ell(x)) \ge h(x)(w(x) - \ell(x)) \end{cases} \quad \text{a.e. on } E^+$$

Therefore, using (8), we obtain

$$0 \ge \int_{E^+} k\cdot(\nabla w - \nabla\ell) + h(w-\ell)\, dx \ge 0$$

since div $k-h \le 0$, so that the latter inequalities are in fact equalities and thus div $k-h \le 0$ is not strict. Using again (8) it follows that

$$0 \ge \int_{E^+} f(x,\nabla w) - f(x,\nabla\ell) - k\cdot(\nabla w - \nabla \ell)\, dx + \int_{E^+} g(x,w) - g(x,\ell) - h(w-\ell)\, dx \ge 0$$

since the two integrands are non negative. Again we deduce that the two integrands vanish a.e. on E^+, proving the claim. □

We will give a set of assumptions (Assumption B) that will ensure the validity of a *Comparison Principle* for minimizers. In order to state Assumption B4 we need a definition. For every x in Ω let $L^x : \mathbf{R}^n \to \mathbf{R}$ be the function defined by

$$\forall p \in \mathbf{R}^n \qquad L^x(p) = L(x,\ell(x),p).$$

Assume that (L,ℓ) satisfies Assumption A. Then, for every x in Ω, to the selection $k(x)$ of $\partial_p L(x,\ell(x),\nabla\ell(x))$ we associate the exposed face $F^L(x)$ of the epigraph of L^x containing $(\nabla\ell(x), L^x(\nabla\ell(x)))$ defined by

$$F^L(x) = \operatorname{epi} L^x \cap \{(p,\zeta)\in \mathbf{R}^n\times\mathbf{R} : \zeta - L^x(\nabla\ell(x)) = k(x)\cdot(p - \nabla\ell(x))\}.$$

We denote by $F^L_{\mathbf{R}^n}(x)$ the vector space spanned by the projection of $F^L(x)$ onto the first n coordinates of $\mathbf{R}^n\times\mathbf{R}$. We notice that if $L(x,z,p) = f(x,p) + g(x,z)$ then, for every x, the function L^x is given by $L^x(p) = f(x,p) + g(x,\ell(x))$ so that the faces of L^x are obtained by translating the faces of the map $p \mapsto f(x,p)$.

Definition 3.6 (The cone property). We say that the vector field $S : \Omega \to \mathbf{R}^n$ satisfies the *cone property* if S is Lipschitz continuous in Ω and there exist a vector u in $\mathbf{R}^n$ and $\alpha > 0$ such that $S(x)\cdot u \ge \alpha$ for every x in Ω.

Remark 3.7. Geometrically this condition implies that the vector field S does not vanish in Ω and that, for every x in Ω, the vector $S(x)$ belongs to a cone $\{y \in \mathbf{R}^n : y\cdot u \ge \beta|y|\}$ for some non negative β. It is easy to show that the solution of the autonomous system $\dot{x} = S(x)$, $x(0) = \bar{x} \in \Omega$ reaches the boundary of Ω in a finite time.

The next result, that we prove in [6, Thm. 4.6], will be used in the proof of the Main Theorem under Assumption B4.

Theorem 3.8. *Let v be a non negative function in $W_0^{1,q}(\Omega)$. Assume that there exists a vector field $S : \Omega \to \mathbf{R}^n$ satisfying the cone property such that $S\cdot\nabla v \le 0$ a.e. on Ω. Then $v = 0$ a.e. on Ω.*

Assumption B. We say that the pair (L,ℓ) satisfies Assumption B if at least one of the following conditions B1, B2, B3 or B4 holds.

B1. The pair (L,ℓ) satisfies A1 – i.e. the function $L(x,z,p)$ is convex in p, nondecreasing in z and there exists k in $L^{q'}(\Omega;\mathbf{R}^n)$ such that $k(x) \in \partial_p L(x,\ell(x),\nabla\ell(x))$ a.e. and div $k \le 0$ – and (just) one of the following items B1.1 or B1.2 holds:

B1.1. The inequality div $k \le 0$ is *strict*;

B1.2. The map $z \longmapsto L(x,z,p)$ is increasing for almost every x in Ω and every p in $\mathbf{R}^n$.

B2. The pair (L,ℓ) satisfies A2 – i.e. the function $L(x,z,p) = f(x,p) + g(x,z)$ is convex in (z,p); there exist k in $L^{q'}(\Omega;\mathbf{R}^n)$ and h in $L^{q'}(\Omega)$ such that $k(x) \in \partial_p f(x,\nabla\ell(x))$ a.e., $h(x) \in \partial_z g(x,\ell(x))$ a.e. and div $k - h \le 0$ – and (just) one of the following items B2.1, B2.2 or B2.3 holds:

B2.1. The inequality div $k - h \le 0$ is *strict*;

B2.2. The faces of $z \mapsto g(x,z)$ containing $(\ell(x), g(x,\ell(x)))$ are reduced to a point for almost every x;

B2.3. The function L does not depend on p, i.e. $L(x,z,p) = g(x,z)$, and the lines containing the non trivial faces of $z \mapsto g(x,z)$ have positive slope.

B3. The pair (L,ℓ) satisfies Assumption A (i.e. A1 or A2) and the faces of the map $p \longmapsto L(x,\ell(x),p)$ containing $(\nabla\ell(x), L(x,\ell(x),\nabla\ell(x)))$ are reduced to a point for almost every x.

B4. The pair (L,ℓ) satisfies Assumption A and there exists $S : \Omega \to \mathbf{R}^n$ with the *cone property* such that, for almost every x in Ω, $S(x)$ is orthogonal to $F^L_{\mathbf{R}^n}(x)$.

Remark 3.9. The strict convexity of $z \mapsto g(x,z)$ implies B2.2 and B2.3 and the strict convexity of $p \longmapsto L(x,z,p)$ implies B3.

We give some examples to illustrate Assumption B4.

Example 3.10. Assumption B3 is a particular case of B4: under B3 the exposed face $F^L(x)$ is reduced to the point $(\nabla\ell(x), L(x,\ell(x),\nabla\ell(x)))$ so that $F^L_{\mathbf{R}^n}(x) = 0$ and thus any constant vector u of $\mathbf{R}^n$ is orthogonal to $F^L_{\mathbf{R}^n}(x)$ for every x. It is obvious that if $u \neq 0$ the constant vector field $S(x) = u$ satisfies the *cone property*. However we formulate Assumption B3 separately since, in this case, the proof of the Main Theorem does not require Theorem 3.8 involving the *cone property*.

Example 3.11. Assumption B4 is fulfilled if $L(x,z,p) = f(p)$ where f is convex and lower semicontinuous, the epigraph of f does not contain n–dimensional faces, ℓ is affine, i.e. $\ell(x) = a \cdot x + b$ ($a \in \mathbf{R}^n$, $b \in \mathbf{R}$) and f is subdifferentiable in a. In fact fix k in $\partial f(a)$: Assumption A1 is fulfilled since the divergence of the constant k is zero; the exposed face $F^L(x)$ of epi f corresponding to k is fixed and has dimension less or equal than $n-1$, therefore there exists a non zero vector u in $\mathbf{R}^n$ that is orthogonal to $F^L_{\mathbf{R}^n}(x)$ for every x in Ω. The constant vector field $S(x) = u$ satisfies the *cone property*.

Example 3.12. Assumption B4 is fulfilled if (L,ℓ) satisfies Assumption A and the vector spaces of the faces of the epigraph of L^x are contained in a prescribed vector subspace of $\mathbf{R}^{n+1}$ of dimension less or equal than $n-1$. In fact, as in the Example 3.11, we can choose a constant non zero vector field S such that $S(x)$ is orthogonal to $F^L_{\mathbf{R}^n}(x)$ for every x.

Example 3.13. Let $n = 2$, $L(x,z,p) = f(x,p) + g(x,z)$ and f be smooth. Assume that, for every x in Ω, the vector spaces of the projections of the faces of the function $p \mapsto f(x,p)$ lie in the cone $\{y \in \mathbf{R}^2 : |y \cdot u| \geq \alpha|y|\}$, ($\alpha > 0$, $u \in \mathbf{R}^2 \setminus \{(0,0)\}$). Then, for every function ℓ, the pair (L,ℓ) fulfills Assumption B4. In fact let $u^\perp$ be a non zero vector orthogonal to u and Σ be a smooth vector field of unit vectors such that $\Sigma(x)$ is orthogonal to $F^L(x)$ for every x: then either $\Sigma(x) \cdot u^\perp \geq \alpha$ on Ω or $-\Sigma(x) \cdot u^\perp \geq \alpha$ on Ω. We set, respectively, $S = \Sigma$ or $S = -\Sigma$.

Example 3.14 (The radial case). Let $L(p) = f(|p|)$ for some superlinear convex function $f : [0,+\infty[\to \mathbf{R}$ and let Ω be an annulus centered at the origin. The set of points where f is locally affine is a countable union of intervals $N = \cup_n [a_n, b_n]$. Let $\ell(x) = \varphi(|x|)$ where φ is absolutely continuous ($\varphi'(0) = 0$ if $0 \in \Omega$). Assume that the set $C = \{r : \varphi'(r) \in N\}$ is at most countable. Then the pair (L,ℓ) fulfills Assumption B3. In fact the faces of L at $(p, L(p))$ are not reduced to a point if and only if $|p|$ belongs to N. Now $\nabla\ell(x) = \varphi'(|x|)\frac{x}{|x|}$ if $x \neq 0$ ($\nabla\ell(0) = 0$ if $0 \in \Omega$) and therefore the faces of L at $\nabla\ell(x)$ are not reduced to a point if and only if $|x| \in C$.

We are now in the position to state the *Comparison Principle* for minimizers.

Main Theorem 3.15. *Let (L,ℓ) satisfy Assumption B and w be a minimizer of I in $W^{1,q}_{\bar u}(\Omega)$. Assume that $w \leq \ell$ on $\partial\Omega$. Then $w \leq \ell$ a.e. in Ω.*

Proof. As in Proposition 3.5 we introduce the set

$$E^+ = \{x \in \Omega : w(x) > \ell(x)\};$$

we assume by contradiction that E^+ has positive measure. By Proposition 3.5 the inequalities (2) (under B1) and (3) (under B2) are not strict, proving the claim under Assumptions B1.1 and B2.1.
If (L,ℓ) satisfies B1.2 then the equality (5) implies that $\ell(x) = w(x)$ a.e. on E^+; analogously, if (L,ℓ) satisfies B2.2 then the equality (6) implies again that $\ell(x) = w(x)$ a.e. on E^+, a contradiction. The same equality (6) shows that for almost every x in E^+ the set

$$\{(z,\zeta) \in \mathbf{R} \times \mathbf{R} : \zeta - g(x,\ell(x)) = h(x)(z - \ell(x))\} \cap \operatorname{epi}(z \mapsto g(x,z))$$

is a non trivial face of epi $(z \mapsto g(x,z))$. Assumption B2.3 then implies that $h > 0$ a.e. on E^+ and thus

$$0 \geq I(w) - I(w \wedge \ell) = \int_{E^+} h(x)(w(x) - \ell(x))\,dx > 0,$$

a contradiction since $w \wedge \ell$ belongs to $W^{1,q}_{\bar u}(\Omega)$.
Under Assumption B3 the equality (4) implies that $\nabla w = \nabla\ell$ on E^+ and therefore $\nabla(w-\ell)^+ = 0$ on Ω, yielding $w = \ell$ on E^+; proving the claim in all cases but B4.
Finally assume that B4 holds. By Proposition 3.5 we have

$$L(x,\ell(x),\nabla w(x)) - L(x,\ell(x),\nabla\ell(x)) = k(x) \cdot (\nabla w(x) - \nabla\ell(x)) \quad \text{a.e. on } E^+$$

so that, for almost every x in E^+, the points $(\nabla\ell(x), L^x(\nabla\ell(x)))$ and $(\nabla w(x), L^x(\nabla w(x)))$ belong to the face $F^L(x)$ and therefore the vector $\nabla w(x) - \nabla\ell(x)$ is in the projection $F^L_{\mathbf{R}^n}(x)$ of $F^L(x)$ onto the first n-coordinates of $\mathbf{R}^n \times \mathbf{R}$. Now, Assumption B4 implies the existence of a vector field $S : \Omega \to \mathbf{R}^n$ with the *cone property* satisfying $S \cdot (\nabla w - \nabla\ell) = 0$ a.e. on E^+. Since $(w-\ell)^+ \in W^{1,q}_0(\Omega)$ and its gradient is given by

$$\nabla(w-\ell)^+ = \begin{cases} \nabla w - \nabla\ell & \text{on } E^+ \\ 0 & \text{on } \Omega \setminus E^+ \end{cases}$$

then the previous equality yields $S \cdot \nabla(w-\ell)^+ = 0$ a.e. on Ω. Obviously, the function $(w-\ell)^+$ is non negative in Ω; Theorem 3.8 implies that $(w-\ell)^+ = 0$ on Ω and therefore E^+ is negligible, proving the claim. □

Remark 3.16 (Assumption B′). A similar result holds, with the obvious changes, in the case where $\ell \leq w$ on $\partial\Omega$. Namely it is enough to reformulate Assumption B by reversing the inequalities, by replacing Assumption A by Assumption A′ (Remark 3.3) and the term "nondecreasing" (resp. "non negative") by "nonincreasing" (resp. "non positive"). We will refer in the sequel to this dual set of hypotheses as Assumption B′ (B′1,...).

Remark 3.17. We point out that in Theorem 3.15 we do not assume any growth assumption on L or on its derivatives. In particular the minimizers of I do not necessarily satisfy the *Euler–Lagrange* equation and our claim does not follow from the classical *Comparison Principle* for variational equations [5, Thm. 10.7].

We have to underline that Assumption B seems to be rather technical. In fact a straightforward application of our Main Theorem yields the following result.

Theorem 3.18. *Let ℓ be such that $\ell = \inf_{\lambda\in\Lambda} \ell_\lambda$ for some family of functions ℓ_λ in $W^{1,q}(\Omega)$. Assume that (L,ℓ_λ) satisfies* Assumption B *for every λ in Λ and that w is a minimizer of I on $W^{1,q}_{\bar u}(\Omega)$ satisfying $w \leq \ell_\lambda$ on $\partial\Omega$ for every λ in Λ. Then $w \leq l$ a.e. on Ω. In particular, if ℓ belongs to $W^{1,q}(\Omega)$ and $w \leq \ell$ on $\partial\Omega$ then $w \leq \ell$ a.e. on Ω.*

Remark 3.19. Again a dual result yielding a bound from below holds under the obvious changes formulated in Remark 3.16.

Remark 3.20. The previous theorem applies to the case where L is a function of the variable p, i.e. $L(x,z,p) = f(p)$, where $f : \mathbf{R}^n \to \mathbf{R}$ is a lower semicontinuous convex function whose epigraph has no n–dimensional faces and ℓ is concave, upper semicontinuous on $\overline{\Omega}$. In fact, in this case, ℓ is the infimum of a family of affine functions ℓ_λ [4]; Example 3.11 yields the conclusion. We point out that, in this situation, Assumption B is not necessarily fulfilled, as in the following example (showing that Assumption B is not necessary for the validity of the *Comparison Principle*).

Example 3.21. Let Ω be the subset of $\mathbf{R}^2$ defined by $\Omega = \{ x = (x_1, x_2) : x_1^2 + x_2^2 < 1 \}$; we consider the functions $L(x,z,p) = f(p) = |p|$ and $\ell(x) = \min\{\ell^1(x), \ell^2(x)\}$ where we set $\ell^1(x_1,x_2) = \frac{1}{4}x_1 + \frac{1}{2}$, $\ell^2(x_1,x_2) = 1 - x_1^2 - x_2^2$. Here (L,ℓ^1) and (L,ℓ^2) satisfy Assumption B. In fact ℓ^1 is affine and f is subdifferentiable at $(1/4, 0) = \nabla\ell^1$: the validity of Assumption B for (L,ℓ^1) follows from Example 3.11. Concerning the pair (L,ℓ^2) we consider the selection k of the subdifferential of f in $\nabla\ell^2$ given by

$$k(x) = \begin{cases} f_p(\nabla\ell^2(x)) = -\frac{x}{|x|} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Following Remark 3.4 the divergence of $k(x)$ (for $x \neq 0$) is the trace of the product of the Hessian of f at $\nabla\ell^2(x)$ with the Hessian of ℓ^2 at x given by

$$\text{Hess}\, f(\nabla\ell^1(x))\, \text{Hess}\,\ell^1(x) = \begin{pmatrix} \frac{x_2^2}{2|x|^3} & \frac{-x_1x_2}{2|x|^3} \\ \frac{-x_1x_2}{2|x|^3} & \frac{x_1^2}{2|x|^3} \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

so that div $k(x) = -\frac{1}{|x|} < 0$ $(x \neq 0)$ showing that (L,ℓ^2) fulfills Assumption B1.1. We are thus in the position to apply Theorem 3.18. We prove now that (L,ℓ) does not satisfy Assumption B. Let E, F be the sets defined by

$$E = \{x \in \Omega : \ell^2(x) < \ell^1(x)\}, \quad F = \{x \in \Omega : \ell^1(x) < \ell^2(x)\};$$

then $E \cup \overline{F} = \Omega$, E contains a circular annulus $E_r = \{(x_1,x_2) : r \le x_1^2 + x_2^2 < 1\}$ for some non negative r and F is non negligible. The subdifferential of L in $(x, \ell(x), \nabla\ell(x))$ is given by

$$k(x) = \partial_p L(x,\ell(x),\nabla\ell(x)) = \begin{cases} -\dfrac{x}{|x|} & \text{if } x \in E, \\ (1,0) & \text{if } x \in F \end{cases}$$

It follows that

$$\text{div}\ k(x) = \begin{cases} -\dfrac{1}{|x|} & \text{if } x \in E, \\ 0 & \text{if } x \in F. \end{cases}$$

It can be easily proved that div $k(x) \le 0$ in Ω and that the inequality div $k(x) \le 0$ is not *strict*, since it equals zero on a non negligible set. Therefore both Assumptions A1 and A2 hold, but not B1.1 nor B2.1. It is obvious that B1.2, B2.2 and B2.3 do not hold. Moreover, for almost every x in Ω, the face of the map $p \mapsto |p|$, containing the point $(\nabla\ell(x), |\nabla\ell(x)|)$ is the halfline through the origin and the point itself, showing that B3 is not fulfilled. For every x in E_r the gradient of ℓ is given by $\nabla\ell(x) = -2x$ so that the set $F^L_{\mathbf{R}^2}(x)$ is parallel to x and thus each non zero vector field $S(x)$ such that $S(x)\cdot x = 0$, for x varying in the annulus E_r, spans the whole space $\mathbf{R}^2$, thus violating B4.

Theorem 3.15 together with Theorem 3.8 yield Theorem 1 of [1]

Theorem 3.22. *Let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be convex, lower semicontinuous and assume that the epigraph of f does not contain n–dimensional faces. Let $\ell(x) = a \cdot x + b$ be an affine function and w be a minimizer of $I(u) = \int_\Omega f(\nabla u(x))\,dx$ in $W^{1,1}_{\bar{u}}(\Omega)$ satisfying $w \le \ell$ on $\partial\Omega$. Then $w \le \ell$ a.e. on Ω.*

Proof. If we assume that f is subdifferentiable in a, then Assumption B4 is fulfilled (see Example 3.11) and, thus, our Main Theorem yields the result. Assume that f is not subdifferentiable in a and that $w > l$ on a set of non negative measure. Then a does not belong to the interior of the domain of f and following [1] the point a can be separated by a hyperplane K from the closed convex set Dom(f), i.e. there exists a non zero vector h such that $0 \ge \sup\{h \cdot (d - a) : d \in \text{Dom}(f)\}$. Now since $I(w)$ is finite then the gradient ∇w of w belongs to Dom(f) a.e. so that the scalar product

$$\nabla(w - \ell)^+ \cdot h = \begin{cases} h \cdot (\nabla w - a) & \text{if } w > \ell, \\ 0 & \text{otherwise} \end{cases}$$

is non positive a.e. on Ω. Then Theorem 3.8 implies that $(w - \ell)^+ = 0$ on Ω, and thus $w \le \ell$ a.e. on Ω, a contradiction. ☐

4. Variational Equations

We consider here an application of our Main Theorem to the *variational equation*

$$\text{div}\ L_p(x,u,\nabla u) - L_z(x,u,\nabla u) = 0, \tag{9}$$

i.e.

$$\forall \varphi \in C_c^\infty(\Omega) \qquad \int_\Omega L_p(x,u,\nabla u)\cdot\nabla\varphi + L_z(x,u,\nabla u)\varphi\,dx = 0$$

which is the *Euler–Lagrange equation* associated to the functional

$$I(u) = \int_\Omega L(x,u,\nabla u)\,dx.$$

The classical *Comparison Principle* for divergence form operators [5, Thm. 10.7] in the case of *variational equations* can be rewritten as follows.

Theorem 4.1. *Let L be regular, elliptic and convex in (z,p); let ℓ in $C^1(\overline{\Omega})$ be a supersolution and w be a weak solution to the Euler–Lagrange equation (9). If $w \le \ell$ on $\partial\Omega$ then $w \le \ell$ on Ω.*

Theorem 3.15 provides some new conditions ensuring the validity of the *Comparison Principle* for *variational equations*.

Theorem 4.2. *Let $L(x,z,p)$ be convex in (z,p) for almost every x in Ω, ℓ be such that the pair (L,ℓ) satisfy* Assumption B *and let w in $W^{1,q}(\Omega)$ be a solution to the Euler–Lagrange equation* (9) *such that the map $x \longmapsto L(x,w(x),\nabla w(x))$ belongs to $L^1(\Omega)$ and $w \le \ell$ on $\partial\Omega$. Then $w \le \ell$ on Ω.*

Proof. The convexity of L in (z,p) and the integrability assumption ensure that w is a minimizer of I in $w + W_0^{1,q}(\Omega)$; Theorem 3.15 yields the result. □

Remark 4.3. We point out here what are the main differences between Theorem 4.2 and the classical result. First we allow L, ℓ and w to be nonsmooth; moreover we do not impose the ellipticity assumption, allowing thus the epigraph of the function $p \longmapsto L(x,z,p)$ to have non trivial faces. Instead we require that L is either monotonic in z (in Assumptions B1, B3, B4) or the sum of two functions (in Assumptions B2, B3, B4); in this situation Theorem 4.2 under Assumption B3 generalizes Theorem 10.7 of [5]. We do not cover however the case where L is elliptic but is neither monotonic in z nor the sum of two functions.

It is interesting to apply our result to the case of linear, possibly degenerate, elliptic equations of the type

$$\operatorname{div}\,(A(x)\nabla u) - c(x)u = 0. \tag{10}$$

where, for every x in Ω, $A(x)$ is a $n \times n$ symmetric matrix whose coefficients belong to $L^\infty(\Omega)$ and c is a function in $L^\infty(\Omega)$. We consider the following assumption.

Assumption C. We say that (A,c,ℓ) satisfy Assumption C if the matrix $A(x)$ is positive semidefinite and $c \ge 0$ a.e. in Ω, ℓ in $W^{1,2}(\Omega)$ is a *supersolution* of the equation $\operatorname{div}\,(A(x)\nabla u) - c(x)u = 0$, i.e.:

$$\forall \varphi \in C_c^\infty(\Omega),\ \varphi \ge 0 \text{ in } \Omega, \qquad \int_\Omega A(x)\nabla\ell(x)\cdot\nabla\varphi(x) + c(x)\ell(x)\varphi(x)\,dx \ge 0,$$

and (just) one of the following condition holds:

C1. The function ℓ is a *strict supersolution* of $\operatorname{div}\,(A(x)\nabla u) - c(x)u = 0$, i.e.:

$$\forall \varphi \in C_c^\infty(\Omega),\ \varphi \ge 0 \text{ in } \Omega,\ \varphi \ne 0, \quad \int_\Omega A(x)\nabla\ell(x)\cdot\nabla\varphi(x) + c(x)\ell(x)\varphi(x)\,dx > 0.$$

C2. The function c is positive a.e. in Ω.

C3. The matrix $A(x)$ is positive definite for almost every x in Ω.

C4. There exists a function $S : \Omega \to \mathbf{R}^n$ satisfying the *cone property* such that, for almost every x in Ω, $S(x)$ is orthogonal to the kernel $\ker A(x)$ of $A(x)$.

Example 4.4. As in Example 3.13 it is easy to prove that the Assumption C4 is fulfilled if, for instance, $n = 2$ and the kernels $\ker A(x)$ for x varying in Ω are contained in the cone $\{y \in \mathbf{R}^2 : |y \cdot u| \ge \beta|y|\}$ for some $\beta > 0$ and u in $\mathbf{R}^2 \setminus \{(0,0)\}$.

Remark 4.5. Assumption C4 is equivalent to the existence, for every x in Ω, of a $n \times n$ matrix $T(x)$ such that $S(x) = T(x)A(x)$.

The application of Theorem 4.2 yields the following *Comparison Principle* for elliptic equations in divergence form.

Theorem 4.6. *Let (A,c,ℓ) satisfy* Assumption C *and w in $W^{1,2}(\Omega)$ be a weak solution to $\operatorname{div}(A(x)\nabla u) - c(x)u = 0$ such that $w \le \ell$ in $\partial\Omega$. Then $w \le \ell$ a.e. in Ω.*

Proof. Let $L(x,z,p) = p \cdot A(x)p/2 + c(x)z^2/2$: the *Euler–Lagrange equation* associated to L is (10). It is obvious that the function $x \longmapsto L(x,w(x),\nabla w(x))$ belongs to $L^1(\Omega)$. In this setting the Assumption C1 (resp. C2, C3) yields the validity of Assumption B2.1 (resp. B2.2, B3). It is easy to check that the projection on $\mathbf{R}^n$ of the non trivial faces of the epigraph of the map $p \mapsto p \cdot A(x)p$ is a traslation of $\ker A(x)$; thus C4 implies B4. The conclusion follows from Theorem 4.2. □

For v in $W^{1,2}(\Omega)$ we set

$$\sup_{\partial\Omega} v = \inf\{k \in \mathbf{R} : v \le k \text{ in } \partial\Omega\}.$$

We deduce here a *weak Maximum Principle* for elliptic equations in divergence form.

Theorem 4.7. *Let A be positive definite, $c \ge 0$ a.e. in Ω and assume that (just) one of the* Assumptions C2, C3 *or* C4 *hold. Let w in $W^{1,2}(\Omega)$ be a weak solution to*

$$\operatorname{div}\,(A(x)\nabla u) - c(x)u = 0.$$

Then $w \le \sup_{\partial\Omega} w^+$ in Ω.

Proof. Let ℓ be the constant function equal to $\sup_{\partial\Omega} w^+$. Then for every non negative φ in $C_c^\infty(\Omega)$ we have

$$\int_\Omega A(x)\nabla\ell\cdot\nabla\varphi + c(x)\ell\varphi\,dx = \int_\Omega c(x)\ell\varphi\,dx \ge 0,$$

so that ℓ is a supersolution of $\operatorname{div}\,(A(x)\nabla u) - c(x)u = 0$. Theorem 4.6 yields the conclusion. □

Remark 4.8. We point out that the classical *weak Maximum Principle* [5, Thm. 8.1] requires that the equation is strictly elliptic, i.e. that $p \cdot A(x)p \ge \lambda|p|^2$ $(\lambda > 0)$ for all x, p. This condition is contained in our Assumption C3, where we just require ellipticity. Further, our conditions C2 and C4 allow the equation to be degenerate.

5. On the Lipschitz continuity of minimizers

In this section we fix a Lipschitz continuous function $\bar{u} : \Omega \to \mathbf{R}$ and we denote by K its Lipschitz constant. We are concerned here with the existence of Lipschitz continuous minimizers of the functional

$$I(u) = \int_\Omega L(x,u,\nabla u)\,dx \qquad \text{on } W_{\bar{u}}^{1,q}(\Omega).$$

A slight modification of the contents of [9] yields a result in this direction when the minimizer of I lies between two Lipschitz continuous functions.

Assumption D. We say that L satisfies Assumption D if

$$L(x,z,p) = f(p) + g(x,z)$$

is convex in (z,p), the function $g(x,z)$ is continuously differentiable in z and the derivative g_z is Lipschitz continuous (and thus $\nabla_x g_z$ and g_{zz} exist a.e.). Moreover either

$$\inf\{g_{zz}(x,z) : (x,z) \in \Omega \times \mathbf{R}\} > 0 \tag{11}$$

or

$$f \text{ is strictly convex}, \quad g(x,z) = h(z), \quad f(p) \geq \alpha|p|^q + \beta \quad (\alpha > 0, \beta \in \mathbf{R}). \tag{12}$$

We set $i = \inf\{g_{zz}(x,z) : (x,z) \in \Omega \times \mathbf{R}\}$, $s = \frac{1}{K}\sup\{|\nabla_x g_z(x,z)| : (x,z) \in \Omega \times \mathbf{R}\}$ and

$$M(g,K) = \begin{cases} \max\{s/i,\ K\} & \text{if (11) holds;} \\ K & \text{if (12) holds.} \end{cases}$$

Theorem 5.1. *Let L satisfy* Assumption D *and w be a minimizer of I on $W^{1,q}_{\bar{u}}(\Omega)$. Let u^1, u^2 be two Lipschitz continuous functions in $W^{1,q}_{\bar{u}}(\Omega)$ with Lipschitz constant K satisfying $u^1 \leq w \leq u^2$ a.e. in Ω. Then w is Lipschitz continuous and $M(g,K)$ is a Lipschitz constant of w.*

Proof. Assume that (11) holds. Let

$$L_\varepsilon = \max\{s/i,\ K\} + \varepsilon, \qquad C = B(0, L_\varepsilon].$$

The Minkowski function γ_C of C is given by

$$\forall \xi \in \mathbf{R}^n \qquad \gamma_C(\xi) = \inf\{\lambda : \xi \in \lambda B(0,L_\varepsilon]\} = \frac{|\xi|}{L_\varepsilon}$$

and thus $\sup\{\gamma_C(-\nabla_x g_z(x,z)) : (x,z) \in \Omega \times \mathbf{R}\} = s/L_\varepsilon < i$ showing that the hypothesis of Theorem 3.1 of [9] is fulfilled. It follows that $\nabla w \in -C$ a.e. or equivalently that $|\nabla w| \leq L_\varepsilon$ and therefore w is Lipschitz continuous with Lipschitz constant L_ε for every $\varepsilon > 0$. If (12) holds the only non trivial case arises when (11) does not hold, i.e. if $\inf\{h_{zz}(z) : z \in \mathbf{R}\} = 0$: since in this case g does not depend on x then the conditions of Theorem 3.3 in [9] are satisfied, proving the claim. □

We give some conditions ensuring that the minimizers of I are bounded by Lipschitz continuous functions in $W^{1,q}_{\bar{u}}(\Omega)$.

Definition 5.2 (The Generalized Bounded Slope Condition (G.B.S.C.)$_K$). We say that the pair $(L,\bar{u})$ satisfies the $(G.B.S.C.)_K$ if $\bar{u} : \Omega \to \mathbf{R}$ is Lipschitz continuous with Lipschitz constant K and for every $\bar{x}$ in $\partial\Omega$ there exist two Lipschitz continuous functions $\ell^1_{\bar{x}}, \ell^2_{\bar{x}} : \Omega \longrightarrow \mathbf{R}$ with Lipschitz constant K such that the pair $(L, \ell^2_{\bar{x}})$ (resp. $(L, \ell^1_{\bar{x}})$) satisfies Assumption A (resp. A′, see Remark 3.3) and moreover

$$\forall x \in \partial\Omega \qquad \ell^1_{\bar{x}}(x) \leq \bar{u}(x) \leq \ell^2_{\bar{x}}(x) \qquad \text{and} \qquad \ell^1_{\bar{x}}(\bar{x}) = \bar{u}(\bar{x}) = \ell^2_{\bar{x}}(\bar{x}). \tag{13}$$

Remark 5.3. The $(G.B.S.C.)_K$ on $(L,\bar{u})$ is a generalization of the *Bounded Slope Condition* $(B.S.C.)_K$ introduced in the variational context by Stampacchia in [8] for functions L of the form $L(x,z,p) = f(p)$ and where it is merely required that for every $\bar{x}$ in $\partial\Omega$ there exist vectors $k^+(\bar{x})$ and $k^-(\bar{x})$ bounded in norm by K such that

$$\forall x \in \partial\Omega \qquad \bar{u}(\bar{x}) + k^-(\bar{x}) \cdot (x - \bar{x}) \leq \bar{u}(x) \leq \bar{u}(\bar{x}) + k^+(\bar{x}) \cdot (x - \bar{x}). \tag{14}$$

In fact setting $\ell^1_{\bar{x}}(x) = \bar{u}(\bar{x}) + k^-(\bar{x}) \cdot (x - \bar{x})$ and $\ell^2_{\bar{x}}(x) = \bar{u}(\bar{x}) + k^+(\bar{x}) \cdot (x - \bar{x})$ the inequalities (14) give (13). Moreover $\nabla\ell^1_{\bar{x}}(x) = k^-(\bar{x})$ and $\nabla\ell^2_{\bar{x}}(x) = k^+(\bar{x})$ are constants; therefore the selections of the subdifferentials of f in $\nabla\ell^1_{\bar{x}}$ and in $\nabla\ell^2_{\bar{x}}$ are constants too so that their divergence is zero, showing that the Assumptions A and A′ are fulfilled.

Theorem 5.4. *Let L satisfy* Assumption D *and assume that the pair $(L,\bar{u})$ fulfills the $(G.B.S.C.)_K$. Let w be a solution to the problem*

$$\textit{minimize} \quad I(u) = \int_\Omega L(x,u,\nabla u)\,dx \qquad \textit{on } W^{1,q}_{\bar{u}}(\Omega). \tag{P}$$

Then w is Lipschitz continuous, a Lipschitz constant being $M(g,K)$.

Proof. Fix $\bar{x}$ in $\partial\Omega$. By definition the pair $(L, \ell^2_{\bar{x}})$ (resp. $(L, \ell^1_{\bar{x}})$) satisfies Assumption A (resp. A′). Moreover Assumption D on L ensures the validity of Assumption B (resp. B′) on $(L, \ell^2_{\bar{x}})$ (resp. $(L, \ell^1_{\bar{x}})$); in fact under (11) g is strictly convex in z (condition B2.2) whereas (12) implies B3 (resp. B′3). Theorem 3.18 together with Remark 3.19 yield

$$\forall x \in \Omega \quad \ell^1(x) = \sup_{\bar{x}\in\partial\Omega} \ell^1_{\bar{x}}(x) \leq w(x) \leq \inf_{\bar{x}\in\partial\Omega} \ell^2_{\bar{x}}(x) = \ell^2(x).$$

Now ℓ^1, ℓ^2 are Lipschitz continuous and belong to $W^{1,q}_{\bar{u}}(\Omega)$. Theorem 5.1 yields the conclusion. □

Remark 5.5. Theorem 5.4 is a generalization of Theorem 2 of [1] where the same conclusion is obtained for a more restrictive class of integrands of the form $L(x,z,p) = f(p)$ and for data satisfying the $(B.S.C.)_K$.

Corollary 5.6. *Under the assumptions of* Theorem 5.4 *the problem* (P) *and the problem*

$$\textit{minimize} \quad I(u) = \int_\Omega L(x,u,\nabla u)\,dx \quad \textit{on } W^{1,q}_{\bar{u}}(\Omega), \quad |\nabla u| \leq M(g,K) \ \textit{a.e.} \tag{P′}$$

have the same set of solutions.

Following the proof of Theorem 3 in [1] we obtain the following result ensuring the validity of the *Euler–Lagrange equation* for minimizers.

Theorem 5.7. *Let L satisfy* Assumption D, *f be continuously differentiable and assume that $(L,\bar{u})$ fulfills the $(G.B.S.C.)_K$. Let w be a solution to Problem* (P). *Then w is Lipschitz continuous and it satisfies the Euler–Lagrange equation i.e.*

$$\int_\Omega f_p(\nabla w(x)) \cdot \nabla\varphi(x) + g_z(x, w(x))\varphi(x)\,dx = 0$$

for every function φ in $C^\infty_c(\Omega)$.

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