Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 243 (2007) 388–413
Journal of Differential Equations
www.elsevier.com/locate/jde

Lipschitz regularity for minima without strict convexity of the Lagrangian

Carlo Mariconda *, Giulia Treu

Dipartimento di Matematica Pura e Applicata, Università di Padova, 63 via Trieste, I-35121 Padova, Italy

Received 22 May 2007; revised 22 May 2007

Available online 6 June 2007

Abstract

We give, in a non-smooth setting, some conditions under which (some of) the minimizers of $\int_\Omega f(\nabla u(x))\,dx + g(x,u(x))\,dx$ among the functions in $W^{1,1}(\Omega)$ that lie between two Lipschitz functions are Lipschitz. We weaken the usual strict convexity assumption in showing that, if just the faces of the epigraph of a convex function $f:\mathbb{R}^n \to \mathbb{R}$ are bounded and the boundary datum u_0 satisfies a generalization of the Bounded Slope Condition introduced by A. Cellina then the minima of $\int_\Omega f(\nabla u(x))\,dx$ on $u_0 + W_0^{1,1}(\Omega)$, whenever they exist, are Lipschitz. A relaxation result follows.

MSC: 49J30; 49J10

Keywords: Subdifferential; Convexity; Strictly convex; Face; Epigraph; Calculus of variations; Lipschitz; Regularity; Relaxation; Bounded Slope Condition; Non-smooth; Lavrentiev; Legendre; Domain; Polar; Demi-coercivity

1. Introduction

Few months ago we had the opportunity to read the paper [4] of Arrigo Cellina. One of the main novelties contained in that paper was a new kind of *Bounded Slope Condition*. The classical (BSC) of constant K, introduced by Hartman and Stampacchia prescribes, for a real valued function defined on an open bounded subset Ω of $\mathbb{R}^n$, at every point x^0 of the boundary $\partial\Omega$ of Ω, the existence of two affine functions of slope less than K that bound u_0 from above

* Corresponding author.
E-mail addresses: maricond@math.unipd.it (C. Mariconda), treu@math.unipd.it (G. Treu).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.05.028

and from below on $\partial\Omega$ and that coincide with u_0 at x^0. The (BSC) was used to study the problem of minimizing the functional

$$I(u)=\int_{\Omega} f\big(\nabla u(x)\big)\,dx, \qquad u=u_0 \quad \text{on } \partial\Omega$$

in various functional spaces. Under the assumption that f is strictly convex Stampacchia proved in [15] that if u_0 satisfies the (BSC) then I has a minimizer among the Lipschitz functions that coincide with u_0 in $\partial\Omega$. More recently (for instance in [5,11]), the (BSC) was used to prove that if I has a minimizer in $u_0+W_0^{1,1}(\Omega)$ then every such minimizer is Lipschitz; F.H. Clarke even proved in [7] the local Lipschitz regularity under a Bounded Slope Condition just from below or from above.

All of the proofs of these results use a sort of *Comparison Principle* which allows, under the assumption that f is strictly convex, to pass from an inequality like $u_0\geqslant \ell$ on $\partial\Omega$ to $\bar u\geqslant\ell$ on Ω, $\bar u$ being the minimizer of I in the class of interest and ℓ an affine function. The Comparison Principle between a minimizer and an affine function does not hold in general if the Lagrangian f is not strictly convex. For instance, as it is pointed out in [4], in the case where $f(\xi)=0$ on $[-1,1]$ and $f(\xi)=(|\xi|^2-1)^2$ on $|\xi|>1$, the functions $v(x)=0$ and $w(x)=-|x|+1$ both minimize $\int_{-1}^{1} f(x'(t))\,dt$ among the absolutely continuous functions that vanish on $\{-1,1\}$; however $w\leqslant v$ on $\{-1,1\}$ but $w>v$ on $]-1,1[$.

In trying to get rid of the strict convexity assumption on f, Cellina defines in [4] a new class of functions, depending on the *polar* or *Legendre transform* f^* of the Lagrangian f and establishes a new Comparison Principle between these new functions and the minimizers of I, even in the case where f is not strictly convex; the analogous of the (BSC), using these new functions instead of affine functions is then introduced.

This condition, that we call here *Cellina Bounded Slope Condition* (CBSC) of constant K requires that, for every x^0 in $\partial\Omega$, u_0 is bounded in $\partial\Omega$ from above and from below by a new class of functions, depending on f (not affine in general, but which turn out to be affine if f is strictly convex), both Lipschitz of constant less than K and coinciding with u_0 at x^0; both the (BSC) and the (CBSC) are recalled in Section 5 of the paper. The paper [4] ends with the following regularity result.

Theorem *(Cellina). Let Ω be an open, bounded, and convex set; let $f:\mathbb{R}^n\to\mathbb{R}$ be convex. Assume that* (u_0,f) *satisfies the* (*CBSC*) *of constant K, that the* domain of f^* is open*, and f is* strictly convex *at every $|\xi|>K$. Let $\bar u$ be a* continuous *minimizer of*

$$I(u)=\int_{\Omega} f\big(\nabla u(x)\big)\,dx$$

in $u_0+W_0^{1,1}(\Omega)$. Then $\bar u$ is Lipschitz and its Lipschitz constant is bounded by K.

We just mention that the assumption on the polar of f in the theorem simply means that the set of vectors p of $\mathbb{R}^n$ such that an affine function $p\cdot x+b$ bounds f from below is open.

We prove here that the conclusion of this regularity result does still hold if the *epigraph* of f has no unbounded faces. Moreover there's no need to assume neither the continuity of the minimizer nor that the domain of f^* is open; actually our condition on the faces of the epigraph of f implies that f is non-constant on straight lines so that the domain of f^* has a non-empty interior, and this is enough to prove the claim, which applies for instance to functions like $f(\xi) = \sqrt{1+|\xi|^2}$. We just mention that the condition that the domain of f^* has a non-empty interior has already been used in some papers concerning BV functions and arises for instance, in the form of the equivalent growth condition *demi-coercivity* defined in [1], in studying the equilibrium of elastic structures with unilateral constraints on the stress. The result seems to be optimal since, for instance, the problem of minimizing $\int_0^1 |x'(t)|\,dt$ among the functions that assume a prescribed value at 0 and 1 has a non-Lipschitz minimizer.[1]

Quite surprisingly for us, the assumption that the faces of the epigraph of f are bounded is actually a reformulation of the growth condition (CGA)—*Conical Growth Assumption* that we introduced in [13] in order to weaken the classical superlinearity condition in a Lipschitz regularity result; this is somewhat analogous to the demi-coercivity growth condition which is also equivalent to the fact that the function involved is not constant on straight lines.

The condition that the faces of the epigraph of a function are bounded appeared in [6] under the name of *Bounded Intersection Property* as a condition that is satisfied by the convex functions that verify some growth conditions (even weaker than superlinearity, like the *Growth Assumption* (GA) introduced in the same paper); there it was just used to establish a representation formula for the points of the epigraph of the convexified function in terms of the points of the epigraph of the original function.

The proof of our new regularity result is based on the Comparison Principle of [4] (that holds if f is just convex): if we assume the (CBSC), the minimizer $\bar{u}$ of I lies between two Lipschitz functions u_1 and u_2 of Lipschitz constant K that coinciding with u_0 on the boundary of Ω and such that $u_1 \leqslant \bar{u} \leqslant u_2$ a.e. on Ω. We show, in a more general setting, that in this situation there is a further minimizer $\bar{v}$ of I that is Lipschitz, with a Lipschitz constant bounded by K. Now, the points $(\nabla\bar{u}(x), f(\nabla\bar{u}(x)))$ and $(\nabla\bar{v}(x), f(\nabla\bar{v}(x)))$ belong a.e. to the same face of the epigraph of f. Hence, if the faces of the epigraph are bounded, the gradient of $\bar{u}$ turns out to be bounded as well by a quantity depending on K and f; thus $\bar{u}$ is Lipschitz too. Therefore, under the (CGA), the (BSC) is a particular case of (CBSC); we point out the two conditions are not equivalent.

Under the assumption that the faces of the epigraph of f are bounded, we give a condition on f under which the classical (BSC) of constant K for a boundary datum u_0 implies the (CBSC) for (u_0, f) of an explicit constant depending on f and K.

In our proof, the existence of a Lipschitz minimizer in the class $W^{1,p}_{u_1,u_2}(\Omega)$ of functions of $u_0 + W^{1,p}_0(\Omega)$ $(p \geqslant 1)$ satisfying $u_1 \leqslant u \leqslant u_2$ a.e. on Ω is essentially obtained by means of a result of [16], stating that if f is convex and $g(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is convex in u, twice differentiable in both variables and the derivatives satisfy a suitable inequality on $\Omega \times \mathbb{R}$ then every minimizer of $I(u) = \int_\Omega f(\nabla u(x)) + g(x, u(x))\,dx$ in $W^{1,p}_{u_1,u_2}(\Omega)$ is Lipschitz. The first part of this paper deals with a generalization of the latter result, allowing $g(x,u)$ not to be differentiable but, instead, just convex in u and with a partial subdifferential that is monotonic with respect to the order induced by an appropriate cone of $\mathbb{R}^n \times \mathbb{R}$.

[1] In presenting the paper to us, Arrigo Cellina seemed not to be fully satisfied of his regularity result, due to the presence of the strict convexity assumption on f. What a best challenge for us to try to drop out this assumption in the occasion of the meeting for Cellina's 65th birthday. The book *Differential Inclusions* by J.P. Aubin and A. Cellina begins with an introduction called "Epigraph"; the letter "E" among two "putti" writing a derivative; the draw was also used for the poster of the meeting. It is a nice coincidence that the conclusion of the regularity result of Cellina does still hold if simply the *Epigraph* of f has no unbounded faces.

Finally, under the assumption that the *bipolar* of a function fulfills the (CGA) (i.e. the bipolar's faces of the epigraph are bounded), we apply the representation result of [14] to obtain some relationships between the problems of minimizing the functionals

$$I(u)=\int\limits_{\Omega} f\big(\nabla u(x)\big)\,dx, \qquad I^{**}(u)=\int\limits_{\Omega} f^{**}\big(\nabla u(x)\big)\,dx$$

in various functional spaces. In particular we deduce that if I^{**} has a minimizer in $u_0+W_0^{1,1}(\Omega)$ that is Lipschitz then the value of the minimum of I^{**} equals the infimum of I in the same class of functions; more precisely the minimum of I^{**} coincides with the infimum of I in the smaller class of Lipschitz functions with a suitable Lipschitz constant, so no *Lavrentiev phenomenon* occurs. These conclusions are usually obtained under some suitable conditions that bound the Lagrangian both from below and from above. The first result in this direction was obtained by Cellina in [6] for functionals defined on $W^{1,1}([a,b],\mathbb{R}^n)$ with Lagrangian $f(x,x')$ depending on the state and the velocity, under the growth assumption (GA) mentioned above.

2. Notation, basic assumptions and preliminary results

In this paper Ω is an open, bounded subset of $\mathbb{R}^n$, endowed with the usual scalar product "$\cdot$" and the euclidian norm $|\cdot|$.

In Sections 3 and 4 of the paper we fix three Lipschitz functions u_0, u_1, u_2 defined on Ω that coincide on the boundary $\partial\Omega$ of Ω and we set

$$K=\max\big\{\mathrm{Lip}(u_i)\colon\ i=1,2\big\}$$

where, by $\mathrm{Lip}(u)$, we denote the Lipschitz constant of u.

Let $p\geqslant 1$; we consider the closed subset $W^{1,p}_{u_1,u_2}(\Omega)$ of $u_0+W_0^{1,p}(\Omega)$ defined by

$$W^{1,p}_{u_1,u_2}(\Omega)=\big\{u\in W^{1,p}(\Omega)\colon\ u_1\leqslant u\leqslant u_2 \text{ a.e.}\big\}.$$

We will often refer to the following *Basic Assumption*.

Basic Assumption. The function $f:\mathbb{R}^n\to\mathbb{R}$ is convex, the function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $u\mapsto g(x,u)$ is convex for a.e. x in Ω. There exist $a\in L^{p'}(\Omega)^n$ $(1/p+1/p'=1)$ and $b\in L^1(\Omega)$ such that $f(\xi)+g(x,u)\geqslant a(x)\cdot\xi+b(x)$ for all $(x,u,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$.

The functional I is defined on $W^{1,p}(\Omega)$ by

$$I(u)=\int\limits_{\Omega} f\big(\nabla u(x)\big)+g\big(x,u(x)\big)\,dx.$$

For every x such that $u\mapsto g(x,u)$ is convex, we will denote by $\partial_u g(x,u)$ the (partial) subdifferential of $g(x,\cdot)$; in this context there exist a measurable $p(x,u)$ such that $p(x,\cdot)\in\partial_u g(x,\cdot)$ for a.e. x.

The cone C_K and the order relation "$\prec_K$". For every $K \geqslant 0$ we consider the cone of $\mathbb{R}^n \times \mathbb{R}$ defined by

$$C_K = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}\colon u > K|x|\}$$

and the order relation $\prec_K$ induced by C_K defined on $\mathbb{R}^n \times \mathbb{R}$ by

$$(x, u) \prec_K (y, v) \quad \Leftrightarrow \quad (y - x, v - u) \in C_K \quad \Leftrightarrow \quad v > u + K|y - x|.$$

In what follows we will be concerned with the following monotonicity conditions on the partial subdifferential $\partial_u g(x, u)$ of g.

Monotonicity Condition.

(Strict Monotonicity Condition) There exists $p(x, \cdot) \in \partial_u g(x, \cdot)$ for a.e. x such that

$$(x, u) \prec_K (y, v) \quad \Rightarrow \quad p(x, u) < p(y, v)$$

or

(Large Monotonicity Condition) There exists $p(x, \cdot) \in \partial_u g(x, \cdot)$ for a.e. x such that

$$(x, u) \prec_K (y, v) \quad \Rightarrow \quad p(x, u) \leqslant p(y, v).$$

We state here some facts that clarify the Monotonicity Condition.

Proposition 2.1.

(i) *Assume that g satisfies the Strict* (*respectively Large*) *Monotonicity Condition. Then the function $u \mapsto g(x, u)$ is strictly convex* (*respectively convex*) *for a.e. x*; *the converse holds true if $g(x, u)$ does not depend on the first variable x.*
(ii) *Assume that $g(x, u)$ admits a locally Lipschitz partial derivative $g_u(x, u)$ with respect to the second variable*: *by Rademacher's theorem $g_u(x, u)$ admits partial derivatives g_{ux} and g_{uu} on $\Omega \times \mathbb{R} \setminus N$, where N is negligible. Assume moreover that for every (x, u) in $\Omega \times \mathbb{R}$ there exists a neighborhood I of (x, u) such that*

$$\sup\{|g_{ux}(y, v)|\colon (y, v) \in I \setminus N\} < (\textit{respectively} \leqslant)\, K \inf\{g_{uu}(y, v)\colon (y, v) \in I \setminus N\}.$$

Then g fulfills the Strict (*respectively Large*) *Monotonicity Condition.*
(iii) *In particular the Strict* (*respectively Large*) *Monotonicity Condition is satisfied if $g(x, u)$ admits a partial derivative $g_u(x, u)$ of class $\mathcal{C}^1$ and moreover*

$$\forall (x, u) \in \Omega \times \mathbb{R} \quad |g_{ux}(x, u)| < (\textit{respectively} \leqslant)\, K g_{uu}(x, u).$$

Proof. (i) follows from the fact that a convex function of a real variable is strictly convex if and only if it admits a strictly monotonic subdifferential. Assume now that g fulfills the assumption stated in (ii) (strict version) and let $(x, u) \prec_K (y, v)$. *Lebourg's mean value theorem* for Lipschitz functions [8, Theorem 2.4] yields the existence of (ξ, η) in the *Clarke generalized gradient* of g_u at a point (z, w) of the segment joining (x, u) to (y, v) such that

$$g_u(y, v) - g_u(x, u) = \xi \cdot (y - x) + \eta(v - u).$$

The generalized gradient formula [8, Theorem 8.1] shows that (ξ, η) belongs to the convex hull $\mathrm{co}(S)$ of the set

$$S = \Big\{ \lim_{i\to+\infty} \big(g_{ux}(z_i, w_i), g_{uu}(z_i, w_i)\big)\colon\ (z_i, w_i) \in \Omega \times \mathbb{R} \setminus N,\ \lim_{i\to+\infty} (z_i, w_i) = (z, w) \Big\}.$$

Let I be a neighborhood of (z, w), $\rho \in \mathbb{R}$, $\varepsilon > 0$ be such that

$$\sup\big\{\big|g_{ux}(z', w')\big|\colon\ (z', w') \in I \setminus N\big\} \leqslant \rho - \varepsilon \leqslant \rho + \varepsilon \leqslant K \inf\big\{g_{uu}(z', w')\colon\ (z', w') \in I \setminus N\big\}.$$

Thus S is contained in the convex set

$$\big\{(\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}\colon\ |\alpha| \leqslant \rho - \varepsilon \leqslant \rho + \varepsilon \leqslant K\beta\big\} \subset C_K$$

so that in particular $|\xi| < K\eta$. Therefore $\eta > 0$ and, since $v - u > K|y - x|$,

$$\begin{aligned} g_u(y, v) - g_u(x, u) &\geqslant -|\xi||y - x| + \eta(v - u) \\ &> -|\xi||y - x| + \eta K|y - x| = \big(K\eta - |\xi|\big)|y - x| \geqslant 0 \end{aligned}$$

proving that $g_u(y, v) > g_u(x, u)$. The case where g satisfies the assumption (ii) in the large sense can be treated more easily and in a similar way; (iii) follows immediately. □

3. Lipschitzianity of the minima that satisfy an a priori bound: The strictly convex case

Theorem 3.1 *(Lipschitz continuity of the minimizers in $W^{1,p}_{u_1,u_2}(\Omega)$). Assume that f, g satisfy the Basic Assumption and that one of the following conditions holds*:

(a) *g satisfies the Strict Monotonicity Condition i.e. there exists $p(x, \cdot) \in \partial_u g(x, \cdot)$ for a.e. x such that*

$$v > u + K|y - x| \quad \Rightarrow \quad p(x, u) < p(y, v)$$

or

(b) *f is strictly convex and g satisfies the Large Monotonicity Condition i.e. there exists $p(x, \cdot) \in \partial_u g(x, \cdot)$ for a.e. x such that*

$$v > u + K|y - x| \quad \Rightarrow \quad p(x, u) \leqslant p(y, v).$$

Assume that the functional

$$I(u) = \int_\Omega f\big(\nabla u(x)\big) + g\big(x, u(x)\big)\, dx$$

admits a minimizer $\bar u$ in $W^{1,p}_{u_1,u_2}(\Omega)$. Then $\bar u$ is Lipschitz and its Lipschitz constant is bounded by $K=\max\{\mathrm{Lip}(u_i)\colon i=1,2\}$.

This first part of the proof of Theorem 3.1 follows the lines of that of Theorem 3.1 in [16], which is itself inspired by [2]. We write it here for the convenience of the reader since, apart some minor changes, Treu and Vornicescu give some conditions that ensure that $\nabla\bar u$ belongs to a prescribed closed convex set whereas we just look for $\bar u$ to be Lipschitz, making our first part of the proof less technical than the one presented in [16].

Proof. Let u_0^* be a Lipschitz continuous extension of u_0 to $\mathbb{R}^n$, with Lipschitz constant bounded by K. We then consider the following Lipschitz continuous extensions of u_1, u_2 to $\mathbb{R}^n$ defined by

$$u_i^*(x)=\begin{cases} u_i(x) & \text{if } x\in\Omega,\\ u_0^*(x) & \text{if } x\notin\Omega,\end{cases}\qquad i=1,2.$$

Let also

$$\bar u^*(x)=\begin{cases} \bar u(x) & \text{if } x\in\Omega,\\ u_0^*(x) & \text{if } x\notin\Omega.\end{cases}$$

Clearly $\bar u^*\in W^{1,p}(\Omega')$ for every open, bounded subset Ω' containing Ω.

Fix h in $\mathbb{R}^n$. We first show that

$$\bigl|\bar u^*(x+h)-\bar u^*(x)\bigr|\leqslant K|h|\quad\text{a.e.}$$

or, equivalently, that

$$\bar u^*(x+h)\leqslant \bar u^*(x)+K|h|,\qquad \bar u^*(x)\leqslant \bar u^*(x-h)+K|h|\quad\text{a.e.}$$

Assume, by contradiction, that one of the above inequalities does not hold on a set of strictly positive measure. Then at least one of the sets

$$E_h^+=\bigl\{x\in\mathbb{R}^n\colon \bar u^*(x+h)-K|h|>\bar u^*(x)\bigr\},\qquad E_h^-=\bigl\{x\in\mathbb{R}^n\colon \bar u^*(x-h)+K|h|<\bar u^*(x)\bigr\}$$

is non-negligible; actually since

$$E_h^+=E_h^--h$$

the sets defined above are both non-negligible. Let u_h^+ and u_h^- be the functions defined by

$$u_h^+(x)=\max\bigl\{\bar u^*(x+h)-K|h|,\bar u^*(x)\bigr\},\qquad u_h^-(x)=\min\bigl\{\bar u^*(x-h)+K|h|,\bar u^*(x)\bigr\}.$$

Notice that $u_h^+(x)=\bar u^*(x+h)-K|h|$ on E_h^+, $u_h^-(x)=\bar u^*(x-h)+K|h|$ on E_h^- and that $u_h^-\leqslant u_h^+$ a.e. in $\mathbb{R}^n$.

As in the proof of [16, Theorem 3.1] we remark that

$$u_1^*\leqslant u_h^-\leqslant u_h^+\leqslant u_2^*,\qquad E_h^-\cup E_h^+\subset\Omega\quad\text{a.e.}$$

In fact if $x \notin E_h^+$ then $u_h^+(x) = \bar{u}^*(x) \leqslant \bar{u}_2(x)$ so that $u_h^+ \leqslant u_2^*$ and, analogously, $u_1^* \leqslant u_h^-$. Moreover, by the Lipschitz continuity of u_2^*, $\bar{u}^*(x+h) - K|h| \leqslant u_2^*(x+h) - K|h| \leqslant u_2^*(x)$ so that, if $x \notin \Omega$, $\bar{u}^*(x+h) - K|h| \leqslant u_2^*(x) = \bar{u}^*(x)$ and $x \notin E_h^+$, proving that $E_h^+ \subset \Omega$; the inclusion $E_h^- \subset \Omega$ follows similarly. It follows that if $x \in E_h^+$ then $x + h \in E_h^- \subset \Omega$ so that $\bar{u}^*(x+h) = \bar{u}(x+h)$; analogously when $x \in E_h^-$ then $x - h \in \Omega$ and thus $\bar{u}^*(x-h) = \bar{u}(x-h)$.

Therefore, for every λ in [0, 1], the functions

$$u_h^+ + \lambda\big(u_h^+ - \bar{u}\big), \qquad u_h^- + \lambda\big(u_h^- - \bar{u}\big)$$

belong to $W^{1,p}_{u_1,u_2}(\Omega)$ and thus

$$I\big(u_h^+ + \lambda\big(u_h^+ - \bar{u}\big)\big) \geqslant I(\bar{u}), \qquad I\big(u_h^- + \lambda\big(u_h^- - \bar{u}\big)\big) \geqslant I(\bar{u})$$

or, equivalently,

$$\int_{E_h^+} f\big(\nabla\bar{u}(x) + \lambda\big(\nabla\bar{u}(x+h) - \nabla\bar{u}(x)\big)\big) - f\big(\nabla\bar{u}(x)\big)\,dx \\ + \int_{E_h^+} g\big(x, \bar{u}(x) + \lambda\big(\bar{u}(x+h) - K|h| - \bar{u}(x)\big)\big) - g\big(x, \bar{u}(x)\big)\,dx \geqslant 0 \tag{1}$$

and

$$\int_{E_h^-} f\big(\nabla\bar{u}(x) + \lambda\big(\nabla\bar{u}(x-h) - \nabla\bar{u}(x)\big)\big) - f\big(\nabla\bar{u}(x)\big)\,dx \\ + \int_{E_h^-} g\big(x, \bar{u}(x) + \lambda\big(\bar{u}(x-h) + K|h| - \bar{u}(x)\big)\big) - g\big(x, \bar{u}(x)\big)\,dx \geqslant 0.$$

The change of variables that maps x in $x+h$ in the left-hand member of the last inequality yields

$$\int_{E_h^+} f\big(\nabla\bar{u}(x+h) + \lambda\big(\nabla\bar{u}(x) - \nabla\bar{u}(x+h)\big)\big) - f\big(\nabla\bar{u}(x+h)\big)\,dx \\ + \int_{E_h^+} g\big(x+h, \bar{u}(x+h) + \lambda\big(\bar{u}(x) + K|h| - \bar{u}(x+h)\big)\big) - g\big(x+h, \bar{u}(x+h)\big)\,dx \geqslant 0. \tag{2}$$

Here begins the original part of the proof for a non-smooth function g. Adding term by term the inequalities (1) and (2) we obtain the inequality

$$B \geqslant A$$

where

$$A = \int_{E_h^+} f\big(\nabla\bar{u}(x)\big) + f\big(\nabla\bar{u}(x+h)\big)$$
$$- f\big(\lambda\nabla\bar{u}(x) + (1-\lambda)\nabla\bar{u}(x+h)\big) - f\big(\lambda\nabla\bar{u}(x+h) + (1-\lambda)\nabla\bar{u}(x)\big)\,dx$$

and

$$B = \int_{E_h^+} b(x)\,dx$$

where

$$b(x) = g\big(x, \bar{u}(x) + \lambda\psi(x)\big) + g\big(x+h, \bar{u}(x+h) - \lambda\psi(x)\big) - g\big(x, \bar{u}(x)\big) - g\big(x+h, \bar{u}(x+h)\big)$$

with

$$\psi(x) = \bar{u}(x+h) - K|h| - \bar{u}(x) > 0 \quad \text{on } E_h^+.$$

The function f is convex and thus

$$f\big(\lambda\nabla\bar{u}(x) + (1-\lambda)\nabla\bar{u}(x+h)\big) \leqslant \lambda f\big(\nabla\bar{u}(x)\big) + (1-\lambda) f\big(\nabla\bar{u}(x+h)\big)$$

and

$$f\big(\lambda\nabla\bar{u}(x+h) + (1-\lambda)\nabla\bar{u}(x)\big) \leqslant \lambda f\big(\nabla\bar{u}(x+h)\big) + (1-\lambda) f\big(\nabla\bar{u}(x)\big).$$

Notice that $\nabla\bar{u}(x) \neq \nabla\bar{u}(x+h)$ on a non-negligible subset of E_h^+, otherwise $\nabla u_h^+ = \nabla\bar{u}$ on Ω; moreover since $u_h^+ - \bar{u} \in W_0^{1,p}(\Omega)$ then $u_h^+ = \bar{u}$ on Ω, contradicting the assumption that E_h^+ is non-negligible. Therefore, if f is strictly convex, the latter inequalities are strict on a non-negligible subset of E_h^+. It follows that

$$A \geqslant 0; \qquad A > 0 \quad \text{if } f \text{ is strictly convex.}$$

Let $p(x,\cdot) \in \partial_u g(x,\cdot)$ be measurable. Then, for a.e. x,

$$g\big(x, \bar{u}(x) + \lambda\psi(x)\big) - g\big(x, \bar{u}(x)\big) \leqslant p\big(x, \bar{u}(x) + \lambda\psi(x)\big)\lambda\psi(x)$$

and

$$g\big(x+h, \bar{u}(x+h) - \lambda\psi(x)\big) - g\big(x+h, \bar{u}(x+h)\big) \leqslant -p\big(x+h, \bar{u}(x+h) - \lambda\psi(x)\big)\lambda\psi(x)$$

and thus

$$b(x) \leqslant \lambda\psi(x)\big[p\big(x, \bar{u}(x) + \lambda\psi(x)\big) - p\big(x+h, \bar{u}(x+h) - \lambda\psi(x)\big)\big].$$

Set

$$y = x+h, \qquad u = \bar{u}(x) + \lambda\psi(x), \qquad v = \bar{u}(x+h) - \lambda\psi(x).$$

Then

$$\begin{aligned} v-u &= \bar{u}(x+h)-\lambda\psi(x)-\bar{u}(x)-\lambda\psi(x) \\ &= \bar{u}(x+h)-2\lambda\bigl(\bar{u}(x+h)-K|h|-\bar{u}(x)\bigr)-\bar{u}(x) \\ &= (1-2\lambda)\bigl(\bar{u}(x+h)-\bar{u}(x)\bigr)+2\lambda K|h| \end{aligned}$$

so that

$$v-u-K|h|=(1-2\lambda)\bigl(\bar{u}(x+h)-K|h|-\bar{u}(x)\bigr).$$

Fix $\lambda<1/2$. Then, since $\bar{u}(x+h)-K|h|-\bar{u}(x)=\psi(x)>0$ a.e. on E_h^+, we obtain $v-u-K|h|>0$, i.e. $(x,u)\prec_K(y,v)$ so that

$$\bigl(x,\bar{u}(x)+\lambda\psi(x)\bigr)\prec_K\bigl(y,\bar{u}(x+h)-\lambda\psi(x)\bigr)\quad\text{a.e. on } E_h^+.$$

Assume that (a) holds. Then

$$p\bigl(x,\bar{u}(x)+\lambda\psi(x)\bigr)<p\bigl(x+h,\bar{u}(x+h)-\lambda\psi(x)\bigr)\quad\text{a.e. on } E_h^+$$

and thus $0>b(x)$ a.e. on E_h^+ whence

$$0>B=\int\limits_{E_h^+} b(x)\,dx\geqslant A\geqslant 0,$$

a contradiction. If, instead, (b) holds then

$$p\bigl(x,\bar{u}(x)+\lambda\psi(x)\bigr)\leqslant p\bigl(x+h,\bar{u}(x+h)-\lambda\psi(x)\bigr)\quad\text{a.e. on } E_h^+.$$

Moreover $A>0$ and thus

$$0\geqslant B\geqslant A>0$$

yielding again a contradiction.

Therefore, in both cases, it follows that E_h^+ is negligible and thus, for all h in $\mathbb{R}^n$, the inequality $|\bar{u}^*(x+h)-\bar{u}^*(x)|\leqslant K|h|$ holds a.e. in Ω.

Let e_i be the ith vector of the canonical basis; it follows from the claim we have just proved that there exists a negligible set Z such that, for every x in $\Omega\setminus Z$, the gradient $\nabla\bar{u}(x)$ exists and, moreover, $|\bar{u}^*(x+\frac{1}{m}e_i)-\bar{u}^*(x)|\leqslant\frac{K}{m}$ for all $m=1,2,\dots$ and $i=1,\dots,n$. Therefore $|\bar{u}^*(x+\frac{1}{m}e_i)-\bar{u}^*(x)|/\frac{1}{m}\leqslant K$ for every m and i so that, passing to the limit, we obtain that $|\nabla\bar{u}(x)|\leqslant K$ on $\Omega\setminus Z$. Thus $\bar{u}-u_0\in W_0^{1,\infty}(\Omega)$ and u_0 is Lipschitz: it follows that $\bar{u}$ is Lipschitz, and its Lipschitz constant is bounded by K. □

Remark 3.2. It follows from Proposition 2.1 that the assumptions of Theorem 3.1 imply that either f or g is strictly convex; therefore there exists at most one minimizer of I in $W_{u_1,u_2}^{1,p}(\Omega)$. We will prove in the sequel that the problem of minimizing I in $W_{u_1,u_2}^{1,p}(\Omega)$ admits existence, even in a non-strictly convex setting.

Remark 3.3. Our Theorem 3.1 generalizes, in a non-smooth setting, Theorem 3.1 of [16] where the authors assume a more restrictive version of our new condition (ii) of Proposition 2.1, namely that $g(x,u)$ admits a Lipschitz partial derivative $g_u(x,u)$ and that

$$\sup\{|g_{ux}(x,u)|:\ (x,u)\in\Omega\times\mathbb{R}\} < K\inf\{g_{uu}(x,u):\ (x,u)\in\Omega\times\mathbb{R}\}.$$

Our new Monotonicity Condition drops out, in Theorem 3.1, two orders of derivatives.

As it is shown in the smooth setting of [16], Theorem 3.1 can be generalized, with some slight technicality, to a result that gives sufficient conditions under which the gradient of the minimum $\bar{u}$ belongs to a prescribed compact convex subset $\mathcal{K}$ of $\mathbb{R}^n$. It is enough to replace, in the Monotonicity Condition, the cone C_K with

$$C_{\mathcal{K}} = \{(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}:\ \gamma_{\mathcal{K}^o}(\xi)<\eta\}$$

where $\gamma_{\mathcal{K}^o}$ is the Minkowski functional of the polar $\mathcal{K}^o$ of $\mathcal{K}$. More precisely the following result holds.

Theorem 3.4. *Let $\mathcal{K}$ be a compact convex subset of $\mathbb{R}^n$; $u_0, u_1, u_2 : \Omega\to\mathbb{R}$ be Lipschitz functions such that $u_1 = u_2 = u_0$ on $\partial\Omega$ and, moreover, $\nabla u_i\in\mathcal{K}$, $i = 0,1,2$. Assume that f, g satisfy the Basic Assumption and that one of the following conditions holds*:

(a) *there exists $p(x,\cdot)\in\partial_u g(x,\cdot)$ for a.e. x such that*

$$v > u + \gamma_{\mathcal{K}^o}(y-x) \quad\Rightarrow\quad p(x,u) < p(y,v)$$

or

(b) *f is strictly convex and there exists $p(x,\cdot)\in\partial_u g(x,\cdot)$ for a.e. x such that*

$$v > u + \gamma_{\mathcal{K}^o}(y-x) \quad\Rightarrow\quad p(x,u) \leqslant p(y,v).$$

Assume that the functional

$$I(u) = \int_\Omega f(\nabla u(x)) + g(x,u(x))\,dx$$

admits a minimizer $\bar{u}$ in $W^{1,p}_{u_1,u_2}(\Omega)$. Then $\nabla\bar{u}\in\mathcal{K}$ a.e.

We just mention that, in this more general situation, an analogue of Proposition 2.1 does hold; for instance (a) is satisfied (with $C_{\mathcal{K}}$ instead of C_K) if $g(x,u)$ admits a locally Lipschitz partial derivative $g_u(x,u)$ with respect to the second variable and, for every (x,u) in $\Omega\times\mathbb{R}$, there exists a neighborhood I of (x,u) such that

$$\sup\{\gamma_K(-g_{ux}(y,v)):\ (y,v)\in I\setminus N\} < \inf\{g_{uu}(y,v):\ (y,v)\in I\setminus N\}$$

where N is the set in which g_u is not differentiable.

4. Lipschitzianity of the minima that satisfy an a priori bound—The non-strictly convex case

The next result follows directly from Theorem 3.1 and is useful in a non-strictly convex setting. It also gives existence of a minimizer in $W^{1,p}_{u_1,u_2}(\Omega)$.

Theorem 4.1 *(Existence of a Lipschitz minimizer in $W^{1,p}_{u_1,u_2}(\Omega)$). Assume that f, g satisfy the Basic Assumption and the Large Monotonicity Condition, i.e. there exists $p(x,\cdot) \in \partial_u g(x,\cdot)$ for a.e. x such that*

$$v > u + K|y-x| \quad \Rightarrow \quad p(x,u) \leqslant p(y,v).$$

Then the problem of minimizing

$$I(u) = \int_\Omega f\big(\nabla u(x)\big) + g\big(x, u(x)\big)\,dx$$

in $W^{1,p}_{u_1,u_2}(\Omega)$ admits at least one solution; *moreover at least one of the minimizers of I is Lipschitz with a Lipschitz constant bounded by $K = \max\{\mathrm{Lip}(u_i)\colon i = 1,2\}$. In particular, if I is strictly convex, the unique minimizer of I is Lipschitz.*

Proof. For every $k = 1, 2, \ldots$ let I_k be the functional defined on $W^{1,p}(\Omega)$ by

$$I_k(u) = \int_\Omega f_k\big(\nabla u(x)\big) + g\big(x, u(x)\big)\,dx$$

where we set $f_k(\xi) = f(\xi) + \frac{1}{k}|\xi|^p$ if $p > 1$, $f_k(\xi) = f(\xi) + \frac{1}{k}|\xi|^2$ if $p = 1$. Then I_k is coercive and has therefore a minimizer $\bar u_k$ in the closed subset $W^{1,p}_{u_1,u_2}(\Omega)$ of $W^{1,p}(\Omega)$. Moreover f_k is strictly convex so that f_k and g satisfy the assumption (b) of Theorem 3.1. It follows that $\bar u_k$ is Lipschitz with a Lipschitz constant bounded by K: we may assume, up to a subsequence, that $\bar u_k$ converges weakly in $W^{1,p}_{u_1,u_2}(\Omega)$ and uniformly to a function $\bar u$, which is therefore Lipschitz with a Lipschitz constant bounded by K. The functional I being lower semicontinuous we have

$$I(\bar u) = \int_\Omega f\big(\nabla \bar u(x)\big) + g\big(x, \bar u(x)\big)\,dx \leqslant \liminf_{k\to+\infty} \int_\Omega f\big(\nabla \bar u_k(x)\big) + g\big(x, \bar u_k(x)\big)\,dx.$$

Moreover, for every k,

$$\int_\Omega f\big(\nabla \bar u_k(x)\big)\,dx + g\big(x, \bar u_k(x)\big)\,dx \leqslant \int_\Omega f\big(\nabla \bar u_k(x)\big) + \frac{1}{k}|\nabla \bar u_k|^p\,dx + g\big(x, \bar u_k(x)\big)\,dx = I_k(\bar u_k).$$

Let u be any function in $W^{1,p}_{u_1,u_2}(\Omega)$; since $\bar u_k$ is a minimizer of I_k the latter inequalities yield

$$I(\bar u) \leqslant \liminf_{k\to+\infty} I_k(\bar u_k) \leqslant \liminf_{k\to+\infty} I_k(u) = I(u)$$

thus proving that $\bar u$ is a minimizer of I in $W^{1,p}_{u_1,u_2}(\Omega)$. $\square$

In the particular case of $g(x,u)=g(u)$ (not depending on x) Theorem 4.1 becomes particularly attractive.

Corollary 4.2. *Assume that $f:\mathbb{R}^n\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ are convex and that f is bounded from below by an affine function. Then the problem of minimizing*

$$I(u)=\int_{\Omega} f\big(\nabla u(x)\big)+g\big(u(x)\big)\,dx$$

in $W^{1,p}_{u_1,u_2}(\Omega)$ admits at least one solution; moreover at least one of the minimizers is Lipschitz with a Lipschitz constant bounded by $K=\max\{\mathrm{Lip}(u_i)\colon i=1,2\}$.

The proof of this corollary follows from the fact that, by Proposition 2.1, the assumptions of Theorem 4.1 are satisfied.

Remark 4.3. This corollary extends and gives an alternative proof of Theorem 5.1 of [12] where we assumed I to be strictly convex.

The purpose of what follows is to show under the assumptions of Theorem 4.1 that if, instead of f being strictly convex, the faces of the epigraph of f are bounded then *every* minimizer of I in $W^{1,p}_{u_1,u_2}(\Omega)$ is Lipschitz.

We first show that if the fact that the epigraph of f has no unbounded faces is equivalent to a sort of growth condition that we introduced in [13] under the name of *Conical Growth Assumption* (CGA). In [13] this condition appeared to be quite weaker than superlinearity but seemed there to be much more technical than the simple requirement that the faces of the epigraph are bounded.

Conical Growth Assumption (CGA). A convex function $f:\mathbb{R}^n\to\mathbb{R}$ satisfies the (CGA) if and only if, for every positive R_0, there exist $\varepsilon>0$, $c\in\mathbb{R}$ and $R>0$ such that for every $|\xi|\geqslant R$, $|\xi_0|\leqslant R_0$

$$f(\xi)\geqslant f(\xi_0)+p(\xi_0)\cdot(\xi-\xi_0)+\varepsilon|\xi|+c$$

where $p(\xi_0)\in\partial f(\xi_0)$.

Remark 4.4. By definition of subdifferential the convex function f satisfies

$$f(\xi)\geqslant f(\xi_0)+p(\xi_0)\cdot(\xi-\xi_0)$$

for every ξ; if f satisfies the (CGA) then, after a radius depending on $|\xi_0|$, the function grows a little more than that.

Remark 4.5. The (CGA) is fulfilled if, for instance, f is superlinear, i.e. if

$$\lim_{|\xi|\to+\infty}\frac{f(\xi)}{|\xi|}=+\infty.$$

Theorem 4.6. *A convex function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the Conical Growth Assumption (CGA) if and only if the faces of the epigraph of f are bounded.*

Proof. Assume that f satisfies the (CGA) and fix ξ_0 in $\mathbb{R}$. Then, for suitable $R > 0$, $\varepsilon > 0$ and $c \in \mathbb{R}$ the inequality

$$f(\xi) \geqslant f(\xi_0) + p(\xi_0) \cdot (\xi - \xi_0) + \varepsilon|\xi| + c \tag{$*$}$$

holds for every $|\xi| \geqslant R$ and some $p(\xi_0) \in \partial f(\xi_0)$. It follows that if $|\xi| > \max\{R, -c/\varepsilon\}$ then $(\xi, f(\xi))$ does not belong to the same face of the epigraph of f containing $(\xi_0, f(\xi_0))$ otherwise

$$f(\xi) = f(\xi_0) + p(\xi_0) \cdot (\xi - \xi_0)$$

so that, by $(*)$, $\varepsilon|\xi| + c = 0$, a contradiction.

Assume now that the faces of the epigraph of f are bounded. If f does not fulfill the (CGA) there exists $R_0 > 0$ and, for every $k \geqslant 1$, ξ^k and ξ_0^k in $\mathbb{R}^n$ with $|\xi^k| \geqslant k$, $|\xi_0^k| \leqslant R_0$ such that

$$f\left(\xi^k\right) < f\left(\xi_0^k\right) + p\left(\xi_0^k\right) \cdot \left(\xi^k - \xi_0^k\right) + \frac{1}{k}\left|\xi^k\right|$$

where $p(\xi_0^k) \in \partial f(\xi_0^k)$. Thus, for every $t \in [0, 1]$, by convexity we have

$$\begin{aligned} f\left((1-t)\xi_0^k + t\xi^k\right) &\leqslant (1-t) f\left(\xi_0^k\right) + t f\left(\xi^k\right) \\ &< f\left(\xi_0^k\right) + p\left(\xi_0^k\right) \cdot t\left(\xi^k - \xi_0^k\right) + \frac{t}{k}\left|\xi^k\right|. \end{aligned}$$

Set $\xi^k = \lambda_k \nu_k$ with $\lambda_k \geqslant k$ and $|\nu_k| = 1$. The latter inequality can be rewritten as

$$f\left((1-t)\xi_0^k + t\lambda_k\nu_k\right) < f\left(\xi_0^k\right) + tp\left(\xi_0^k\right) \cdot \left(\lambda_k\nu_k - \xi_0^k\right) + \frac{t}{k}\lambda_k$$

so that, for $r > 0$, $t = r/\lambda_k$ and k sufficiently big we obtain

$$f\left(\left(1 - \frac{r}{\lambda_k}\right)\xi_0^k + r\nu_k\right) < f\left(\xi_0^k\right) + \frac{r}{\lambda_k} p\left(\xi_0^k\right) \cdot \left(\lambda_k\nu_k - \xi_0^k\right) + \frac{r}{k}.$$

Up to a subsequence we may assume that $\lim_k \nu_k = \nu_0$, $\lim_k \xi_0^k = \xi_0$. Moreover, the subdifferential of f being bounded on the balls of $\mathbb{R}^n$, we may assume that $\lim_k p(\xi_0^k) = p$; necessarily $p \in \partial f(\xi_0)$. Passing to the limit in the latter inequality we obtain

$$f(\xi_0 + r\nu_0) \leqslant f(\xi_0) + p \cdot (r\nu_0).$$

Moreover, the opposite inequality holds since $p \in \partial f(\xi_0)$ and therefore

$$f(\xi_0 + r\nu_0) = f(\xi_0) + p \cdot \left((\xi_0 + r\nu_0) - \xi_0\right)$$

so that the points $(\xi_0, f(\xi_0))$ and $(\xi_0 + r\nu_0, f(\xi_0 + r\nu_0))$ belong to the same face of the epigraph of f. A contradiction, since r is arbitrary and the faces of the epigraph of f are bounded. It follows that the (CGA) holds. □

We will need to introduce a further notation.

Notation R_K^f. Let $f:\mathbb{R}^n \to \mathbb{R}$ be convex. We denote by R_K^f the maximum modulus of the points of the domain that belong to the projection of a face of the epigraph $\operatorname{epi}(f)$ of f containing the image of a point of modulus less than K; in other words

$$R_K^f = \max\bigl\{|\xi|:\ \exists |\eta| \leqslant K\ \bigl(\xi, f(\xi)\bigr) \text{ and } \bigl(\eta, f(\eta)\bigr) \text{ belong to the same face of } \operatorname{epi}(f)\bigr\}.$$

Proposition 4.7. *Assume that the faces of the epigraph of f are bounded. Then R_K^f is finite.*

Proof. Since, from Theorem 4.6, the (CGA) holds there exist $R > 0$, $\varepsilon > 0$ and c in $\mathbb{R}$ such that

$$f(\xi) \geqslant f(\eta) + p(\eta)\cdot(\xi - \eta) + \varepsilon|\xi| + c$$

for every $|\eta| \leqslant K$ and $|\xi| \geqslant R$, where $p(\eta) \in \partial f(\eta)$. It follows that if $|\xi| > \max\{R, -c/\varepsilon\}$ then the points $(\eta, f(\eta))$ and $(\xi, f(\xi))$ do not belong to the same face of the epigraph of f. $\square$

We are now in the position to weaken the strict convexity assumption in the last part of Theorem 4.1.

Theorem 4.8 *(Existence and Lipschitz continuity of minimizers in $W^{1,p}_{u_1,u_2}(\Omega)$). Assume that f, g fulfill the Basic Assumption and that g satisfies the Large Monotonicity Condition, i.e. there exists $p(x,\cdot) \in \partial_u g(x,\cdot)$ for a.e. x such that*

$$v > u + K|y - x| \quad \Rightarrow \quad p(x,u) \leqslant p(y,v).$$

Assume moreover that the faces of the epigraph of f are bounded. Then the problem of minimizing

$$I(u) = \int_\Omega f\bigl(\nabla u(x)\bigr) + g\bigl(x, u(x)\bigr)\,dx$$

in $W^{1,p}_{u_1,u_2}(\Omega)$ admits at least one solution. Every minimizer of I in $W^{1,p}_{u_1,u_2}(\Omega)$ is Lipschitz with a Lipschitz constant bounded by

$$R_K^f = \max\bigl\{|\xi|:\ \exists |\eta| \leqslant K\ \bigl(\xi, f(\xi)\bigr) \textit{ and } \bigl(\eta, f(\eta)\bigr) \textit{ belong to the same face of } \operatorname{epi}(f)\bigr\}$$

where $K = \max\{\operatorname{Lip}(u_i):\ i = 1, 2\}$. Moreover, one of these minimizers has a Lipschitz constant bounded by K.

The proof of the theorem is based on the following lemma.

Lemma 4.9. *Let f, g satisfy the Basic Assumptions and $\bar{u}$, $\bar{v}$ be two minimizers of*

$$I(u) = \int_\Omega f\bigl(\nabla u(x)\bigr) + g\bigl(x, u(x)\bigr)\,dx$$

on a convex subset of $W^{1,p}(\Omega)$. Then for a.e. x in Ω the points

$$\big(\nabla\bar{u}(x), f\big(\nabla\bar{u}(x)\big)\big), \qquad \big(\nabla\bar{v}(x), f\big(\nabla\bar{v}(x)\big)\big)$$

belong to the same face of the epigraph of f.

Proof. Let Σ be the subset of Ω where $(\nabla\bar{u}(x), f(\nabla\bar{u}(x)))$ and $(\nabla\bar{v}(x), f(\nabla\bar{v}(x)))$ do not belong to the same face of the epigraph of f. For any measurable function $q:\mathbb{R}^n \to \mathbb{R}^n$ such that $q(\xi) \in \partial f(\xi)$ the set Σ can be expressed by the formula

$$\Sigma = \big\{x \in \Omega:\ f\big(\nabla\bar{v}(x)\big) > f\big(\nabla\bar{u}(x)\big) + q\big(\nabla\bar{u}(x)\big)\cdot\big(\nabla\bar{v}(x) - \nabla\bar{u}(x)\big)\big\}$$

it follows that Σ is a measurable subset of Ω. Let, for any measurable subset A of Ω and u in $W^{1,p}(\Omega)$,

$$I_A(u) = \int_A f\big(\nabla u(x)\big) + g\big(x, u(x)\big)\,dx.$$

By decomposing Ω as the union of Σ with its complement we obtain

$$I\left(\frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}\right) = I_{\Omega\setminus\Sigma}\left(\frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}\right) + I_\Sigma\left(\frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}\right).$$

The convexity of f and g (on the last variable) yields, for a.e. x,

$$f\left(\frac{1}{2}\nabla\bar{u}(x) + \frac{1}{2}\nabla\bar{v}(x)\right) \leqslant \frac{1}{2}f\big(\nabla\bar{u}(x)\big) + \frac{1}{2}f\big(\nabla\bar{v}(x)\big)$$

and, analogously

$$g\left(x, \frac{1}{2}\bar{u}(x) + \frac{1}{2}\bar{v}(x)\right) \leqslant \frac{1}{2}g\big(x, \bar{u}(x)\big) + \frac{1}{2}g\big(x, \bar{v}(x)\big)$$

and thus

$$I_{\Omega\setminus\Sigma}\left(\frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}\right) \leqslant \frac{1}{2}I_{\Omega\setminus\Sigma}(\bar{u}) + \frac{1}{2}I_{\Omega\setminus\Sigma}(\bar{v}).$$

Moreover, on Σ, $(\nabla\bar{u}(x), f(\nabla\bar{u}(x)))$ and $(\nabla\bar{v}(x), f(\nabla\bar{v}(x)))$ do not belong to the same face of the epigraph of f and thus we have

$$f\left(\frac{1}{2}\nabla\bar{u} + \frac{1}{2}\nabla\bar{v}\right) < \frac{1}{2}f(\nabla\bar{u}) + \frac{1}{2}f(\nabla\bar{v}) \quad \text{a.e. on } \Sigma$$

so that, if Σ is not negligible,

$$I_\Sigma\left(\frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}\right) < \frac{1}{2}I_\Sigma(\bar{u}) + \frac{1}{2}I_\Sigma(\bar{v}).$$

Therefore, if Σ is not negligible, by adding together the functionals $I_{\Omega\setminus\Sigma}$ and I_Σ we obtain

$$I\left(\frac{1}{2}\bar{u}+\frac{1}{2}\bar{v}\right)<\frac{1}{2}I(\bar{u})+\frac{1}{2}I(\bar{v})=I(\bar{u})=I(\bar{v})$$

since $\bar{u}$ and $\bar{v}$ are both minimizers of I in a convex subset of $W^{1,p}_{u_1,u_2}(\Omega)$, a contradiction. It follows that Σ is negligible, so that $\nabla\bar{u}$ and $\nabla\bar{v}$ belong a.e. to the same face of the epigraph of f. □

Proof of Theorem 4.8. Theorem 4.1 yields the existence of a minimizer $\bar{u}$ of I that is Lipschitz with a Lipschitz constant bounded by K; therefore $|\nabla u(x)|\leqslant K$ a.e. If $\bar{v}$ is any other minimizer of I in $W^{1,p}_{u_1,u_2}(\Omega)$ then, by the lemma, the points $(\nabla\bar{u}(x), f(\nabla\bar{u}(x)))$ and $(\nabla\bar{v}(x), f(\nabla\bar{v}(x)))$ belong a.e. to the same face of the epigraph of f. Then $\nabla\bar{v}(x)$ belongs a.e. to the projection of the face of the epigraph of f containing $(\nabla\bar{u}(x), f(\nabla\bar{u}(x)))$ and in particular, for a.e. x in Ω, $|\nabla v(x)|$ is bounded above by

$$\max\bigl\{|\xi|\colon \exists|\eta|\leqslant K\ \bigl(\xi, f(\xi)\bigr)\text{ and }\bigl(\eta, f(\eta)\bigr)\text{ belong to the same face of epi}(f)\bigr\}=R_K^f$$

which is finite, by Proposition 4.7. Now $\bar{v}-\bar{u}\in W^{1,p}_0(\Omega)$ and $|\nabla(\bar{v}-\bar{u})|$ is essentially bounded; it follows that $\bar{v}$ is Lipschitz and its Lipschitz constant is bounded by R_K^f. □

5. Lipschitz regularity under the Bounded Slope Condition of Cellina

In this part of the paper we let $g=0$ and thus consider the functional

$$I(u)=\int_\Omega f\bigl(\nabla u(x)\bigr)\,dx$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is convex, bounded below by an affine function. Let u_0 be a Lipschitz function on Ω.

There are many recent results concerning the Lipschitz regularity of the minima of the functional I on $u_0+W^{1,p}(\Omega)$ under the assumption that the boundary datum u_0 satisfies the *Bounded Slope Condition* (BSC) of Hartman–Stampacchia, that we recall here; we refer to [10] for some classical results involving the (BSC).

Bounded Slope Condition (BSC). The function u_0 satisfies the (BSC) of constant $K\geqslant 0$ if for every $x^0\in\partial\Omega$ there exist $k^+(x^0)$ and $k^-(x^0)$ in $\mathbb{R}^n$ such that, for every $x\in\partial\Omega$,

$$u_0\bigl(x^0\bigr)+k^-\bigl(x^0\bigr)\cdot\bigl(x-x^0\bigr)\leqslant u_0(x)\leqslant u_0\bigl(x^0\bigr)+k^+\bigl(x^0\bigr)\cdot\bigl(x-x^0\bigr).$$

In addition there is $K\geqslant 0$ such that

$$\forall x^0\in\partial\Omega\quad \max\bigl\{\bigl|k^-\bigl(x^0\bigr)\bigr|,\bigl|k^+\bigl(x^0\bigr)\bigr|\bigr\}\leqslant K.$$

We mention [5,13] for some regularity results when the boundary datum satisfies the (BSC); Clarke even proved a local Lipschitz regularity in [7] for boundary data satisfying a partial BSC (just from below, or from above). The proofs of these results rely strongly on the assumption that the Lagrangian is strictly convex; they are in fact based on the Comparison Principle between a minimum and an affine function (a special class of minima) which may be false if the Lagrangian is not strictly convex.

More recently Cellina introduced a new class of minimizers of I and formulated a new Comparison Principle dropping out the requirement of strict convexity of f.

We recall briefly these results for the convenience of the reader. By f^* we denote the *polar* of f. The *effective domain* of a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is

$$\mathrm{Dom}(\varphi) = \bigl\{\xi \in \mathbb{R}^n \colon \varphi(\xi) < +\infty\bigr\}.$$

Definition 5.1. For $\theta \in \mathrm{Dom}(f^*)$, $x^0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we define the functions

$$h^+_{\theta,x^0,r}(x) = \sup_{k \in \partial f^*(\theta)} k \cdot \bigl(x - x^0\bigr) + r, \qquad h^-_{\theta,x^0,r}(x) = \inf_{k \in \partial f^*(\theta)} k \cdot \bigl(x - x^0\bigr) + r.$$

The function $h^+_{\theta,x^0,r}$ is convex and $h^-_{\theta,x^0,r}$ is concave; they both coincide with r at x^0. We mention that if f is strictly convex then $\partial f^*(\theta)$ is single valued and thus in this case the maps just defined are affine.

It turns then out that, in what concerns the minimizers of I, under a suitable "growth" assumption on the convex function f (unnecessarily strictly convex) the functions $h^+_{\theta,x^0,r}$ and $h^-_{\theta,x^0,r}$ do the same job as do affine functions when f is strictly convex.

For instance it is proved in [5, Theorem 1] that if f is convex (even extended valued), lower semicontinuous and $\mathrm{Dom}(f^*)$ has a non-empty interior then, for θ in the interior of $\mathrm{Dom}(f^*)$, $x^0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$ the map $h^+_{\theta,x^0,r}$ (respectively $h^-_{\theta,x^0,r}$) is a minimizer of I among the functions in

$$h^+_{\theta,x^0,r} + W^{1,1}_0(\Omega) \quad \bigl(\text{respectively } h^-_{\theta,x^0,r} + W^{1,1}_0(\Omega)\bigr).$$

We recall here some characterizations of the latter condition on the effective domain of f^*; we refer to [1] for the simple proof of the following equivalence result.

Proposition 5.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. The following conditions are equivalent*:

(i) *The interior of* $\mathrm{Dom}(f^*)$ *is non-empty*;
(ii) f *is* demi-coercive*, i.e. there exist $a > 0$, $p \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that*

$$f(\xi) \geqslant a|\xi| + p \cdot \xi + c$$

for all ξ in $\mathbb{R}^n$;
(iii) f *is non-constant on straight lines.*

In what follows by $u \leqslant v$ on $\partial\Omega$ we mean that the positive part $(u - v)^+$ of $u - v$ belongs to $W^{1,1}_0(\Omega)$. We will use the result of Cellina, that we state here for the convenience of the reader in a slightly general version.

Comparison Principle *(Cellina). (See [5, Theorem 2].) Let Ω be convex, $f:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be convex, lower semicontinuous and non-constant on straight lines. Let $\bar u$ be a minimizer of*

$$I(u) = \int_{\Omega} f\big(\nabla u(x)\big)\,dx$$

on $u_0 + W_0^{1,1}(\Omega)$. Assume that, for θ in the interior of $\mathrm{Dom}(f^*)$, *$x^0 \in \mathbb{R}^n \setminus \Omega$ and $r \in \mathbb{R}$, we have $h^+_{\theta,x^0,r} \geqslant u_0$ (respectively $h^-_{\theta,x^0,r} \leqslant u_0$) on $\partial\Omega$. Then $h^+_{\theta,x^0,r} \geqslant \bar u$ (respectively $h^-_{\theta,x^0,r} \leqslant \bar u$) a.e. on Ω.*

Remark 5.3. The true formulation of the Comparison Principle is actually slightly different. Cellina assumes in [5] that the domain of f^* is open; however the proofs of these results still hold if the domain of f^* has a non-empty interior or equivalently (Proposition 5.2) that f is non-constant on straight lines. This is the case, for instance, of $f(\xi) = \sqrt{1+\xi^2}$, $\xi \in \mathbb{R}$, where the domain of f^* is $[-1,1]$. As it is pointed out in [5], the domain of f^* is open if f has a superlinear growth.

Lemma 5.4. *Assume that the faces of the epigraph of f are bounded. Then if $\xi_0 \in \mathbb{R}^n$ and $p \in \partial f(\xi_0)$ there is a neighborhood of p that is contained in the domain of f^*; in particular* $\mathrm{Dom}(f^*)$ *has a non-empty interior. Moreover the norm of every element of $\partial f^*(p)$ is bounded by $R^f_{|\xi_0|}$ given by*

$$R^f_{|\xi_0|} = \max\big\{|\xi|:\ \exists|\eta| \leqslant |\xi_0|\ \big(\xi, f(\xi)\big) \textit{ and } \big(\eta, f(\eta)\big) \textit{ belong to the same face of } \mathrm{epi}(f)\big\}.$$

Proof. By Theorem 4.6 the (CGA) holds: there exist $\varepsilon > 0$, $R \geqslant |\xi_0|$ and $c \in \mathbb{R}$ such that, for every $|\xi| \geqslant R$,

$$f(\xi) \geqslant f(\xi_0) + p\cdot(\xi - \xi_0) + \varepsilon|\xi| + c$$

so that, for every $\nu \in \mathbb{R}^n$ with $|\nu| = 1$, $|\xi| \geqslant \xi\cdot\nu$ and thus

$$f(\xi) \geqslant f(\xi_0) + p\cdot\xi + \varepsilon\nu\cdot\xi - p\cdot\xi_0 + c$$

or, equivalently,

$$\forall|\xi| \geqslant R \quad f(\xi) \geqslant f(\xi_0) + (p+\varepsilon\nu)\cdot\xi - p\cdot\xi_0 + c.$$

Moreover, f being bounded from below by an affine function, $f(\xi) - (p+\varepsilon\nu)\cdot\xi$ is bounded from below too on the ball of radius R centered in the origin. Therefore there exists $r \in \mathbb{R}$ such that

$$f(\xi) \geqslant (p+\varepsilon\nu)\cdot\xi + r$$

for every ξ in $\mathbb{R}^n$, proving that $p + \varepsilon\nu \in \mathrm{Dom}(f^*)$ for every unitary vector ν. The convexity of $\mathrm{Dom}(f^*)$ implies that the entire ball of radius ε and centered in p is contained in $\mathrm{Dom}(f^*)$. To prove the last part of the claim it is enough to remark that if $\xi \in \partial f^*(p)$ then $p \in \partial f(\xi)$ so that

$f(\xi)-f(\xi_0)=p\cdot(\xi-\xi_0)$ and thus the points $(\xi, f(\xi))$ and $(\xi_0, f(\xi_0))$ belong to the same face of the epigraph of f. $\square$

Remark 5.5. Lemma 5.4 provides also a direct proof that the interior of the domain of f^* is non-empty if the faces of the epigraph of f are bounded. Alternatively, this conclusion could be directly obtained by the equivalence stated in Proposition 5.2.

The following condition on the boundary datum analogous to the classical (BSC) was introduced in [5]; we call it the *Cellina Bounded Slope Condition*.

(CBSC). We say that (u_0, f) satisfies the (CBSC) of constant $K \geqslant 0$ if for every $x^0 \in \partial\Omega$ there exist $\theta^+(x^0)$ and $\theta^-(x^0)$ in the interior of $\mathrm{Dom}(f^*)$ such that, for every $x \in \partial\Omega$,

$$u_0(x^0) + \inf_{k\in\partial f^*(\theta^-(x^0))} k\cdot(x-x^0) \leqslant u_0(x) \leqslant u_0(x^0) + \sup_{k\in\partial f^*(\theta^+(x^0))} k\cdot(x-x^0).$$

In addition there is $K \geqslant 0$ such that

$$K \geqslant \sup_{x^0\in\partial\Omega} \sup\bigl\{|k|:\ k\in\partial f^*\bigl(\theta^-\bigl(x^0\bigr)\bigr)\cup\partial f^*\bigl(\theta^+\bigl(x^0\bigr)\bigr)\bigr\}.$$

We notice that when θ belongs to the interior of $\mathrm{Dom}(f^*)$, the subdifferential $\partial f^*(\theta)$ is bounded and therefore $\sup\{|k|:\ k\in\partial f^*(\theta)\}$ is finite. The (CBSC) requires thus a uniformity in this bound. We also mention that if the (CBSC) of constant K holds then u_0 is Lipschitz on $\partial\Omega$ and its Lipschitz constant is bounded by K.

Remark 5.6. The definition of (CBSC) does not actually require that f be convex. We will use it for non-necessarily convex function only in the last section of this paper.

We give here criteria under which the classical (BSC) implies the new (CBSC), that is always fulfilled if the faces of the epigraph are bounded.

Proposition 5.7.

(i) *Let $\bar{k}, \theta \in \mathbb{R}^n$ be such that $\theta \in \partial f(\bar{k})$. Then, for every $x^0 \in \mathbb{R}^n$, $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$ the following inequalities hold*

$$h^-_{\theta,x^0,r}(x) \leqslant \bar{k}\cdot(x-x^0)+r \leqslant h^+_{\theta,x^0,r}(x).$$

(ii) *Assume that the Lipschitz function u_0 satisfies the (BSC) of constant K and that the epigraph of f has no unbounded faces. Then (u_0, f) fulfills the (CBSC) of constant R^f_K.*

Proof. Claim (i) follows immediately from the fact that $\bar{k}\in\partial f^*(\theta)$ so that

$$\inf\bigl\{k\cdot(x-x^0):\ k\in\partial f^*(\theta)\bigr\} \leqslant \bar{k}\cdot(x-x^0) \leqslant \sup\bigl\{k\cdot(x-x^0):\ k\in\partial f^*(\theta)\bigr\}.$$

If the faces of the epigraph of f are bounded then Lemma 5.4 shows that, every θ is in $\partial f(k)$ belongs to the interior of the domain of f^* and that moreover, every element of $\partial f^*(\theta)$ is bounded by R_K^f; the application of (i) yields the conclusion. □

Remark 5.8. The (BSC) and (CBSC) are not equivalent. It can be shown for instance that if $f(\xi)=0$ for $|\xi|<1$, $f(\xi)=\frac{1}{2}(|\xi|-1)^2$ for $|\xi|\geqslant 1$ then every Lipschitz function u_0 of Lipschitz constant 1 satisfies the (CBSC), no matter what the domain Ω is. Instead the (BSC) implies the convexity of the domain and affinity of the boundary datum on the flat parts of the boundary Ω. More details on the subject will be given in a forthcoming paper.

Example 5.9. Let $n=1$, $\Omega=[a,b]$, $u_0(a)=A$, $u_0(b)=B$. Assume that $\partial f(\frac{B-A}{b-a})$ has a value that is interior to the domain of f^*. Then (u_0,f) satisfies the (CBSC). In fact $u_0(x)=\frac{B-A}{b-a}\times(x-a)+A$ satisfies the (BSC); the claim follows from Proposition 5.7.

We are now able to state the following regularity results without any strictly convexity assumption on f; the first one is an application of Theorem 4.1, the second one follows from Theorem 4.8

Theorem 5.10 *(Existence of a Lipschitz minimizer under the (CBSC)). Let Ω be an open, bounded and convex set; let $f:\mathbb{R}^n\to\mathbb{R}$ be convex. Assume that (u_0,f) satisfies the (CBSC) of constant K and that f is non-constant on straight lines. Assume that*

$$I(u)=\int_{\Omega} f\big(\nabla u(x)\big)\,dx$$

has a minimizer $\bar u$ in $u_0+W_0^{1,1}(\Omega)$. Then there are two Lipschitz functions u_1, u_2 satisfying

$$u_1\leqslant\bar u\leqslant u_2\quad a.e.\qquad \operatorname{Lip}(u_i)\leqslant K,\qquad u_i=u_0\quad on\ \partial\Omega\quad (i=1,2)$$

and therefore there exists a Lipschitz minimizer of I that is Lipschitz with a Lipschitz constant is bounded by K.

Proof. Cellina Comparison Principle yields

$$u_0\big(x^0\big)+\inf_{k\in\partial f^*(\theta^-(x^0))}k\cdot\big(x-x^0\big)\leqslant\bar u(x)\leqslant u_0\big(x^0\big)+\sup_{k\in\partial f^*(\theta^+(x^0))}k\cdot\big(x-x^0\big)$$

for a.e. x in Ω. Set

$$u_1(x)=\sup\Big\{u_0\big(x^0\big)+\inf_{k\in\partial f^*(\theta^-(x^0))}k\cdot\big(x-x^0\big):\ x^0\in\partial\Omega\Big\}$$

and

$$u_2(x)=\inf\Big\{u_0\big(x^0\big)+\sup_{k\in\partial f^*(\theta^+(x^0))}k\cdot\big(x-x^0\big):\ x^0\in\partial\Omega\Big\}.$$

Then u_1 and u_2 are Lipschitz; their Lipschitz constant is bounded by K, moreover $u_1 = u_2 = u_0$ on $\partial\Omega$. It follows that the minimizers of I in $u_0 + W_0^{1,1}(\Omega)$ are minimizers of I in $W^{1,p}_{u_1,u_2}(\Omega)$. Theorem 4.1 yields the conclusion. □

Remark 5.11. As it is pointed out in [7] the existence of a Lipschitz minimizer under the classical (BSC) may be obtained easily by considering a strictly convex perturbation of the functional, like $\int_\Omega f(\nabla u) + \frac{1}{k}|\nabla u|^2\,dx$ and passing to the limit. The same reasoning would not hold here since it may not be true that $(u_0, f(\xi) + \frac{1}{k}|\xi|^2)$ satisfies the (CBSC).

Theorem 5.12 *(Lipschitz continuity of minimizers under the (CBSC) and the (CGA)). Let Ω be an open, bounded and convex set; let $f:\mathbb{R}^n \to \mathbb{R}$ be convex. Assume that (u_0, f) satisfies the (CBSC) of constant K and that the epigraph of f does not contain an unbounded face. Let $\bar{u}$ be a minimizer of*

$$I(u) = \int_\Omega f\big(\nabla u(x)\big)\,dx$$

in $u_0 + W_0^{1,1}(\Omega)$. Then $\bar{u}$ is Lipschitz and its Lipschitz constant is bounded by

$$R_K^f = \max\big\{|\xi|\colon \exists|\eta| \leqslant K,\ \big(\xi, f(\xi)\big) \text{ and } \big(\eta, f(\eta)\big) \text{ belong to the same face of } \operatorname{epi}(f)\big\}.$$

Moreover, I has a minimizer whose Lipschitz constant is bounded by K.

Proof. Since the faces of the epigraph of f are bounded then f is non-constant on straight lines: Theorem 5.10 yields the existence of two Lipschitz functions u_1 and u_2, with a Lipschitz constant bounded by K, such that $u_1 = u_2 = u_0$ on $\partial\Omega$; therefore $\bar{u}$ is a minimizer of I in $W^{1,p}_{u_1,u_2}(\Omega)$. The assumption (again) that the faces of the epigraph of f are bounded together with Theorem 4.8 yield the conclusion. □

Remark 5.13. Theorem 5.12 extends Theorem 3 of [5] where it is assumed that $\bar{u}$ is continuous and, instead of the fact that the faces of the epigraph of f are bounded, that f is strictly convex out of the ball of radius K and that $\operatorname{Dom}(f^*)$ is open. Our result applies for instance to functions like $f(\xi) = \sqrt{1 + |\xi|^2}$.

Remark 5.14. The assumption that the epigraph of f does not contain any unbounded face is somewhat optimal. For instance, any increasing absolutely continuous function on $[-1, 1]$ with prescribed boundary values is a minimizer of $I(u) = \int_0^1 |u'(t)|\,dt$ so that I admits among its minimizers some functions that are not Lipschitz; here the (BSC) is obviously satisfied.

Remark 5.15. We will prove in a forthcoming paper that the convexity assumption on the domain Ω can be weakened in the claims of Theorems 5.10 and 5.12.

6. Relaxation and Lipschitz regularity in the non-convex case

In this section we will consider the case of a Lagrangian $f:\mathbb{R}^n \to \mathbb{R}$, that we assume again to be bounded below by an affine function but not to be convex; u_0 is a Lipschitz function on Ω.

For $r > 0$ we denote by $\mathrm{Lip}_r(\Omega)$ the set of Lipschitz functions on Ω whose Lipschitz constant is less than r; $\mathrm{Lip}_r(\Omega, u_0)$ is the subset of $\mathrm{Lip}_r(\Omega)$ of those functions that coincide with u_0 on $\partial\Omega$; the space of Lipschitz functions on Ω is denoted by $\mathrm{Lip}(\Omega)$. The *bipolar* of f is denoted by f^{**}.

Theorem 6.1. *Assume that $u_0 \in \mathrm{Lip}_K(\Omega)$ and that the epigraph of f^{**} has no unbounded faces. There exists $K' \geqslant K$ such that*

$$\inf\left\{ \int_\Omega f^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_K(\Omega, u_0)\right\} \geqslant \inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0)\right\}.$$

Proof. Given a function $g : \mathbb{R}^n \to \mathbb{R}$ and $r > 0$ let us denote by g_r the function defined by

$$g_r(\xi) = \begin{cases} g(\xi) & \text{if } |\xi| \leqslant r, \\ +\infty & \text{if } |\xi| > r. \end{cases}$$

We obviously have $f^{**}(\nabla u(x)) = (f^{**})_K(\nabla u(x))$ for every u in $\mathrm{Lip}_K(\Omega, u_0)$.

Now $(f^{**})_K \leqslant (f_K)^{**}$, the equality being false in general. However, if the faces of the epigraph of f are bounded, a sort of reverse inequality holds true. In fact, by [14, Theorem 3.2], there exists $K' > K$ such that, for every $\varepsilon > 0$, any $|\xi| \leqslant K$ may be written as a convex combination

$$\xi = \sum_{i=1}^{2n+2} \lambda_i \xi_i, \quad \lambda_i \geqslant 0,\ \sum_{i=1}^{2n+2} \lambda_i = 1,\ |\xi_i| \leqslant K'$$

and

$$\sum_{i=1}^{2n+2} \lambda_i f(\xi_i) \leqslant f^{**}(\xi) + \varepsilon.$$

Therefore, by convexity,

$$(f_{K'})^{**}(\xi) = (f_{K'})^{**}\left(\sum_{i=1}^{2n+2} \lambda_i \xi_i\right) \leqslant \sum_{i=1}^{2n+2} \lambda_i (f_{K'})^{**}(\xi_i) \leqslant f^{**}(\xi) + \varepsilon$$

so that, ε being arbitrary,

$$\forall |\xi| \leqslant K \quad (f_{K'})^{**}(\xi) \leqslant \big(f^{**}\big)_K(\xi).$$

It follows that

$$\inf\left\{ \int_\Omega f^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_K(\Omega, u_0)\right\} \geqslant \inf\left\{ \int_\Omega (f_{K'})^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_K(\Omega, u_0)\right\}$$

and thus, since $\mathrm{Lip}_K(\Omega, u_0) \subset \mathrm{Lip}_{K'}(\Omega, u_0)$,

$$\inf\left\{ \int_\Omega f^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_K(\Omega, u_0) \right\} \geqslant \inf\left\{ \int_\Omega (f_{K'})^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \right\}.$$

The relaxation result [9, Proposition X.3.6] states that

$$\inf\left\{ \int_\Omega (f_{K'})^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \right\} = \inf\left\{ \int_\Omega f_{K'}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \right\}$$

the conclusion follows. □

Remark 6.2. The results hold true if $|\nabla u_0| \leqslant K$ a.e. instead of assuming that u_0 is Lipschitz; in this case it is enough to replace in the claim of Theorem 6.1 the sets $\mathrm{Lip}_r(\Omega, u_0)$ with $\mathcal{W}_r = \{u \in u_0 + W_0^{1,1}(\Omega)\colon\ |\nabla u| \leqslant r \text{ a.e.}\}$.

The next relaxation result is straightforward. As usual, I and I^{**} are the functionals defined by

$$I(u) = \int_\Omega f\big(\nabla u(x)\big)\,dx, \qquad I^{**}(u) = \int_\Omega f^{**}\big(\nabla u(x)\big)\,dx.$$

Theorem 6.3. *Assume that the epigraph of f^{**} has no unbounded faces. Assume moreover that*

$$I^{**}(u) = \int_\Omega f^{**}\big(\nabla u(x)\big)\,dx$$

admits a minimizer on $u_0 + W_0^{1,1}(\Omega)$ that is Lipschitz. Then

$$\min\big\{I^{**}(u)\colon\ u \in u_0 + W_0^{1,1}(\Omega)\big\} = \inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \right\}$$

for some K'; in particular there is no Lavrentiev phenomenon for I, i.e.

$$\inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\,dx\colon\ u \in u_0 + W_0^{1,1}(\Omega) \right\} = \inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}(\Omega, u_0) \right\}.$$

Proof. Let K be greater than the Lipschitz constant of the minimizer of I. Since

$$\min I^{**} = \inf\left\{ \int_\Omega f^{**}\big(\nabla u(x)\big)\,dx\colon\ u \in \mathrm{Lip}_K(\Omega, u_0) \right\}$$

then by Theorem 6.1 we have

$$\min I^{**} \geqslant \inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\, dx \colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \right\}$$

for some K'. Now

$$\inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\, dx \colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \right\} \geqslant \inf\left\{ \int_\Omega f\big(\nabla u(x)\big)\, dx \colon\ u \in u_0 + W_0^{1,1}(\Omega) \right\}$$

and $f \geqslant f^{**}$ on $\mathbb{R}^n$ so that $\inf I \geqslant \inf I^{**}$; it follows that the above inequalities are in fact equalities. □

The proofs of the next corollaries follow then directly from the results of the previous sections.

Corollary 6.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that the epigraph of f^{**} has no unbounded faces. Let Ω be an open bounded and convex set and (u_0, f) satisfy the (CBSC). Assume that I^{**} has a minimizer in $u_0 + W_0^{1,1}(\Omega)$. Then there exists K' such that*

$$\min\big\{ I^{**}(u)\colon\ u \in u_0 + W_0^{1,1}(\Omega) \big\} = \inf\big\{ I(u)\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \big\}$$

and thus, in particular,

$$\min I^{**} = \inf I \quad \text{on } u_0 + W_0^{1,1}(\Omega).$$

Moreover, the minimizers of I, if they exist, are Lipschitz.

The assumption that I^{**} has a minimizer is satisfied when f^{**} has superlinear growth; the popularity of this growth condition leads us to reformulate explicitly Corollary 6.4 in this situation as follows.

Corollary 6.5. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ has a superlinear growth. Let Ω be an open, bounded, and convex set and (u_0, f) satisfy the (CBSC). There exists K' such that*

$$\min\big\{ I^{**}(u)\colon\ u \in u_0 + W_0^{1,1}(\Omega) \big\} = \inf\big\{ I(u)\colon\ u \in \mathrm{Lip}_{K'}(\Omega, u_0) \big\}$$

and thus, in particular,

$$\min I^{**} = \inf I \quad \text{on } u_0 + W_0^{1,1}(\Omega).$$

Moreover, the minimizers of I, if they exist, are Lipschitz.

Remark 6.6. We underline the fact that, in Corollary 6.5, we do not need any condition that bounds the Lagrangian f from above, as instead is assumed in the classical relaxation results (see for example [3, Theorem 10.8.3]).

References

[1] G. Anzellotti, G. Buttazzo, G. Dal Maso, Dirichlet problem for demi-coercive functionals, Nonlinear Anal. 10 (1986) 603–613.

[2] H. Brézis, M. Sibony, Équivalence de deux inéquations variationnelles et applications, Arch. Ration. Mech. Anal. 41 (1971) 254–265.

[3] L. Carbone, R. De Arcangelis, Unbounded Functionals in the Calculus of Variations, Chapman & Hall/CRC, New York, 2001.

[4] A. Cellina, Comparison results and estimates on the gradient without strict convexity, SIAM J. Control Optim. 46 (2007) 738–749.

[5] A. Cellina, On the bounded slope condition and the validity of the Euler Lagrange equation, SIAM J. Control Optim. 40 (2001/2002) 1270–1279.

[6] A. Cellina, The classical problem of the calculus of variations in the autonomous case: Relaxation and Lipschitzianity of solutions, Trans. Amer. Math. Soc. 356 (2004) 415–426.

[7] F.H. Clarke, Continuity of solutions to a basic problem in the calculus of variations, Ann. Sci. Norm. Super. Pisa Cl. Sci. 4 (2005) 511–530.

[8] F.H. Clarke, Yu.S. Ledyaev, R.J. Stern, P.R. Wolenski, Nonsmooth Analysis and Control Theory, Grad. Texts in Math., vol. 178, Springer-Verlag, New York, 1998.

[9] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, Classics Appl. Math, vol. 28, SIAM, Philadelphia, 1999.

[10] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[11] C. Mariconda, G. Treu, Existence and Lipschitz regularity for minima, Proc. Amer. Math. Soc. 130 (2002) 395–404.

[12] C. Mariconda, G. Treu, Gradient maximum principle for minima, J. Optim. Theory Appl. 112 (2002) 167–186.

[13] C. Mariconda, G. Treu, Lipschitz regularity of the minimizers of autonomous integral functionals with discontinuous non-convex integrands of slow growth, Calc. Var. Partial Differential Equations 29 (2007) 99–117.

[14] C. Mariconda, G. Treu, A relaxation result for autonomous integral functionals with discontinuous non-coercive integrand, ESAIM Control Optim. Calc. Var. 10 (2004) 201–210.

[15] G. Stampacchia, On some regular multiple integral problems in the calculus of variations, Comm. Pure Appl. Math. 16 (1963) 383–421.

[16] G. Treu, M. Vornicescu, On the equivalence of two variational problems, Calc. Var. Partial Differential Equations 11 (2000) 307–319.