Calc. Var. (2007) 29:99–117
DOI 10.1007/s00526-006-0059-4

Calculus of Variations

Lipschitz regularity of the minimizers of autonomous integral functionals with discontinuous non-convex integrands of slow growth

Carlo Mariconda · Giulia Treu

Received: 9 July 2003 / Accepted: 11 July 2006 /
Published online: 21 October 2006

Abstract Let $L(x,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a Borelian function and let (P) be the problem of minimizing

$$\int_a^b L(y(t), y'(t))\, \mathrm{d}t$$

among the absolutely continuous functions with prescribed values at a and b. We give some sufficient conditions that weaken the classical superlinear growth assumption to ensure that the minima of (P) are Lipschitz. We do not assume convexity of L w.r. to ξ or continuity of L.

Mathematics Subject Classification (1991) 49-XX

1 Introduction

In this paper we are concerned with the Lipschitz regularity of the solutions of the problem

$$\min \int_a^b L(y(t), y'(t))\, \mathrm{d}t : \; y \in AC([a,b], \mathbb{R}^N), \quad y(a) = A, \;\; y(b) = B. \tag{P}$$

A classical result states that if $L(x,\xi)$ is continuous, convex in ξ and superlinear then every minimum of (P) is Lipschitz continuous. Some recent results show that the

C. Mariconda · G. Treu (✉)
Dipartimento di Matematica pura e applicata, Università di Padova,
63 via Trieste, 35121 Padova, Italy
e-mail: treu@math.unipd.it

C. Mariconda
e-mail: maricond@math.unipd.it

same result holds true by weakening some assumption: continuity is not assumed in [1], a weaker growth assumption and no convexity in [2], no continuity and no convexity in [3]. Ambrosio, Ascenzi, Buttazzo in [1] and Dal Maso, Frankowska in [3] do strongly use some inequalities involving the superlinearity condition. Instead, Cellina in [2] introduces a new growth assumption (GA) and the result is obtained through a relaxation result and a reparametrization technique. Following [2], a function $L(x,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is said to satisfy the GA if the intersection of the supporting hyperplanes of $L^{**}(x,\xi)$ (the convex l.s.c. envelope of $L(x,\cdot)$) with the axis $\xi = 0$ in $\mathbb{R}^{N+1}$ tends to $-\infty$ as $|\xi|$ goes to infinity. This condition is fulfilled when L is superlinear and there are functions, like $|t| - \sqrt{|t|}$, that satisfy (GA) and are not superlinear.

Our starting point is a key inequality in the proof of [3] which involves the behavior of the sections $(x,\lambda) \mapsto L(x,\lambda u)$, $(\lambda \geq 0, |u| = 1)$. This leads us to introduce the condition (GA_σ) that is fulfilled if the sections of the function L satisfy condition (GA), with some uniformity with respect to x and u. We show then in Sect. 3 that, if L is a Borelian function that satisfies (GA_σ), every minimizer of (P) is Lipschitz. In Sect. 4 we study the properties of the functions that satisfy (GA) or (GA_σ). In particular we give a geometrical characterization of assumption (GA). Opposite to the superlinear case, where superlinear functions have superlinear sections, the sections of a function that satisfies the GA do not necessarily satisfy (GA), as it is shown in Example 4.1. In Sect. 5 we introduce a global growth assumption (G) involving the bipolar of L (instead of the bipolar of the sections, as in (GA_σ)) and L itself which is stronger than (GA); most of the section is devoted to the proof that (G) is stronger than (GA_σ); yielding a new sufficient condition for the Lipschitz regularity of the minima of (P). Finally, in the last section, we show that in the one dimensional case, the assumptions (GA), (GA_σ) and (G) are all equivalent.

We just mention that, since assumption (GA) does not imply (GA_σ), the regularity result of [2] for continuous lagrangians does not follow directly from our result; we will extend it in a forthcoming paper concerning a relaxation result, as a consequence of Corollary 3.13.

2 Notation, main assumptions and preliminary results

In what follows $|\cdot|$ is the euclidean norm and "$\cdot$" the scalar product in $\mathbb{R}^N$, $B(0,R]$ is the ball of radius R. Let $f : \mathbb{R}^N \to \mathbb{R}$. We denote by f^{**} the bipolar of f, by epi$f$ the epigraph of f. The derivative of f with respect to η is $df(\xi;\eta)$ and its gradient is ∇f; if $N = 1$ $d_\ell f(\xi)$ and $d_r f(\xi)$ are, respectively, the left and right derivatives of f in ξ; in this case the derivative of f is f'. If u belongs to the unitary sphere ∂B_N of $\mathbb{R}^N$, $f_u : \mathbb{R}^+ \to \mathbb{R}$ is the $u-$section of f defined by $f_u(\lambda) = f(\lambda u)$. If f is convex, ∂f is the subdifferential of f. For a function $L(x,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ we denote by $L^{**}(x,\xi)$ the bipolar of $\xi \mapsto L(x,\xi)$ and by $dL^{**}((x,\xi);\eta)$ (resp. $\partial_\xi L(x,\xi)$) the directional derivative with respect to η (resp. the subdifferential at ξ) of $\xi \mapsto L^{**}(x,\xi)$. Moreover if u is in ∂B_N, the u-section of $L(x,\cdot)$ is denoted by $L_u(x,\cdot)$. When the variable ξ is in $\mathbb{R}$, $d_\ell L^{**}(x,\xi)$ and $d_r L^{**}(x,\xi)$ are respectively the left and right derivatives of $\xi \mapsto L(x,\xi)$ w.r.t. ξ. If $L(x,\xi)$ is differentiable, $\nabla_\xi L(x,\xi)$ is the gradient of L w.r.t. the variable ξ. Finally, $AC([a,b],\mathbb{R}^N)$ is the space of absolutely continuous functions on $[a,b]$ with values in $\mathbb{R}^N$.

In this paper $L : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is just a Borelian function. We assume, moreover, that the bipolar of every section L_u is never equal to $-\infty$. Notice that $(L_u)^{**}$, the bipolar of $L_u(x,\cdot)$, may be different from the u-section $(L^{**})_u$ of the bipolar of L; in general, since $L^{**} \le L$, we have $(L^{**})_u \le (L_u)^{**}$.

The following result will be widely used in the paper.

Monotonicity Lemma 2.1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex, $u \in \partial B_N, p(\xi)$ in $\partial f(\xi)$. Then:*

(i) *for every $t, p(tu)\cdot u \in \partial f_u(t)$;*
(ii) *the map $t \mapsto p(tu)\cdot tu - f(tu)$ is increasing.*

Proof For $t, t_0 \in \mathbb{R}$ we have

$$f_u(t) - f_u(t_0) = f(tu) - f(t_0u) \ge p(t_0u)\cdot(tu - t_0u) = p(t_0u)\cdot u(t-t_0),$$

proving (i). Let $s < t$; we have

$$f(su) - f(tu) \ge p(tu)\cdot(s-t)u = p(tu)\cdot su - p(tu)\cdot tu.$$

By (i) and the monotonicity of the subdifferential of convex functions it follows that $p(tu)\cdot u \ge p(su)\cdot u$ so that

$$f(su) - f(tu) \ge p(tu)\cdot(s-t)u = p(su)\cdot su - p(tu)\cdot tu$$

i.e. $p(su)\cdot su - f(su) \le p(tu)\cdot tu - f(tu)$. □

3 Lipschitzianity of the minima

We introduce here the condition that will imply the lipschitzianity of the minima of (P).

Condition (H). We say that L satisfies (H) if for every compact subset C of $\mathbb{R}^N$ and $M \in \mathbb{R}$ there exists $R > 0$ such that, for every $\lambda \ge R, u \in \partial B_N$ and $x \in C$, one of the following conditions holds:

(a) $(L_u)^{**}(x,\lambda) < L(x,\lambda u)$
or
(b) $d_\ell (L_u)^{**}(x,\lambda)\lambda - (L_u)^{**}(x,\lambda) \ge M$.

Remark 3.1 The interpretation of (a) is straightforward; the explanation of (b) is postponed after the definition of the growth assumption (GA_σ).

Theorem 3.2 *Assume that L satisfies (H). Every minimizer of (P) in $AC([a,b],\mathbb{R}^N)$ is Lipschitz.*

Proof Let y be a minimizer of (P). It is shown in [3, formula (3.3)] that, if we set

$$g(t,v) = \begin{cases} L(y(t), vy'(t)) & \text{if } v \in]0,2[\\ +\infty & \text{elsewhere} \end{cases}$$

then $g^{**}(t,1) = L(y(t), y'(t))$ and, without assuming any growth assumption, there exists c in $\mathbb{R}$ such that

$$d_\ell g^{**}(t,1) \le L(y(t),y'(t)) + c, \quad g^{**}(t,1) = g(t,1) \text{ for a.e. } t.$$

The proof of these results is based on the fact that $\varphi(t) = t$ minimizes the functionals

$$I(\varphi) = \int_a^b g(t, \varphi'(t))\,\mathrm{d}t, \quad I^{**}(\varphi) = \int_a^b g^{**}(t, \varphi'(t))\,\mathrm{d}t$$

among the absolutely continuous functions satisfying $\varphi(a) = a$ and $\varphi(b) = b$. We point out that the same result holds true if g is replaced by the function h defined by

$$h(t, v) = \begin{cases} L(y(t), vy'(t)) & \text{if } v > 0 \\ +\infty & \text{elsewhere} \end{cases}$$

In fact, by [4, Corollary 20.5 and Theorem 18.25], the change of variable formula in the key Lemma 2.3 of [3] does still hold for an absolutely continuous function ψ whose derivative is strictly positive a.e. Therefore

$$d_\ell h^{**}(t, 1) \le L(y(t), y'(t)) + c, \quad h^{**}(t, 1) = h(t, 1) \text{ for a.e. } t. \tag{3.1}$$

Set, for every t such that $y'(t) \neq 0$,

$$u(t) = \frac{y'(t)}{|y'(t)|}.$$

With our notations we have

$$h(t, v) = L_{u(t)}(y(t), v|y'(t)|)$$

and therefore

$$h^{**}(t, v) = (L_{u(t)})^{**}(y(t), v|y'(t)|) \quad d_\ell h^{**}(t, v) = d_\ell (L_{u(t)})^{**}(y(t), v|y'(t)|)|y'(t)|.$$

By (3.1) we obtain that

$$(L_{u(t)})^{**}(y(t), |y'(t)|) = L_{u(t)}(y(t), y'(t)) \text{ a.e.} \tag{3.2}$$

and moreover, for a.e. t such that $y'(t) \neq 0$,

$$d_\ell (L_{u(t)})^{**}(y(t), v|y'(t)|)|y'(t)| - (L_{u(t)})^{**}(y(t), |y'(t)|) \le c \tag{3.3}$$

Let $C = \{y(t) : t \in [a, b]\}$. By (H) there exists R such that, for $x \in C, u \in \partial B_N$ and $\lambda \ge R$, either $(L_u)^{**}(x, \lambda) < L(x, \lambda u)$ or $d_\ell (L_u)^{**}(x, \lambda)\lambda - (L_u)^{**}(x, \lambda) > c$. Therefore, if $|y'(t)| > R$, conditions (3.2) and (3.3) are not fulfilled at t. It follows that $|y'(t)| \le R$ a.e. □

The next growth assumption will be strictly related to case (b) of assumption (H).

Growth assumption on the sections (GA_σ). We say that L satisfies (GA_σ) if, for every $u \in \partial B_N$, there exists $q_u(x, \lambda)$ in $\partial (L_u)^{**}(x, \lambda)$ such that

$$\lim_{\lambda \to +\infty} q_u(x, \lambda)\lambda - (L_u)^{**}(x, \lambda) = +\infty \tag{3.4}$$

uniformly with respect to x in compact sets and to u in ∂B_N.

Remark 3.3 Fix u in ∂B_N and x in $\mathbb{R}^N$. The set

$$\{(\tau, z) \in [0, +\infty[\times \mathbb{R} : z = (L_u)^{**}(x, \lambda) + q_u(x, \lambda)(\tau - \lambda)\}$$

is a tangent plane to the epigraph of the map $(L_u)^{**}(x,\cdot)$ in λ; the ordinate of its intersection with the axis $\tau = 0$ is $z(x,\lambda) = (L_u)^{**}(x,\lambda) - q_u(x,\lambda)\lambda$. Therefore condition (GA_σ) is equivalent to $\lim\limits_{\lambda\to+\infty} z(x,\lambda) = -\infty$ uniformly with respect to x in compact sets and to u in ∂B_N.

Proposition 3.4 *Assumption* (GA_σ) *is equivalent to each of the following conditions:*

(i) *For every $q_u(x,\lambda)$ in $\partial(L_u)^{**}(x,\lambda)$ $(u \in \partial B_N)$,*

$$\lim_{\lambda\to+\infty} q_u(x,\lambda)\lambda - (L_u)^{**}(x,\lambda) = +\infty$$

uniformly with respect to x in compact sets and to u in ∂B_N.

(ii) $\lim\limits_{\lambda\to+\infty} d_\ell(L_u)^{**}(x,\lambda)\lambda - (L_u)^{**}(x,\lambda) = +\infty$ *uniformly w.r. to x in compact sets and to u in ∂B_N.*

In particular, in the case where $(L_u)^{**}$ is of class $\mathcal{C}^1$, condition (GA_σ) *is equivalent to* $\lim\limits_{\lambda\to+\infty} \partial_\lambda(L_u)^{**}(x,\lambda)\lambda - (L_u)^{**}(x,\lambda) = +\infty$ *uniformly w.r. to x in compact sets and to u in ∂B_N.*

Proof (i) Assume that (3.4) holds for some $q_u(x,\lambda)$ in $\partial(L_u)^{**}(x,\lambda)$ and let $a_u(x,\lambda) \in \partial(L_u)^{**}(x,\lambda)$. Let C be a compact subset of $\mathbb{R}^N$, $M \in \mathbb{R}$ and let R be such that

$$q_u(x,R)R - (L_u)^{**}(x,R) \geq M$$

for every $x \in C, u \in \partial B_N$. The monotonicity Lemma 2.1 ensures that for every $\lambda > R$

$$a_u(x,\lambda)\lambda - (L_u)^{**}(x,\lambda) \geq M.$$

To prove (ii) it is enough to remark that there exists $q_u(x,\lambda) \in \partial(L_u)^{**}(x,\lambda)$ such that $d_\ell(L_u)^{**}(x,\lambda) = q_u(x,\lambda)$; (i) yields the conclusion. □

Remark 3.5 It turns out from the previous proposition that, in the case where the u-sections of L are convex, condition (H) is equivalent to (GA_σ).

Condition (GA_σ) generalizes the classical superlinearity condition.

Proposition 3.6 *Assume that for every compact subset C of $\mathbb{R}^N$, there exists $r > 0$ such that L is bounded on $C \times B(0,r]$. Assume moreover that L is superlinear, i.e. that*

$$L(x,\xi) \geq \Theta(|\xi|), \qquad \lim_{r\to+\infty} \Theta(r) = +\infty.$$

Then L satisfies (GA_σ).

Proof If L does not satisfy (GA_σ) there exists a compact subset C of $\mathbb{R}^N$, a sequence (x_n) in C, a sequence (λ_n) in $\mathbb{R}$, with $\lambda_n \to +\infty$ and a sequence (u_n) in $B(0,1]$ such that

$$q_{u_n}(x_n,\lambda_n)\lambda_n - (L_{u_n})^{**}(x_n,\lambda_n) \leq h$$

for some $q_u(x,\lambda) \in \partial(L_u)^{**}(x,\lambda)$ and h in $\mathbb{R}$. Therefore, if k and r are such that $L \leq k$ on $C \times B(0,r]$, we have

$$\begin{aligned} k \geq (L_{u_n})^{**}(x_n,r) &\geq (L_{u_n})^{**}(x_n,\lambda_n) + q_{u_n}(x_n,\lambda_n)(r-\lambda_n) \\ &\geq -h + r q_{u_n}(x_n,\lambda_n) \end{aligned} \tag{3.5}$$

so that $q_{u_n}(x_n, \lambda_n) \le (h+k)/r$. From (3.5) we deduce that, for $\lambda_n \ge r$,

$$(L_{u_n})^{**}(x_n, \lambda_n) \le k + \frac{h+k}{r}(\lambda_n - r),$$

contradicting the superlinear growth of L. □

The result obtained recently by Dal Maso and Frankowska in [3, Theorem 2.1] follows then directly from Theorem 3.2 and Proposition 3.6.

Corollary 3.7 *Assume that for every compact subset C of $\mathbb{R}^N$, there exists $r > 0$ such that L is bounded on $C \times B(0,r]$. Assume moreover that L is superlinear, i.e. that*

$$L(x,\xi) \ge \Theta(|\xi|), \qquad \lim_{r\to+\infty} \Theta(r) = +\infty.$$

Then every minimizer of (P) *is Lipschitz.*

Notice that, if $L(x,\xi)$ is convex with respect to ξ, then every bipolar $q_u(x,\lambda)$ of L_u at $(x, \lambda u)$ is of the form

$$q_u(x,\lambda) = p(x,\lambda u)\cdot u,$$

where $p(x,\xi) \in \partial L(x,\xi)$. In this case condition (H) turns then out to be equivalent to the following growth condition.

Growth assumption (GA). We say that L satisfies (GA) if there exists $p(x,\xi)$ in $\partial L^{**}(x,\xi)$ such that

$$\lim_{|\xi|\to+\infty} p(x,\xi)\cdot\xi - L^{**}(x,\xi) = +\infty \tag{3.6}$$

uniformly w.r. to x in a compact set.

Remark 3.8 In what follows we also will be concerned with functions of a real variable ξ that take the value $+\infty$ in $]-\infty,0]$; in this case the above limit is intended as ξ goes to $+\infty$. Therefore a function L satisfies (GA$_\sigma$) if every section L_u satisfies (GA) uniformly w.r. to x in compact sets and to u in ∂B_N.

Remark 3.9 The set

$$\{(\eta,z)\in\mathbb{R}^N\times\mathbb{R} : z = L^{**}(x,\xi) + p(x,\xi)\cdot(\eta-\xi)\}$$

is a tangent plane to the epigraph of the map $L^{**}(x,\cdot)$ in ξ; the ordinate of its intersection with the axis $\eta = 0$ is $z(x,\xi) = L^{**}(x,\xi) - p(x,\xi)\cdot\xi$. Therefore condition (GA) is equivalent to $\lim_{|\xi|\to+\infty} z(x,\xi) = -\infty$ uniformly with respect to x in compact sets.

Proposition 3.10 *Assumption* (GA) *is equivalent to each of the following conditions:*

(i) *For every $q(x,\xi)$ in $\partial L^{**}(x,\xi)$,*

$$\lim_{|\xi|\to+\infty} q(x,\xi)\cdot\xi - L^{**}(x,\xi) = +\infty$$

uniformly w.r. to x in compact sets.

(ii) $\lim_{|\xi|\to+\infty} dL^{**}((x,\xi);\xi) - L^{**}(x,\xi) = +\infty$ *uniformly w.r. to x in a compact set.*

In particular, if L^{**} is of class $\mathcal{C}^1$, condition (GA) *is equivalent to*

$$\lim_{|\xi|\to+\infty} \nabla_\xi L^{**}(x,\xi)\cdot\xi - L^{**}(x,\xi) = +\infty$$

uniformly w.r. to x in compact sets.

Proof (i) Assume that (GA) holds for some $p(x,\xi) \in \partial L^{**}(x,\xi)$ and let $q(x,\xi) \in \partial L^{**}(x,\xi)$. Let C be a compact subset of $\mathbb{R}^N$, $M \in \mathbb{R}$ and let R be such that

$$p(x,\xi)\cdot\xi - L^{**}(x,\xi) \geq M$$

for every $x \in C$ and $|\xi| \geq R$. The monotonicity Lemma 2.1 ensures that for every $t > 1, x \in C$ and $|\xi| \geq R$,

$$q(x,t\xi)\cdot t\xi - L^{**}(x,t\xi) \geq p(x,\xi)\cdot\xi - L^{**}(x,\xi) \geq M$$

and therefore $q(x,\eta)\cdot\eta - L^{**}(x,\eta) \geq M$ for every $x \in C$ and $|\eta| > R$. (ii) It is enough to remark that there exists $p(x,\xi) \in \partial L^{**}(x,\xi)$ such that $dL^{**}((x,\xi);\xi) = p(x,\xi)\cdot\xi$; (i) yields the conclusion. □

Remark 3.11 By Proposition 4 in [2] every convex function $f : \mathbb{R}^N \to \mathbb{R}$ that is superlinear, i.e. $\lim_{|\xi|\to+\infty} f(\xi)/\xi = +\infty$ fulfills (GA). An example of a function that satisfies (GA) without being superlinear is $|t| - \sqrt{|t|}$.

Remark 3.12 The GA was first presented in [2] as a generalization of the superlinearity condition where it is proven that the conclusion of Theorem 3.2 does hold by assuming the continuity of L in both variables and (GA) instead of (H). Since, as we show in the next section, neither (H) implies (GA) nor (GA) implies (H), the result of Cellina does not follow directly from our Theorem 3.2. We will generalize it in a forthcoming paper concerning a relaxation result; the result will follow from the next direct consequence of Theorem 3.2.

Corollary 3.13 *Assume that $L(x,\xi)$ is convex in ξ and satisfies* (GA). *Then every minimizer of* (P) *in $AC([a,b],\mathbb{R}^N)$ is Lipschitz.*

4 The growth assumptions (GA) and (GA$_\sigma$)

Opposite to the superlinear case a function that satisfies the GA does not, in general, satisfy the same growth assumption on its sections. The following examples show that, in general, neither (GA) implies (GA$_\sigma$) nor does (GA$_\sigma$) imply (GA).

Example 4.1 We show here that, in general, (GA) does not imply (GA$_\sigma$). Let

$$L(\xi_1,\xi_2) = \begin{cases} 0 & \text{if } |\xi| \leq 1 \\ \sqrt{1+\xi_1^2} & \text{if } |\xi| > 1,\ \xi_2 = 0 \\ |\xi| - \sqrt{|\xi|} & \text{if } |\xi| > 1,\ \xi_2 \neq 0 \end{cases}$$

Then $L^{**}(\xi) = \max\{|\xi| - \sqrt{|\xi|}, 0\}$ so that, for $|\xi| > 1, \nabla L^{**}(\xi)\cdot\xi - L^{**}(\xi) = \sqrt{|\xi|}/2$ tends to $+\infty$ as $|\xi|$ tends to $+\infty$; therefore L fulfills (GA). However the sections of L are convex and, for $u = (1,0)$ and $\lambda > 1$, the function

$$L_u'(\lambda)\lambda - L_u(\lambda) = -\frac{1}{\sqrt{\lambda^2+1}} + 1$$

is bounded, so that L does not satisfy (GA$_\sigma$). We point out that with a slight modification of the definition of L one can obtain even a continuous function satisfying (GA) but not (G).

Example 4.2 In general condition (GA$_\sigma$) does not imply (GA). For instance the function $L(\xi) = -\sqrt{|\xi|}$ fulfills (GA$_\sigma$) but its bipolar is identically equal to $-\infty$.

In some of the next results we assume that $L(x, \xi)$ is bounded on the compact subsets of its domain; this obviously implies that any subdifferential of $L^{**}(x, \xi)$ is bounded on compact sets too. We give here a geometrical characterization of condition (GA).

Theorem 4.3 *Assume that L is bounded on compact sets. Then L satisfies* (GA) *if and only if for every compact subset C of* $\mathbb{R}^N, p(x, \xi)$ *in* $\partial L^{**}(x, \xi)$ *and* $R_0 > 0$ *there exists* $R > 0$ *such that, setting* $\varphi_{\xi_0}(t, \xi) = L^{**}(x, t\frac{\xi}{|\xi|}) - L^{**}(x, \xi_0) - p(x, \xi_0) \cdot (t\frac{\xi}{|\xi|} - \xi_0) \geq 0$, *we have*

$$L^{**}(x, \xi) \geq L^{**}(x, \xi_0) + p(x, \xi_0) \cdot (x - \xi_0) + \frac{M}{R}(|\xi| - R) + \frac{\varphi_{\xi_0}(R, \xi)}{R}|\xi| \qquad (4.7)$$

for every $|\xi_0| \leq R_0, x$ *in C and* $|\xi| \geq R$.

Proof We assume that $L^{**}(x, \xi) = f(\xi)$, the general case is not much different; it is not restrictive to assume that f is convex. Assume that (GA) holds; let $R_0 > 0, x \in C$, $|\xi_0| \leq R_0$ and $p(\xi) \in \partial f(\xi)$. Let $u \in \partial B_N$. Then $q(t) = p(tu) \cdot u - p(\xi_0) \cdot u$ belongs to the subdifferential of the map $t \mapsto \varphi_{\xi_0}(t, u)$ and

$$\begin{aligned} tq(t) - \varphi_{\xi_0}(t, u) &= p(tu) \cdot tu - f(tu) + f(\xi_0) - p(\xi_0) \cdot \xi_0 \\ &\geq p(tu) \cdot tu - f(tu) + c \end{aligned}$$

where $c = \sup\{f(\xi_0) - p(\xi_0) \cdot \xi_0 : |\xi_0| \leq R_0\}$ is finite by the boundedness condition. It follows from (GA) that $\lim\limits_{t \to +\infty} tq(t) - \varphi_{\xi_0}(t, u) = +\infty$ uniformly with respect to $u \in \partial B_N$ and $|\xi_0| \leq R_0$. Therefore if we fix $M \in \mathbb{R}$ there exists $R > 0$ such that $tq(t) - \varphi_{\xi_0}(t, u) \geq M$ for every $t \geq R, u \in \partial B_N, |\xi_0| \leq R_0$; in particular, for every $u \in \partial B_N, q(R) \geq (M + \varphi_{\xi_0}(R, u))/R$, so that

$$\varphi_{\xi_0}(t, u) - \varphi_{\xi_0}(R, u) \geq \frac{M + \varphi_{\xi_0}(R, u)}{R}(t - R) \qquad (4.8)$$

for every $t > R$. If we set $t = |\xi|$ and $u = \frac{\xi}{|\xi|}$, (4.8) is equivalent to

$$\begin{aligned} f(\xi) &\geq f(\xi_0) + p(\xi_0) \cdot (x - \xi_0) + \frac{M}{R}(|\xi| - R) + \varphi_{\xi_0}(R, u) + \frac{\varphi_{\xi_0}(R, u)}{R}(|\xi| - R) \\ &= f(\xi_0) + p(\xi_0) \cdot (x - \xi_0) + \frac{M}{R}(|\xi| - R) + \frac{\varphi_{\xi_0}(R, u)}{R}|\xi|, \end{aligned}$$

proving (4.7). Conversely assume that (4.7) holds for $\xi_0 = 0$, then the last part of the previous proof shows that

$$\varphi_0(\tau, \xi) - \varphi_0(R, \xi) \geq \frac{M + \varphi_0(R, \xi)}{R}(\tau - R)$$

for every $\tau > R$. It follows that $d_r\varphi_0(R, \xi)$, the right derivative of the map $t \mapsto \varphi_0(t, \xi)$ at $t = R$, is greater than $\frac{M+\varphi_0(R,\xi)}{R}$. Let $q(t)$ be a subdifferential of the map $t \mapsto \varphi_0(t, \xi)$ and $\tau > R$; by the monotonicity Lemma 2.1 for every $t > \tau$ we have

$$tq(t) - \varphi_0(t, \xi) \geq \tau q(\tau) - \varphi_0(\tau, \xi) \geq \tau \frac{M + \varphi_0(R, \xi)}{R} - \varphi_0(\tau)$$

so that, letting τ tend to R, we obtain that

$$tq(t) - \varphi_0(t,\xi) \geq R\frac{M + \varphi_0(R,\xi)}{R} - \varphi_0(R,\xi) = M$$

for every $t > R$. Again by Lemma 2.1 we may choose $q(t) = \Big(p(t\frac{\xi}{|\xi|}) - p(0)\Big) \cdot \frac{\xi}{|\xi|}$ so that, if in the latter inequality we set $t = |\xi|$, we obtain

$$p(\xi) \cdot \xi - f(\xi) \geq M + f(0)$$

for every $|\xi| \geq R$, proving the validity of (GA). □

The following condition will be exploited later.

Conical growth assumption (CGA). For every compact subset C of $\mathbb{R}^N$ and $R_0 \geq 0$ there exist $\varepsilon > 0, R > 0$ and $c \in \mathbb{R}$ such that

$$\forall \xi \in \mathbb{R}^N \quad |\xi| \geq R, \quad L^{**}(x,\xi) \geq L^{**}(x,\xi_0) + p(x,\xi_0) \cdot (\xi - \xi_0) + \varepsilon|\xi| + c$$

for every $x \in C, |\xi_0| \leq R_0$ and $p(x,\xi_0)$ in $\partial L^{**}(x,\xi_0)$.

The following result is a direct consequence of Theorem 4.3; it actually appeared first in the proof of Theorem 2 of [2].

Corollary 4.4 *Assume that L is bounded on compact sets and satisfies the* GA. *Then L satisfies* (CGA).

Proof It is enough to remark that the function φ_{ξ_0} defined in Theorem 4.3 is positive; the inequality (4.7) yields the conclusion. □

Example 4.5 There are functions satisfying (CGA) but not (GA). Let, for instance

$$L(\xi) = \frac{3}{4} \quad \text{if } |\xi| \leq 1; \quad L(\xi) = -\frac{3}{4}\left(1 - \frac{1}{2^{n-1}}\right) + \left(1 - \frac{1}{2^{2n}}\right)|\xi| \quad \text{if } 2^{n-1} \leq |\xi| < 2^n.$$

Then L is continuous, convex, piecewise affine. Moreover L satisfies (CGA): in fact it is enough to remark that if $\xi_0 \in]2^{m-1}, 2^m[, \xi \in]2^{n-1}, 2^n[$ then $p(\xi_0) = (1 - \frac{1}{2^{2m}})$ and

$$L(\xi) - [L(\xi_0) + p(x_0)(\xi - \xi_0)] = \left(\frac{1}{2^{2m}} - \frac{1}{2^{2n}}\right) + \frac{1}{2^{n+1}} - \frac{1}{2^m}.$$

However L does not satisfy (GA) since, if $\xi_n = 2^{n-1}$ and $p(\xi_n) = \Big(1 - \frac{1}{2^{2n}}\Big)$, then

$$\lim_{n\to+\infty} p(\xi_n)\xi_n - L(\xi_n) = \lim_{n\to+\infty} \frac{3}{4}\left(1 - \frac{1}{2^{n-1}}\right) = \frac{3}{4}.$$

A powerful consequence of (CGA) was established in [2] in the continuous case; we state it here in a more general setting. For every (x,ξ) we set

$$\overline{L}(x,\xi) = \liminf_{\eta\to\xi} L(x,\eta),$$

i.e. $\overline{L}(x,\xi)$ denotes the lower semicontinuous envelope of the map $\eta \mapsto L(x,\eta)$. The proof of the following result is based on the fact that if $f : \mathbb{R}^N \to \mathbb{R}$ is convex and satisfies (CGA) then the intersection of its epigraph with any supporting hyperplane is bounded. This condition is referred in [2] as the *Bounded Intersection Property*.

Theorem 4.6 *Assume that L satisfies* (CGA) *and let $p(x,\xi)$ in $\partial L^{**}(x,\xi)$. Then given $R_0 > 0$ and a compact subset C of R^N there exists $R > 0$ (depending only on R_0 and C) such that for every $x \in C$; for every ξ, with $|\xi| \le R_0$, there exists at most $\nu \le N+1$ points ξ_i, with $|\xi_i| \le R$, and coefficients of a convex combination α_i, such that*

$$\begin{pmatrix} \xi \\ L^{**}(x,\xi) \end{pmatrix} = \sum_{i=1}^{\nu} \alpha_i \begin{pmatrix} \xi_i \\ \overline{L}(x,\xi_i) \end{pmatrix}$$

and $L^{**}(x,\xi) = \overline{L}(x,\xi_i) = L^{**}(x,\xi) + p(x,\xi)\cdot(\xi - \xi_i)$.

Proof It is enough to remark that Theorem 1 in [2] holds for functions that are lower semicontinuous instead of continuous and that the bipolar of a function coincides with the bipolar of its lower semicontinuous envelope. □

Conditions (GA) and (GA_σ) are equivalent if $L(x,\xi)$ is convex in ξ.

Proposition 4.7 *Assume that $L(x,\cdot)$ is convex for every x in $\mathbb{R}^N$. Then L satisfies* (GA) *if and only if L satisfies* (GA_σ).

Proof If $L(x,\cdot)$ is convex then $(L_u)^{**}(x,\lambda) = L_u(x,\lambda) = L(x,\lambda u)$; moreover if $p(x,\xi) \in \partial L(x,\xi)$ then, by Lemma 2.1, $q_u(x,\lambda) = p(x,\lambda u)\cdot u \in \partial L_u(x,\lambda)$; the equivalence of the two conditions follows immediately. □

In Example 4.5 we showed that condition (CGA) does not imply (GA). We end this section by showing that (CGA) together with (GA_σ) imply (GA).

Proposition 4.8 *Assume that L satisfies* (CGA) *and that the function $\xi \mapsto L(x,\xi)$ is lower semicontinuous and $L_u(x,\cdot)$ is convex for every x in $\mathbb{R}^N$. If L satisfies* (GA_σ) *then L satisfies* (GA).

We postpone the proof of Proposition 4.8 after the following Lemma.

Lemma 4.9 *Assume that $L_u(x,\cdot)$ is convex for every x and u in ∂B_N. Let ξ and ξ_0 in $\mathbb{R}^N$ be such that, for some $p(x,\xi)$ in $\partial L^{**}(x,\xi)$,*

$$L(x,\xi_0) = L^{**}(x,\xi) + p(x,\xi)\cdot(\xi_0 - \xi). \tag{4.9}$$

Then $p(x,\xi)\cdot u \ge d_\ell L_u(x,|\xi_0|)$, where $u = \xi_0/|\xi_0|$.

Proof Set $\lambda_0 = |\xi_0|$. For every $x \in \mathbb{R}^N$ and $\lambda \ge 0$ we have

$$L(x,\lambda u) \ge L^{**}(x,\lambda u) \ge L^{**}(x,\xi) + p(x,\xi)\cdot(\lambda u - \xi).$$

It follows from (4.9) that, for every $\lambda \ge 0$,

$$L(x,\lambda u) - L(x,\lambda_0 u) \ge p(x,\xi)\cdot u(\lambda - \lambda_0)$$

and thus $p(x,\xi)\cdot u \in \partial L_u(x,\lambda_0)$; in particular $p(x,\xi)\cdot u$ is greater than $d_\ell L_u(x,|\xi_0|)$, the minimum of $\partial L_u(x,|\xi_0|)$; proving the claim. □

Proof of Proposition 4.8 For every $u \in \partial B_N$ we denote by $q_u(x,\lambda)$ the left derivative of $L_u(x,\cdot)$ at λ. Fix M in $\mathbb{R}$ and C be a compact subset in $\mathbb{R}^N$; from (GA_σ) and the convexity assumption it follows that there exists $R > 0$ such that, for every u and $x \in C$,

$$q_u(x,\lambda)\lambda - L(x,\lambda u) \ge M \quad \text{for every } \lambda \ge R.$$

Fix ξ with $|\xi| \geq R$. By Theorem 4.6 we have $\xi = \sum_{i=1}^{\nu} \alpha_i \xi_i$ for some $\xi_1, \dots, \xi_\nu$ in $\mathbb{R}^N$ and coefficients of a convex combination α_i satisfying $L(x, \xi_i) = L^{**}(x, \xi) + p(x, \xi) \cdot (\xi_i - \xi)$. Let i be such that $|\xi_i| \geq |\xi|$. The latter equality yields

$$p(x, \xi) \cdot \xi - L^{**}(x, \xi) = p(x, \xi) \cdot \xi_i - L(x, \xi_i);$$

moreover by Lemma 4.9, with ξ_i instead of ξ_0, we have

$$p(x, \xi) \cdot \xi_i \geq q_u(x, |\xi_i|)|\xi_i|$$

where $u = \xi_i / |\xi_i|$; it follows that

$$p(x, \xi) \cdot \xi - L^{**}(x, \xi) \geq q_u(x, |\xi_i|)|\xi_i| - L_u(x, |\xi_i|) \geq M,$$

proving the claim $\square$

5 A global growth condition

We introduce here a new global condition that turns out to be stronger than both (GA$_\sigma$) and (GA).

Example 5.1 It is pointed out in [2] that the function $L(\xi) = (2 + \sin \xi)\xi^2$ satisfies (GA) but, if $p(\xi) \in \partial L^{**}(\xi)$, the limit $\lim\limits_{|\xi| \to +\infty} p(\xi)\xi - L(\xi)$ does not exist. In fact, setting $t_k = -\frac{\pi}{2} + 2k\pi$ $(k \in \mathbb{Z})$ we have

$$\partial L^{**}(\xi) = \begin{cases} 2t_k & \text{if } \xi \in]t_k, t_{k+1}[\\ [2t_{k-1}, 2t_k] & \text{if } \xi = t_k \end{cases}$$

so that $p(\xi) \sim 2\xi$ as $|\xi|$ tends to $+\infty$. Then

$$p(\pi/2 + 2k\pi)(\pi/2 + 2k\pi) - L(\pi/2 + 2k\pi) \sim -(\pi/2 + 2k\pi)^2$$

tends to $-\infty$ as $k \to +\infty$. However,

$$p(2k\pi)(2k\pi) - L(2k\pi) = 2t_k(2k\pi) \sim 2(2k\pi)^2 - (2k\pi)^2$$

tends to $+\infty$ as $k \to +\infty$. It follows that

$$\sup_{t>0}\{p(t)t - L(t)\} = +\infty$$

and, analogously,

$$\sup_{t<0}\{p(t)t - L(t)\} = +\infty.$$

These last remarks lead us to introduce the next growth condition (G).

Growth condition (G). We say that L satisfies (G) if , for some $p(x, \xi)$ in $\partial L^{**}(x, \xi)$,

$$\lim_{R \to +\infty} \sup\{p(x, \lambda u) \cdot \lambda u - L(x, \lambda u) : \lambda \leq R\} = +\infty$$

uniformly w.r. to x in compact sets and to u in ∂B_N, i.e. if for every compact subset C of $\mathbb{R}^N$ and $M \in \mathbb{R}$ there exists $R > 0$ such that, for every $u \in \partial B_N$ and $x \in C$,

$$p(x, \lambda_u u) \cdot \lambda_u u - L(x, \lambda_u u) \geq M \tag{5.10}$$

for some $\lambda_u \leq R$.

Remark 5.2 In the one dimensional case L satisfies (G) if and only if

$$\sup_{t>0}\{p(x,t)t - L(x,t)\} = \sup_{t<0}\{p(x,t)t - L(x,t)\} = +\infty$$

uniformly w.r. to x in compact sets. In particular the function defined in Example 5.1 satisfies (G).

Remark 5.3 Since $dL^{**}((x,\xi);\xi) = \max_{p\in\partial L^{**}(x,\xi)} p(x,\xi)\cdot\xi$, then condition (G) can be restated with $dL^{**}((x,\lambda_u u));u)$ instead of $p(x,\lambda_u u)\cdot u$.

The following result shows that condition (G) is stronger than (GA).

Proposition 5.4 *If L satisfies (G) then L fulfills* (GA). *The converse holds true if $\xi \mapsto L(x,\xi)$ is convex for every x.*

Proof Let $p(x,\xi)\in\partial L^{**}(x,\xi)$ and C be a compact subset in $\mathbb{R}^N$. Fix $M\in\mathbb{R}$ and let R,λ_u be such that (5.10) holds. Let $u\in\partial B_N$. By Lemma 2.1, for every $t\geq R$, we have

$$p(x,tu)\cdot tu - L^{**}(x,tu) \geq p(x,\lambda_u u)\cdot\lambda_u u - L^{**}(x,\lambda_u u) \geq p(x,\lambda_u u)\cdot\lambda_u u - L(x,\lambda_u u)\geq M$$

and therefore

$$p(x,\xi)\cdot\xi - L^{**}(x,\xi)\geq M$$

for every $|\xi|\geq R$. The last part of the claim is immediate. □

Example 5.5 We show here that, in general, (GA) does not imply (G). Let

$$L(\xi_1,\xi_2) = \begin{cases} 0 & \text{if } |\xi|\leq 1 \\ \sqrt{1+\xi_1^2} & \text{if } |\xi|>1,\ \xi_2=0 \\ |\xi|-\sqrt{|\xi|} & \text{if } |\xi|>1,\ \xi_2\neq 0 \end{cases}$$

Then $L^{**}(\xi)=\max\{|\xi|-\sqrt{|\xi|},0\}$ so that, for $|\xi|>1, \nabla L^{**}(\xi)\cdot\xi - L(\xi) = \sqrt{|\xi|}/2$ tends to $+\infty$ as $|\xi|$ tends to $+\infty$ whereas, for $\xi=(\xi_1,0)$ with $|\xi|>1, \nabla L^{**}(\xi)\cdot\xi - L^{**}(\xi)$ is asymptotic to $-\sqrt{|\xi_1|}/2$ as $|\xi_1|$ tends to $+\infty$.

Theorem 5.6 *Assume that $L(x,\xi)$ is bounded on compact sets and that L satisfies* (G). *Then L fulfills* (GA_σ).

The proof of Theorem 5.6 is much simpler in the case where L does not depend on x; we sketch it here for the convenience of the reader.

Sketch of the proof of Theorem 5.6 for $L(x,\xi)=L(\xi)$. Assume that $L(\xi)$ fulfills (G). Let $p\in\partial L^{**}$ and $M\in\mathbb{R}$; there exists $R_0>0$ such that, for every unitary vector u, $p(\lambda_u u)\cdot\lambda_u u - L(\lambda_u u)\geq M$, for some $\lambda_u\leq R_0$. Obviously $(L_u)^{**}(\lambda_u)\leq L_u(\lambda_u) = L(\lambda_u u)$ and therefore $p(\lambda_u u)\cdot\lambda_u u - (L_u)^{**}(\lambda_u)\geq M$. Let $q_u\in\partial(L_u)^{**}$, we claim that there exists $\overline{\lambda}$ such that, for every $\lambda\geq\overline{\lambda}$ and unitary vector u,

$$q_u(\lambda)\lambda - (L_u)^{**}(\lambda)\geq p(R_0 u)\cdot R_0 u - (L_u)^{**}(R_0). \tag{5.11}$$

In fact, if (5.11) does not hold then there exist a divergent increasing sequence λ_n and a sequence of unitary vectors u_n such that

$$q_{u_n}(\lambda_n)\lambda_n - (L_{u_n})^{**}(\lambda_n)\leq p(R_0 u_n)\cdot R_0 u_n - (L_{u_n})^{**}(R_0).$$

It follows by convexity that

$$p(R_0u_n)\cdot R_0u_n - q_{u_n}(\lambda_n)\lambda_n \geq (L_{u_n})^{**}(R_0) - (L_{u_n})^{**}(\lambda_n) \geq q_{u_n}(\lambda_n)(R_0-\lambda_n),$$

implying that

$$q_{u_n}(\lambda_n) \leq p(R_0u_n)\cdot u_n. \tag{5.12}$$

Let us point out that, again by Lemma 2.1, $p(R_0u)\cdot u \in \partial(L^{**})_u(R_0)$. Since $(L^{**})_u \leq (L_u)^{**}$, one expects that for every $\varepsilon > 0, p(R_0u)\cdot u \leq q_u(\lambda)+\varepsilon$ definitely; this however does not contradict (5.12). The point here is that L satisfies (GA) and, a fortiori, the CGA: there exist $R > 0, \varepsilon > 0$ and $c \in \mathbb{R}$ such that, for every unitary vector u,

$$\forall \xi \in \mathbb{R}^N \quad |\xi| \geq R, \quad L^{**}(\xi) \geq L^{**}(R_0u) + p(R_0u)\cdot(\xi - R_0u) + \varepsilon|\xi| + c.$$

In particular, for $\xi = \lambda u$ and $\lambda \geq R$, we obtain

$$L^{**}(\lambda u) \geq (p(R_0u)\cdot u + \varepsilon)\lambda + c(R_0), \tag{5.13}$$

where $c(R_0) = \inf\{L^{**}(R_0u) - p(R_0u)\cdot R_0u + c :\ u \in \partial B_N\} > -\infty, L$ being bounded on compact sets. Now the right hand side of the latter inequality is an affine, whence convex function of λ; moreover $L_u(\lambda) = L(\lambda u) \geq L^{**}(\lambda u)$ for every $\lambda \geq 0$; therefore from (5.13) we obtain

$$(L_u)^{**}(\lambda) \geq (p(R_0u)\cdot u + \varepsilon)\lambda + c(R_0)$$

for $\lambda \geq R$. It follows that there exists $\overline{\lambda}$ such that, for $\lambda \geq \overline{\lambda}$, the subdifferential q_u of $(L_u)^{**}$ at λ is larger than $p(R_0u)\cdot u + \varepsilon/2$; we will show that $\overline{\lambda}$ does not depend on u. In particular, for $\lambda_n \geq \overline{\lambda}$, we have $q_{u_n}(\lambda_n) \geq p(R_0u_n)\cdot u_n + \varepsilon/2$, contradicting (5.12). Hence (5.11) holds true and the monotonicity Lemma 2.1 yields the conclusion. □

We postpone the proof of Theorem 5.6 in the general case after some preliminary results that take into account the problem of the uniformity with respect to the first variable varying in a compact set and to the unitary vectors u.

Lemma 5.7 *Let $\varphi, \psi : [R, +\infty[\to \mathbb{R}$ be convex, with $\varphi \leq \psi$, and let $p(\lambda)$ in $\partial\varphi(\lambda), q(\lambda)$ in $\partial\psi(\lambda)$ and $\varepsilon > 0$. For every $\lambda_1 \geq R$ the inequality*

$$q(\lambda) > p(\lambda_1) - \varepsilon$$

holds for every $\lambda \geq \frac{\psi(\lambda_1)-\varphi(\lambda_1)}{\varepsilon} + \lambda_1$.

Proof Assume that $q(\lambda) \leq p(R) - \varepsilon$ for some $\lambda \geq R$. Since q in $\partial\psi$ and p in $\partial\varphi$ we have

$$\begin{aligned}\psi(\lambda) - \psi(R) &\leq q(\lambda)(\lambda - R)\\ &\leq (p(R)-\varepsilon)(\lambda - R) = p(R)(\lambda - R) - \varepsilon(\lambda - R)\\ &\leq \varphi(\lambda) - \varphi(R) - \varepsilon(\lambda - R)\end{aligned}$$

so that

$$\begin{aligned}\psi(\lambda) &\leq \varphi(\lambda) + \psi(R) - \varphi(R) - \varepsilon(\lambda - R)\\ &\leq \psi(\lambda) + \psi(R) - \varphi(R) - \varepsilon(\lambda - R),\end{aligned}$$

which is equivalent to $\lambda \leq \frac{\psi(\lambda_1)-\varphi(\lambda_1)}{\varepsilon} + \lambda_1$. □

The next result is a consequence of Lemma 5.7.

Lemma 5.8 *Let $\psi : [0,+\infty[\to \mathbb{R}$ be convex, with $\psi(\lambda) \geq a\lambda + b$ for every $\lambda \geq R$ and q in $\partial\psi$. For every $\varepsilon > 0$ there exists $R_1 = R_1(\varepsilon, R, \psi(R) - (aR+b))$, increasing in the third variable, such that*

$$q(\lambda) \geq a - \varepsilon \quad \textit{for every } \lambda \geq R_1.$$

Proof From Lemma 5.7 with $\varphi(\lambda) = a\lambda + b$ we may set $R_1(\varepsilon, R, \eta) = \eta/\varepsilon + R$. □

We need also the following simple result.

Lemma 5.9 *Let $\psi : [0,+\infty[\to \mathbb{R}$ be such that $\psi(\lambda) \geq a\lambda + b$ for every $\lambda \geq R$ and $m = \inf\{\psi(\lambda) : 0 \leq \lambda \leq R\} > -\infty$. Then $\psi^{**}(\lambda) \geq a\lambda + \min\{m - aR, b\}$ for every $\lambda \geq 0$.*

The next result compares the subdifferential of $(L_u)^{**}$ with that of $(L^{**})_u$; it is the main tool in the proof of Theorem 5.6.

Lemma 5.10 *Assume that $L(x,\xi)$ is bounded on compact sets and satisfies* (CGA).

$$q_u(x,\lambda) \in \partial(L_u)^{**}(x,\lambda), \quad p(x,\xi) \in \partial L^{**}(x,\xi) \;\; (u \in \partial B_N).$$

Then for every compact subset C of $\mathbb{R}^N$ and $R_0 \geq 0$ there exist $\varepsilon > 0$ and $\lambda_0 = \lambda_0(R_0, \varepsilon)$ such that, for every unitary vector u and $x \in C$,

$$p(x,\lambda u)\cdot u \geq p(x, R_0 u)\cdot u + \varepsilon/2 \quad \textit{and} \quad q(x,\lambda) \geq p(x,R_0 u)\cdot u + \varepsilon/2 \tag{5.14}$$

for every $\lambda \geq \lambda_0$.

Remark 5.11 We recall that, from Lemma 2.1, $p(x,\lambda u)\cdot u \in \partial(L^{**})_u(x,\lambda)$, so that the first inequality in (5.14) is a strict monotonicity property of the subdifferentials of the sections of L^{**}. We also notice that, from the inequality $L^{**} \leq L$ it follows that $(L^{**})_u \leq (L_u)^{**}$; in this situation the application of Lemma 5.7 with $\varphi(\cdot) = (L^{**})_u(x,\cdot)$ and $\psi(\cdot) = (L_u)^{**}(x,\cdot)$ yields that, definitively, $q_u(x,\lambda) \geq p(x,R_0 u)\cdot u - \varepsilon$. Here the result is much stronger, thanks to (CGA).

Proof of Lemma 5.10 By (CGA) there exists $R = R(R_0)$ such that, for every $\lambda \geq R$, $x \in C$ and unitary vector u,

$$\begin{aligned} L^{**}(x,\lambda u) &\geq L^{**}(x,R_0 u) + p(x,R_0)\cdot(\lambda - R_0)u + \varepsilon\lambda + c \\ &\geq (p(x,R_0 u)\cdot u + \varepsilon)\lambda + c_1(R_0,\varepsilon), \end{aligned} \tag{5.15}$$

where $c_1 = \inf\{L^{**}(x,R_0 u) - p(x,R_0 u)\cdot u + c : x \in C, u \in \partial B_N\} > -\infty$ since L is affinely minorized and L is bounded on compact sets. Fix x in C and set $\psi(\lambda) = L^{**}(x,\lambda u)$. We recall that, by the monotonicity Lemma 2.1, $p(x,\lambda u)\cdot u \in \partial\psi(t)$. Set

$$\Delta_1(x,R_0,u,R) = L^{**}(x,Ru) - [L^{**}(x,R_0 u) + p(x,R_0 u)\cdot(R - R_0)u];$$

by Lemma 5.8 there exists $R_1 = R_1(R,\varepsilon,\Delta_1(x,R_0,u,R))$, increasing in the last variable, such that

$$\forall\lambda \geq R_1(R,\varepsilon,\Delta_1(x,R_0,u,R)) \quad p(x,\lambda u)\cdot u \geq p(x,R_0 u)\cdot u + \varepsilon/2. \tag{5.16}$$

To prove the second inequality of our statement, we notice that, by (5.15) we have

$$L_u(x,\lambda) \geq (p(x,R_0 u)\cdot u + \varepsilon)\lambda + c_1(R_0,\varepsilon)$$

for every $\lambda \geq R$ and unitary vector u. Let $m(R) = \inf\{L_u(x,\lambda) : u \in \partial B_N, 0 \leq \lambda \leq R\}$, which is finite since L is bounded below by an affine function. By Lemma 5.9, for every $\lambda \geq 0$ we have

$$(L_u)^{**}(x,\lambda) \geq (p(x,R_0u)\cdot u + \varepsilon)\lambda + c_2(R_0),$$

where

$$c_2(R_0,\varepsilon) = \inf\{c(R_0,\varepsilon), m - [(p(x,R_0u)\cdot u + \varepsilon]R : x \in C, u \in \partial B_N\}.$$

Again, as in the proof of the first inequality, if we set

$$\Delta_2(x,R_0,u,R) = (L_u)^{**}(x,R) - [L^{**}(x,R_0u) + p(x,R_0u)\cdot(R-R_0)u],$$

the application of Lemma 5.8 yields the existence of $R_2 = R_2(R,\varepsilon,\Delta_2(x,R_0,u,R))$, increasing in the last variable, such that

$$\forall\lambda \geq R_2(R,\varepsilon,\Delta_2(x,R_0,u,R)) \quad q_u(x,\lambda) \geq p(x,R_0u)\cdot u + \varepsilon/2. \tag{5.17}$$

Since $L^{**}(x,Ru) \leq (L_u)^{**}(x,R) \leq L(x,Ru)$, by the local boundedness assumption there exists K, depending only on R_0 and R and therefore only on R_0, such that $\Delta_i(x,R_0,u,R) \leq K$ $(i=1,2)$. The conclusion follows by (5.16) and (5.17) if we set

$$\lambda_0 = \max\{R_1(R,\varepsilon,K), R_2(R,\varepsilon,K)\},$$

the functions R_1 and R_2 being monotonic in the last variable. □

We are now in the position to prove the main result of this Sect. (5b).

Proof of Theorem 5.6 Let C be a compact subset of $\mathbb{R}^N$, $M \in \mathbb{R}$ and $p(x,\xi) \in \partial L^{**}(x,\xi)$. From (G) we deduce that there exists R_0 such that, for every u in ∂B_N, and $x \in C$,

$$p(x,\lambda_u u)\cdot\lambda_u u - L(x,\lambda_u u) \geq M$$

for some $\lambda_u \leq R_0$. Since $(L_u)^{**}(x,\lambda) \leq L(x,\lambda u)$ we deduce that $p(x,\lambda_u u)\cdot\lambda_u u - (L_u)^{**}(x,\lambda_u) \geq M$ and therefore, by the monotonicity Lemma 2.1, we have

$$p(x,R_0u)\cdot R_0u - L(x,R_0u) \geq M. \tag{5.18}$$

Let $q_u(x,\lambda) \in \partial(L_u)^{**}(x,\lambda)$. Since, from Corollary 4.4, L fulfills (CGA), then by Lemma 5.10 there exist $\varepsilon > 0$ and $\lambda_0 = \lambda_0(R_0,\varepsilon)$ such that, for every unitary vector u and $x \in C$,

$$q(x,\lambda) \geq p(x,R_0u)\cdot u + \varepsilon/2 \tag{5.19}$$

whenever $\lambda \geq \lambda_0$. Assume that, for some $x \in C$ and $u \in \partial B_N$, we have

$$q_u(x,\lambda_0)\lambda_0 - (L_u)^{**}(x,\lambda_0) \leq p(x,R_0u)\cdot R_0u - (L_u)^{**}(x,R_0). \tag{5.20}$$

Since, by convexity,

$$q_u(x,\lambda_0)(R_0-\lambda_0) \leq (L_u)^{**}(x,R_0) - (L_u)^{**}(x,\lambda_0)$$

then, from (5.20), we obtain

$$q_u(x,\lambda_0)(R_0-\lambda_0) \leq p(x,R_0u)\cdot R_0u - q_u(x,\lambda_0)\lambda_0$$

which is equivalent to

$$q_u(x,\lambda_0) \leq p(x,R_0u)\cdot u,$$

contradicting (5.19). Therefore from (5.18) we obtain

$$q_u(x,\lambda_0)\lambda_0 - (L_u)^{**}(x,\lambda_0) \geq p(x,R_0u)\cdot R_0u - (L_u)^{**}(x,R_0) \geq M$$

for all $u \in \partial B_N$, and $x \in C$. The monotonicity Lemma 2.1 yields the conclusion. □

The converse of Theorem 5.6 does not hold, as it is shown in the following example.

Example 5.12 Set

$$L(\xi_1,\xi_2) = \begin{cases} |\xi| - \sqrt{|\xi|} & \text{if } \xi_2 \neq 0, \\ \xi_1^2 & \text{if } \xi_2 = 0. \end{cases}$$

Clearly L satisfies (GA_σ): in fact the map $t \to |t| - \sqrt{|t|}$ satisfies (GA) and $t \to t^2$ is superlinear. However L does not satisfy (G) since $L^{**}(\xi) = |\xi| - \sqrt{|\xi|}$ and, for $\xi = (\xi_1, 0)$,

$$\nabla L^{**}(\xi)\cdot\xi - L(\xi) = |\xi_1| - \frac{\sqrt{|\xi_1|}}{2} - \xi_1^2 \to -\infty$$

as $|\xi_1| \to +\infty$.

6 The one dimensional case

The purpose of this section is to show that, in the one dimensional case, the growth assumptions (GA_σ), (GA) and (G) introduced above are all equivalent. Here, following the notation introduced in Sect. 2, $L_1(x,\cdot)$ and $L_{-1}(x,\cdot)$ are the two sections $L_u(x,\cdot)$ of $L(x,\cdot), u = \pm 1$.

Lemma 6.1 *Let $L : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$. If L satisfies* (CGA) *then, for every compact subset C of $\mathbb{R}^N$, there exists $\bar\lambda$ such that $(L_u)^{**}(x,\lambda) = L^{**}(x,\lambda), \partial(L_u)^{**}(x,\lambda) = \partial L^{**}(x,\lambda)$ for every $\lambda \geq \bar\lambda$ and $u = \pm 1$.*

Proof Let $p(x,\lambda) \in \partial L^{**}(x,\lambda)$ and C be a compact subset of $\mathbb{R}^N$. We apply three times Lemma 5.10 with $u = 1$ (starting with $R_0 = 0$): let $0 < \lambda_0 < \lambda_1 < \lambda_2$ and $\varepsilon > 0$ be such that

$$\forall x \in C \qquad \begin{cases} p(x,\lambda_0) \geq p(x,0) + \varepsilon \\ p(x,\lambda_1) \geq p(x,\lambda_0) + \varepsilon \\ p(x,\lambda_2) \geq p(x,\lambda_1) + \varepsilon. \end{cases}$$

Fix x in C and set $f(\xi) = L(x,\xi)$. Let $(\xi_1, f(\xi_1)), (\eta, f(\eta))$ be two points of the graph of f with $\xi_1 \leq 0$, $\eta \geq \lambda_2$; we denote by Δ_1 the segment that joins them. Let $(\xi, y_1) \in \Delta_1$ with $\xi \geq \lambda_2$. We claim that there exists $\xi_2 \geq 0$ such that the ordinate of the point (ξ, y_2) of the segment joining $(\xi_2, f(\xi_2))$ with $(\eta, f(\eta))$ is less than y_1. In fact let φ be the affine function on $[0,\lambda_2]$ whose graph is the segment Δ joining $(0, f(0))$ with $(\lambda_2, f(\lambda_2))$ and set

$$\Gamma = \{(\lambda, y) : \lambda \in [0,\lambda_2],\ y \leq \varphi(\lambda)\}.$$

Notice that Γ is not reduced to the segment Δ itself, otherwise f would be affine on $[0,\lambda_2]$, contradicting the fact that the subgradients $p(x,\lambda_0)$ and $p(x,\lambda_1)$ of f at λ_0 and λ_1 are different. Therefore there exists $\xi_2 \in [0,\lambda_2]$ such that $(\xi_2, f(\xi_2)) \in \Gamma$: let $(\eta_1, f(\eta_1))$ be the intersection of the segment Δ_2 joining $(\eta, f(\eta))$ and $(\xi_2, f(\xi_2))$ with the epigraph of f and let y_2 be such that $(\xi, y_2) \in \Delta_2$; clearly $0 \leq \eta_1 \leq \lambda_2$. Let $g : [\xi_1, \eta] \to \mathbb{R}$ be the function whose graph is the union of the segment joining $(\xi_1, f(\xi_1))$ with $(0, f(0))$, of the graph of f restricted to $[0, \lambda_2]$ and of the segment joining $(\lambda_2, f(\lambda_2))$ with $(\eta, f(\eta))$. Since $(\xi_1, f(\xi_1))$ and $(\eta, f(\eta))$ belong to the epigraph of f then g is convex. It follows that the angular coefficient of Δ_1 is smaller of that of Δ_2 and therefore $y_2 \leq y_1$, proving the claim. As a consequence, for every $\xi \geq \lambda_2$,

$$\begin{aligned} L^{**}(x,\xi) &= \inf\{\alpha L(x,\xi_1) + (1-\alpha)L(x,\eta) : \alpha \geq 0,\ \xi = \alpha\xi_1 + (1-\alpha)\eta\} \\ &= \inf\{\alpha L(x,\xi_2) + (1-\alpha)L(x,\eta) : \alpha \geq 0, \eta \geq 0, \xi_2 \geq 0\ \xi = \alpha\xi_2 + (1-\alpha)\eta\} \\ &= (L_1)^{**}(x,\xi), \end{aligned}$$

proving the first part of the claim. It is obvious that $\partial L^{**}(x,\xi) \subset \partial(L_1)^{**}(x,\xi)$ for every $\xi \geq \lambda_2$. Conversely, let $\xi_0 > \lambda_2$ and $p \in \partial(L_1)^{**}(x,\xi_0)$. Then

$$\forall \xi \in [0,+\infty[\quad (L_1)^{**}(x,\xi) \leq (L_1)^{**}(x,\xi_0) + p(\xi - \xi_0);$$

in particular for every $\xi > \lambda_2$ we have

$$L^{**}(x,\xi) - L^{**}(x,\xi_0) \geq p(\xi - \xi_0)$$

and therefore

$$d_\ell L^{**}(x,\xi_0) \leq p \leq d_r L^{**}(x,\xi_0).$$

Now $\partial L^{**}(x,\xi_0) = [d_\ell L^{**}(x,\xi_0), d_r L^{**}(x,\xi_0)]$ (this interval may be unbounded if L is not bounded on bounded sets) and thus $p \in \partial L^{**}(x,\xi_0)$, proving that $\partial L^{**}(x,\xi_0) = \partial(L_1)^{**}(x,\xi_0)$ for every $x_0 > \lambda_2$. An application of this result to the function $\tilde{L}(x,\xi) = L(x,-\xi)$ yields the statement concerning L_{-1}. □

Theorem 6.2 *The function* $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (GA) *if and only if* L *satisfies* (GA$_\sigma$).

Proof If L satisfies (GA) then (CGA) holds and the conclusion follows from Lemma 6.1. Conversely, assume that L satisfies (GA$_\sigma$). Let $\ell(\xi) = a\xi + b$ be an affine function minorizing L on $\mathbb{R}$ and C be compact subset of $\mathbb{R}$. Our assumption implies that there exists $\alpha < 0$ such that, for every $p_{-1}(x,\xi) \in \partial(L_{-1})^{**}(x,\xi)$ we have

$$\forall x \in C \quad \eta_{-1}(x) = (L_{-1})^{**}(x,\alpha) - p_{-1}(x,\alpha)\alpha \leq b.$$

Analogously, there exists $\beta > 0$ such that for every $p_1(x,\xi) \in \partial(L_1)^{**}(x,\xi)$ we have

$$\forall x \in C \quad \eta_1(x) = (L_1)^{**}(x,\beta) - p_1(x,\beta)\beta \leq b.$$

We notice that $(0,\eta_{-1}(x))$ (resp. $(0,\eta_1(x))$) is the intersection of the supporting line $y = \ell_{-1}(x)$ (resp. $y = \ell_1(x)$) to the epigraph of $L_{-1}(x,\cdot)$ (resp. $L_1(x,\cdot)$) at $(\alpha, (L_{-1})^{**}(x,\alpha))$ (resp. $(\beta, (L_1)^{**}(x,\beta))$) with the ordinate axes. Let $(\xi(x), \eta(x))$ be the intersection of the two tangent lines: necessarily $\eta(x) \leq a\xi(x) + b$. The function

$$h(x,\xi) = \begin{cases} (L_{-1})^{**}(x,\xi) & \text{if } \xi \leq \alpha \\ \ell_{-1}(x,\xi) & \text{if } \xi \in [\alpha, \xi(x)] \\ \ell_1(x,\xi) & \text{if } \xi \in [\xi(x), \beta] \\ (L_1)^{**}(x,\xi) & \text{if } \xi \geq \beta \end{cases}$$

is convex in ξ and lower than $L(x,\xi)$. Therefore $h(x,\xi) \leq L^{**}(x,\xi)$ so that $(L_{-1})^{**}(x,\xi) = L^{**}(x,\xi)$ for $\xi \leq \alpha$ and $(L_1)^{**}(x,\xi) = L^{**}(x,\xi)$ for $\xi \geq \beta$. It follows that L satisfies (GA). □

We have shown in Example 5.5 that, in general, condition (GA) does not imply (G), even if the function L depends only on ξ. However in the one dimensional case (GA) and (G) are equivalent.

Proposition 6.3 *The function* $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (G) *if and only if* L *satisfies* (GA).

Proof From Proposition 5.4 it is enough to prove that (GA) implies (G). Henceforth, assume that L satisfies (GA). Fix $M \in \mathbb{R}$ and let C be a compact subset of $\mathbb{R}^N$; there exists R_0 such that

$$\forall \xi \in \mathbb{R}^N \quad |\xi| \geq R_0, \quad p(x,\xi)\xi - L^{**}(x,\xi) > M$$

for every $p(x,\xi) \in \partial L^{**}(x,\xi)$. Fix $\xi > R$; by Theorem 4.6 there exist ξ_1, ξ_2 and $\alpha \in [0,1]$ satisfying $L^{**}(x,\xi) = \alpha \overline{L}(x,\xi_1) + (1-\alpha)\overline{L}(x,\xi_2)$, $\overline{L}(x,\xi_i) = L^{**}(x,\xi_i) = L^{**}(x,\xi) + p(x,\xi)(\xi_i - \xi)$ $(i = 1,2)$ where, as usual, $\overline{L}(x,\xi) = \liminf_{\eta \to \xi} L(x,\eta)$. It is not restrictive to assume that $\xi_2 > R_0$; let $(\lambda_k)_k$ be such that

$$\lambda_k > R_0, \quad \lim_{k \to +\infty} \lambda_k = \xi_2, \quad \lim_{k \to +\infty} L(x,\lambda_k) = L^{**}(x,\xi_2).$$

Since $(p(x,\lambda_k))_k$ is bounded we may assume that $\lim_k p(x,\lambda_k) = q \in \mathbb{R}$; from the inequality $L^{**}(x,\lambda) - L^{**}(x,\lambda_k) \geq p(x,\lambda_k)(\lambda - \lambda_k)$ we deduce that $L^{**}(x,\lambda) - L^{**}(x,\xi_2) \geq q(x)(\lambda - \xi_2)$ for all $\lambda \in \mathbb{R}$, so that $q(x) \in \partial L^{**}(x,\xi_2)$. It follows that

$$\lim_k p(x,\lambda_k)\lambda_k - L(x,\lambda_k) = q(x)\xi_2 - L^{**}(x,\xi_2) > M;$$

in particular $p(x,\lambda_k)\lambda_k - L(x,\lambda_k) > M$ for some $\lambda_k \leq R$. Analogously there exists $\lambda < 0$ such that $q'(x)\lambda - L^{**}(x,\lambda) > M$, proving that L fulfills (G). □

Lastly, we mention that the limit at infinity of an affinely minorized function satisfying (GA) may be different to $+\infty$.

Example 6.4 Let $L(\xi) = -\log \xi$ if $\xi \geq 1, L(\xi) = (\xi - 1)^2 - \xi + 1$ otherwise. Then L is convex, it satisfies (GA) but $\lim_{\xi \to +\infty} L(\xi) = -\infty$.

Actually we show that the limit at infinity of a function of one variable that satisfies (GA) is equal to $+\infty$ or to $-\infty$.

Proposition 6.5 *Let* $L : \mathbb{R} \to \mathbb{R}$ *satisfy* (GA). *Then either*

$$\lim_{\xi \to +\infty} L^{**}(\xi) = +\infty \quad or \quad \lim_{\xi \to +\infty} L^{**}(\xi) = -\infty.$$

Proof Let $p(\xi) \in \partial L^{**}(\xi)$. If there exists ξ_0 such that $p(\xi_0) > 0$ then

$$L^{**}(\xi) - L^{**}(\xi_0) \geq p(\xi_0)(\xi - \xi_0)$$

so that $\lim_{\xi \to +\infty} L^{**}(\xi) = +\infty$. Assume now that $p(\xi) \leq 0$ for every ξ. Then L^{**} is decreasing on $\mathbb{R}$: let $\ell = \lim_{\xi \to +\infty} L^{**}(\xi)$. If $\ell \in \mathbb{R}$ then, from the equality $p(\xi)\xi = p(\xi)\xi - L^{**}(\xi) + L^{**}(\xi)$, by (GA) we deduce that $\lim_{\xi \to +\infty} p(\xi)\xi = +\infty$; however $p(\xi)\xi \leq 0$ for every $\xi \geq 0$, a contradiction. It follows that $\ell = -\infty$. □

References

1. Ambrosio, L., Ascenzi, O., Buttazzo, G.: Lipschitz regularity for minimizers of integral functionals with highly discontinuous integrands. J. Math. Anal. Appl. **142**, 301–316 (1989)
2. Cellina, A.: The classical problem of the calculus of variations in the autonomous case: relaxation and Lipschitzianity of solutions. Trans. Amer. Math. Soc. **356**, 415–426 (2004)
3. Dal Maso, G., Frankowska, H.: Autonomous integral functionals with discontinuous nonconvex integrands: Lipschitz regularity of minimizers, DuBois-Reymond necessary conditions, and Hamilton-Jacobi equations. Appl. Math. Optim. **48**, 39–66 (2003)
4. Hewitt, E., Stromberg, K.: Real and abstract analysis. Graduate Texts in Mathematics, vol. 25, Springer, Berlin Heidelberg New York (1975)