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LOCAL LIPSCHITZ REGULARITY OF MINIMA FOR A SCALAR PROBLEM OF THE CALCULUS OF VARIATIONS

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We consider a functional $I(u) = \int_\Omega f(\nabla u(x))\,dx$ on $u_0 + W^{1,1}(\Omega)$. Under the assumption that f is just convex, we prove a new Comparison Principle, we improve and give a short proof of Cellina's Comparison result for a new class of minimizers. We then extend a local Lipschitz regularity result obtained recently by Clarke for a wider class of functions f and boundary data u_0 satisfying a new one-sided *Bounded Slope Condition*. A relaxation result follows.

Keywords: Bounded Slope Condition; strict convexity; Lipschitz regularity.

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1. Introduction

The regularity of minimizers of integral functionals is a well-known and widely studied problem of the Calculus of Variations. Starting with the De Giorgi–Nash regularity theorem a huge number of results have been proved both for solution of elliptic equations and for minimizers by means of various developments of their techniques.

In this paper, we focus our attention to the scalar case of the Calculus of Variations and in particular to the minimizers of

$$I(u) = \int_\Omega f(\nabla u(x))\,dx, \quad u \in u_0 + W_0^{1,1}(\Omega, \mathbb{R}). \tag{1.1}$$

The techniques we mentioned above apply to local minimizers and lead to the local Lipschitzianity of the solutions under suitably growth assumptions on the lagrangian and/or ellipticity conditions on the associated Euler equation.

One can ask, anyhow, if the boundary data may have a role and to what extent can substitute the growth assumptions in the regularity of the minimizers.

Stampacchia, in the 60's, considered a special class of boundary data: those satisfying the *Bounded Slope Condition* (BSC). It requires that at every point of the boundary $\partial\Omega$ of Ω there are two affine functions $\ell^-_{x^0}$ and $\ell^+_{x^0}$ that coincide with u_0 in x^0 satisfying $\ell^-_{x^0}(x) \leq u_0(x) \leq \ell^+_{x^0}(x)$ on $\partial\Omega$ and such that their gradients are uniformly bounded, for x^0 varying in $\partial\Omega$. This condition is satisfied if, for instance, u_0 is of class C^2, $\partial\Omega$ is smooth and the principal curvatures of $\partial\Omega$ are strictly positive. Moreover, the validity of the (BSC) forces u_0 to be affine or Ω to be convex; in any case u_0 must be affine on the flat parts of $\partial\Omega$. Assuming that f is strictly convex Stampacchia showed in [1] that the functional I has a minimizer among the Lipschitz functions (instead of the Sobolev ones that we consider here).

It is shown, for instance, in [2], that in this situation the Lavrentiev phenomenon does not occur and therefore if u_0 satisfies the (BSC) and f is strictly convex then the minimum of I in the Sobolev space $u_0 + W^{1,1}_0(\Omega)$ is Lipschitz. We then generalized in [3] the results of [2]; the main tool in the proofs is a Comparison Principle, that allows to pass from an inequality among minimizers in the boundary of Ω to the same inequality in all of Ω. This Comparison Principle was also inspired by the results of Giusti [4] in the Lipschitz setting; the *strict convexity* of the functional is essential for its validity.

By means of the aforementioned Comparison Principle presented in [3], Clarke proved in [5] that just a *one-sided* (BSC) is enough to obtain, again under the assumption that the lagrangian is strictly convex, the local Lipschitz regularity of the minimum of I (the *Lower* (BSC) is the requirement, in the (BSC), of just the existence of the affine functions $\ell^-_{x^0}$ from below; an analogue definition holds for the *Upper* (BSC)). As it is shown in [5] this result is somewhat optimal since there are examples where, under these assumptions, the minima are not globally Lipschitz. We remark here that in all these results no growth conditions are needed.

Many problems however involve a lagrangian that is *not strictly convex*; this is the case for instance of those who arise from a non convex one as one considers the relaxed functional, whose lagrangian has flat parts on its epigraph. In this case of a non strictly convex lagrangian both the minimizers may not be Lipschitz and the Comparison Principle may fail.

In [6], Cellina introduced a class of functions that, for many purposes, do the same job as affine functions: they are minimizers of I and preserve the comparison with the minimizers of I from the boundary of Ω to all of Ω. These functions, originally defined by means of the polar of the function f, are here redefined in terms of the support functions of the faces of the epigraph of f. We give a new and short proof of the fact that they satisfy a Comparison Principle; furthermore we do not require anymore that Ω is convex nor that the domain of the polar of f is open. Actually, it is enough that the interior of the closure of Ω differs from Ω for at most a negligible set; this occurs if, for instance, the boundary of Ω is negligible.

In [7], we introduced a condition on (u_0, f), that we called the Cellina (BSC) or (CBSC) since it appeared for the first time in [6]. This condition is somewhat similar to the classical (BSC), where the new class of minimizers mentioned above replace the affine function. We then obtained in [7], under the (CBSC), a Lipschitz regularity result on the minimizers of I without the requirement that f is strictly convex or that f is superlinear: we just assume that the faces of the epigraph of f are bounded. It is worthwhile to underline the fact that all the superlinear functions satisfy this latter assumption and that the (CBSC) is much more general than the (BSC); some relations between these two conditions are given here.

In this paper, we apply these techniques to extend Clarke's local regularity result mentioned above. More precisely, instead of the *strict convexity* of f we just assume that f is superlinear or that the projections of the faces of the epigraph of f are uniformly bounded; then if the boundary datum and f satisfy a *one-sided* (CBSC), then the minimizers of I turn out to be locally Lipschitz and bounded. Our assumption yields here the existence of the greatest and the smaller of the minimizers of I that share the same boundary datum; the proof of the local regularity result is then based on Clarke's dilation technique together with a new Comparison Principle between these extreme minimizers and any other local minimizer of I.

Finally, we consider the case of a possibly non convex function f that grows more than (or even as) a cone or such that the projections of the faces of the epigraph of its convex envelope f^{**} are uniformly bounded. Assuming that the relaxed associated problem of minimizing $I^{**}(u) = \int_\Omega f^{**}(\nabla u(x)\, dx$ has a locally Lipschitz minimizer in $u_0 + W^{1,1}_0(\Omega)$ then the infima of the original problem I among the locally Lipschitz functions of $u_0 + W^{1,1}_0(\Omega)$ coincides with the minimum of I^{**}. This kind of relaxation results are known to hold under the much stronger assumption that f is superlinear. We then apply it to prove a locally Lipschitz regularity result for a Borel measurable class of non convex lagrangians.

2. A Comparison Principle in the Non Strictly Convex Case

In this paper Ω is an open, bounded subset of $\mathbb{R}^n$, endowed with the usual scalar product "$\cdot$" and the euclidian norm $|\cdot|$; $\overline{\Omega}$ is the closure of Ω and $\mathrm{int}(\overline{\Omega})$ the interior of $\overline{\Omega}$; ∂A is the boundary of a set A.

In this section $L : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is a Carathéodory function satisfying

$$L(x,u,\xi) \geq a(x)\cdot\xi + b(x)$$

for a.e. (almost everywhere) x in Ω, for every $(u,\xi) \in \mathbb{R}\times\mathbb{R}^n$, for some a in $L^\infty(\Omega;\mathbb{R}^n)$ and b in $L^1(\Omega)$. The functional I is defined on $W^{1,1}(\Omega)$ by

$$I(v) = \int_\Omega L(x, v(x), \nabla v(x))\, dx.$$

Definition 2.1. A function u is a *minimum* or a *minimizer* of I in $W^{1,1}(\Omega)$ (or in $u + W^{1,1}_0(\Omega)$) if $x \mapsto L(x,u(x),\nabla u(x)) \in L^1(\Omega)$ and $I(u) \leq I(v)$ for every $v \in u + W^{1,1}_0(\Omega)$.

In what follows by $u \leq v$ on $\partial\Omega$ we mean that the positive part $(u-v)^+$ of $u - v$ belongs to $W^{1,1}_0(\Omega)$. In [3, Theorem 4.1] we proved that if $L(x,u,\cdot)$ is strictly convex then $u \le v$ a.e. on Ω whenever u and v are minima of I such that $u \le v$ a.e. on $\partial\Omega$. The conclusion of this result may be false without the strictly convexity assumption on L. The following example is taken from [6].

Example 2.1. Let

$$f(\xi) = \begin{cases} 0 & \text{if } |\xi| \le 1 \\ (|\xi|-1)^2 & \text{if } |\xi| > 1 \end{cases}.$$

The functions $u(x) = -|x|+1$ and $v(x) = 0$ both minimize $\int_{-1}^{1} f(x'(t))\, dt$ among the absolutely continuous functions that vanish on $\{-1,1\}$; however $u \le v$ on $\{-1,1\}$ but $u > v$ on $]-1,1[$.

In the general case, without assuming that the lagrangian is *strictly convex*, we introduce a class of minima for which the conclusion of the Comparison Principle stated above still applies.

Definition 2.2. We say that a minimum u of I in $W^{1,1}(\Omega)$ satisfies the *Comparison Principle from below* if for every minimum v of I in $W^{1,1}(\Omega)$ the inequality $u \le v$ on $\partial\Omega$ implies $u \le v$ a.e. on Ω. Analogously, a minimum v of I in $W^{1,1}(\Omega)$ satisfies the *Comparison Principle from above* if for every minimum u of I in $W^{1,1}(\Omega)$ the inequality $u \le v$ on $\partial\Omega$ implies $u \le v$ a.e. on Ω.

Remark 2.1. Assume that u satisfies the Comparison Principle from below (respectively above). Then u is the minimum (respectively maximum) element among the minima of I (on $u + W_0^{1,1}(\Omega)$). In fact if v is any other minimum of I on $u + W_0^{1,1}(\Omega)$ then $u \le v$ (respectively $u \ge v$) on $\partial\Omega$ and therefore $u \le v$ (respectively $u \ge v$) a.e. on Ω.

It is interesting for our purposes to note that the converse holds true. We shall denote by $u \wedge v$ (respectively $u \vee v$) the pointwise minimum (respectively maximum) of u and v. Our methods here are inspired by those of Giusti in [4] for Lipschitz functions.

Theorem 2.1 (Comparison Principle for Extreme Minima). (*a*) *Let u, v be two minima of I in $W^{1,1}(\Omega)$ and $u \le v$ on $\partial\Omega$. Then $u \wedge v$ and $u \vee v$ are minima of I in $W^{1,1}(\Omega)$.*
(b) *A minimum u of I satisfies the Comparison Principle from below if and only if u is the minimum of the set of the minima of I on $u + W_0^{1,1}(\Omega)$.*
(c) *A minimum v of I satisfies the Comparison Principle from above if and only if v is the maximum of the set of the minima of I on $v + W_0^{1,1}(\Omega)$.*

Proof. Let u, v be two minima of I such that $u \le v$ on $\partial\Omega$. Since $u \wedge v \in u + W_0^{1,1}(\Omega)$ then $I(u) \le I(u \wedge v)$ or equivalently,

$$\int_{u>v} L(x, u(x), \nabla u(x))\, dx \le \int_{u>v} L(x, v(x), \nabla v(x))\, dx.$$

Analogously, by taking $u \vee v \in v + W_0^{1,1}(\Omega)$ we obtain the reverse inequality and thus

$$\int_{u>v} L(x, u(x), \nabla u(x))\, dx = \int_{u>v} L(x, v(x), \nabla v(x))\, dx.$$

It follows that

$$\begin{aligned} I(u \wedge v) &= \int_{u \le v} L(x, u(x), \nabla u(x))\, dx + \int_{u>v} L(x, v(x), \nabla v(x))\, dx \\ &= \int_{\Omega} L(x, u(x), \nabla u(x))\, dx = I(u) \end{aligned}$$

so that $u \wedge v$ is a minimum of I in $u + W_0^{1,1}(\Omega)$; analogously $u \vee v$ turns out to be a minimum of I, proving (a). Assume now that $u \le v$ on $\partial\Omega$ and that u is the minimum among the minimizers of I in $u + W_0^{1,1}(\Omega)$. Then (a) implies that $u \wedge v$ is a minimizer too; the minimality of u yields $u = u \wedge v$ or equivalently $u \le v$ a.e. Claim (c) follows similarly. □

Remark 2.2. In Proposition 4.2, we will give a sufficient condition for the existence of the greatest and the lowest of the minimizers of I that share the same boundary value.

Remark 2.3. If f is strictly convex then Theorem 2.1 is a generalization of [3, Theorem 4.1] in the case of a comparison between two minima. In fact in this case there exists at most a unique minimum of I having a prescribed boundary datum. In particular the affine functions, being minimizers of I, do satisfy the *Comparison Principle* both from below and from above.

Here is a simple but useful application of the last result.

Theorem 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous and $a \in \mathbb{R}^n$ be such that $f(a) < +\infty$. Let $u_0 \in W^{1,1}(\Omega)$ be such that $m \le u_0 \le M$ on $\partial\Omega$ and assume that the problem of minimizing*

$$I(v) = \int_{\Omega} f(\nabla v(x))\, dx$$

in $u_0 + W_0^{1,1}(\Omega)$ admits at least a solution. Then I has a minimizer w that is essentially bounded and moreover

$$a \cdot x + m - \sup\{a \cdot x : x \in \Omega\} \le w(x) \le a \cdot x + M - \inf\{a \cdot x : x \in \Omega\} \quad \text{for a.e. } x \in \Omega.$$

In particular, if $f(0)$ is finite then $m \le w \le M$ a.e. on Ω.

Proof. Let u be a minimizer of I in $u_0 + W_0^{1,1}(\Omega)$. The affine functions $a \cdot x + b$ are minimizers of I. We set $b_1 = m - \sup\{a \cdot x : x \in \Omega\}$ and $b_2 = M - \inf\{a \cdot x : x \in \Omega\}$; then $a \cdot x + b_1 \le u_0 \le a \cdot x + b_2$ on $\partial\Omega$. By Theorem 2.1 we deduce that $w = (u \wedge (a \cdot x + b_2)) \vee (a \cdot x + b_1)$ is a minimizer too. □

Remark 2.4. We shall give, in a subsequent section, a further condition on f that will ensure that *every* minimum of I is essentially bounded.

The previous results may be formulated for *sub/super-minima* instead of minima; we state them here for the convenience of the reader.

Definition 2.3. A function $u \in W^{1,1}(\Omega)$ is a *sub-minimum* of I if

$$L(x, u(x), \nabla u(x)) \in L^1(\Omega) \quad \text{and} \quad I(u) \leq I(v) \text{ if } v \leq u \quad \text{a.e.}$$

Analogously u is said to be a *super-minimum* of I if

$$L(x, u(x), \nabla u(x)) \in L^1(\Omega) \quad \text{and} \quad I(u) \leq I(v) \text{ if } u \leq v \quad \text{a.e.}$$

Definition 2.4. We say that a *sub-minimum* $u \in W^{1,1}(\Omega)$ of I satisfies the *Comparison Principle from below* if for every super-minimum v of I the inequality $u \leq v$ on $\partial\Omega$ implies $u \leq v$ a.e. on Ω. Analogously, a *super-minimum* $v \in W^{1,1}(\Omega)$ of I satisfies the *Comparison Principle from above* if for every sub-minimum u of I the inequality $u \leq v$ on $\partial\Omega$ implies $u \leq v$ a.e. on Ω.

The proof of Theorem 2.1 yields the following analogue for sub/super-minima.

Theorem 2.3 (Comparison Principle for Extreme Sub/Super-Minima). *The minimum (respectively maximum) of the set of the sub-minima (respectively super-minima) of I among the functions of $W^{1,1}(\Omega)$ having a prescribed boundary value satisfies the Comparison Principle from below (respectively above).*

3. A Class of Minimizers That Do Satisfy the Comparison Principle

We consider here $L(x, u, \xi) = f(\xi)$ for some *lower semicontinuous, convex* function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (not identically equal to $+\infty$) and the functional

$$I(v) = \int_\Omega f(\nabla v(x))\, dx.$$

We point out that we *do not* assume here that f is strictly convex.

In this section we define a class of minimizers other than affine functions and show that they satisfy a *Comparison Principle*. The ideas and many of the arguments that we use here arise actually from the paper [6] of Cellina. There are however some differences and improvements that justify to write them here. The definition of this class of minimizers is written here in terms of the faces of the epigraph of f whereas Cellina extensively uses the subdifferential of the polar of f to define them. Furthermore we weaken the assumptions and give a simpler proof of the *Comparison Principle* stated in [6].

Let us recall the notion of *exposed face* of the epigraph of a convex function, we will use it extensively.

Definition 3.1 (Exposed Faces). An *exposed face* of the epigraph of f is the intersection of the epigraph of f with a supporting hyperplane.

The projection onto $\mathbb{R}^n$ of such a face is a closed, convex subset of $\mathbb{R}^n$.

Remark 3.1. It follows from the definition that F is the projection of an exposed face of the epigraph of f if and only if there exists $p \in \mathbb{R}^n$ such that

$$\forall \xi \in \mathbb{R}^n, \quad \forall \xi_0 \in F, \quad f(\xi) - f(\xi_0) \geq p \cdot (\xi - \xi_0)$$

and the equality holds if and only if $\xi \in F$.

Definition 3.2 (The Functions h^+_{F,x^0} and h^-_{F,x^0}). Let F be the projection of a *bounded* exposed face of the epigraph of f and $x^0 \in \mathbb{R}^n$. We define the functions

$$h^+_{F,x^0}(x) = \max_{k \in F} k \cdot (x - x^0), \quad h^-_{F,x^0}(x) = \min_{k \in F} k \cdot (x - x^0).$$

Remark 3.2. Clearly h^+_{F,x^0} is a translate of the support function of F and moreover $h^-_{F,x^0}(x) = -h^+_{F,-x^0}(-x)$. In particular $h^+_{F,0}(x) = \max\{k \cdot x : k \in F\}$ is the *polar* I^*_F of the indicator function $I_F(x)$ of F that is equal to 0 on F, to $+\infty$ out of F. The function h^+_{F,x^0} is convex and h^-_{F,x^0} is concave; they are both Lipschitz with Lipschitz constant equal to $\max\{|\xi| : \xi \in F\}$ and they are equal to 0 at x^0. We mention that if f is strictly convex then F is reduced to a point and thus in this case the maps just defined are affine.

Remark 3.3. This class of functions was introduced by Cellina in [6]. In that paper however these functions are defined in terms of the subdifferential of the polar of f and depend on a further real parameter. More precisely, for θ in the interior of the effective domain of f^*, $x^0 \in \mathbb{R}^n$ and $r \in \mathbb{R}$ the functions $h^+_{\theta,x^0,r}$ and $h^-_{\theta,x^0,r}$ are there defined by

$$h^+_{\theta,x^0,r}(x) = \sup_{k \in \partial f^*(\theta)} k \cdot (x - x^0) + r, \quad h^-_{\theta,x^0,r}(x) = \inf_{k \in \partial f^*(\theta)} k \cdot (x - x^0) + r.$$

The fact of defining them here in terms of the faces of the epigraph of f is only an attempt to simplify the notations; in fact the projection F of an exposed face of the epigraph of f is the subdifferential $\partial f^*(\theta)$ of the polar of f at a point θ that is interior of the domain of f^*.

The next result follows immediately from the fact that the functions defined above arise from the support function of a closed convex set.

Proposition 3.1. *Let F be the projection onto $\mathbb{R}^n$ of a bounded exposed face of the epigraph of f and $x^0 \in \mathbb{R}^n$. Then h^+_{F,x^0} and h^-_{F,x^0} are Lipschitz and thus differentiable almost everywhere. Moreover their gradients belong to F a.e. and*

$$h^+_{F,x^0}(x) = \nabla h^+_{F,x^0}(x) \cdot (x - x^0), \quad h^-_{F,x^0}(x) = \nabla h^-_{F,x^0}(x) \cdot (x - x^0) \quad a.e.$$

Proof. It is not restrictive to assume that $x^0 = 0$. By Remark 3.2, it is enough to prove the result for the function $h^+_{F,0}$, that we call here for simplicity h_F. The function h_F is convex with values in $\mathbb{R}$ and thus admits a non empty subdifferential ∂h_F at every point. Since $h_F(x) = \max\{k \cdot x : k \in F\} = I^*_F(x)$ then $\overline{k} \in \partial h_F(x)$ if and only if $x \in \partial I_F(\overline{k})$; this occurs exactly when $\overline{k} \in F$ and $\overline{k} \cdot x = \max\{k \cdot x : k \in F\} = h_F(x)$. Moreover, h_F is Lipschitz so that h_F is differentiable a.e. and, at every point x of differentiability of h_F, $\partial h_F(x) = \{\nabla h_F(x)\}$; it follows that $\nabla h_F(x) \in F$ and $h_F(x) = \nabla h_F(x) \cdot x$ a.e. □

Corollary 3.1. *Let F be the projection onto $\mathbb{R}^n$ of a bounded exposed face $\widehat{F}$ of the epigraph of f and $\{(x,z) : z = p \cdot x + b\}$ be a supporting hyperplane to $\widehat{F}$. Then, for every $x^0 \in \mathbb{R}^n$,*

$$p \in \partial f(\nabla h^+_{F,x^0}(x)), \quad p \in \partial f(\nabla h^-_{F,x^0}(x)) \quad a.e.$$

Proof. Since, by Proposition 3.1, $\nabla h^+_{F,x^0} \in F$ a.e. then $(\nabla h^+_{F,x^0}(x), f(\nabla h^+_{F,x^0}(x)))$ belong a.e. to $\widehat{F}$ and thus $p \in \partial f(\nabla h^+_{F,x^0}(x))$ a.e. The other case is analogous. □

Theorem 3.1. *Let F be the projection onto $\mathbb{R}^n$ of a bounded exposed face of the epigraph of f and $x^0 \in \mathbb{R}^n$. The functions h^+_{F,x^0} and h^-_{F,x^0} are minima of I.*

Proof. Here again we prove the result for $h_F = h^+_{F,x^0}$, the other case being similar. Let $v \in W^{1,1}(\Omega)$ be such that $v - h_F \in W^{1,1}_0(\Omega)$. By Corollary 3.1 there is $p \in \mathbb{R}^n$ such that, for every $\xi \in \mathbb{R}^n$ and a.e. x in Ω,

$$f(\xi) - f(\nabla h_F(x)) \geq p \cdot (\xi - \nabla h_F(x))$$

so that in particular

$$f(\nabla v(x)) \geq f(\nabla h_F(x)) + p \cdot \nabla(v - h_F)(x) \quad \text{a.e.}$$

and thus, by integration on Ω,

$$I(v) - I(h_F) \geq \int_\Omega p \cdot \nabla(v - h_F)(x)\,dx = 0$$

since $v - h_F \in W^{1,1}_0(\Omega)$. The conclusion follows. □

Remark 3.4. Theorem 3.1 is a reformulation of [6, Theorem 1] that makes no use of the notion of the polar of f; the result stated in Proposition 3.1 is proven here just by relating the functions $h^\pm_{F,x^0}$ to the support functions of a convex set. We stated and proved them here for the convenience of the reader.

Theorem 3.2 (Comparison Principle between Minimizers and $h^\pm_{F,x^0}$). *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous and F be the projection onto $\mathbb{R}^n$ of a bounded exposed face of the epigraph of f. Let w be a minimizer of*

$$I(v) = \int_\Omega f(\nabla v(x))\,dx$$

on $u_0 + W^{1,1}_0(\Omega)$. Assume that, for some $x^0 \notin \mathrm{int}(\overline{\Omega})$, $h^+_{F,x^0} \geq u_0$ on $\partial\Omega$; then $h^+_{F,x^0} \geq w$ a.e. on Ω. Analogously, if $h^-_{F,x^0} \leq u_0$ on $\partial\Omega$ then $h^-_{F,x^0} \leq w$ a.e. on Ω.

Remark 3.5. The conclusion of Theorem 3.2 does still hold if for some $r \in \mathbb{R}$, the functions $h^\pm_{F,x^0}$ are replaced by $h^\pm_{F,x^0} + r$; this is due to the fact that $w(x) - r$ is still a minimizer of I whenever w is a minimizer of I.

Remark 3.6. This result is an extension of [6, Theorem 2] although it is stated here in terms of the *faces of the epigraph* of F instead of the *subdifferentials of the polar* of f^*. The main novelties however are that we do not assume here that the domain of f^* is open (allowing thus to work with functions like $\sqrt{1+|\xi|^2}$), nor that Ω is convex. Further, the proof of the result is much simpler and shorter than that of [6, Theorem 2].

Remark 3.7. As it is pointed out in [6], the claim of Theorem 3.2 does not hold, in general, if $x_0 \in \mathrm{int}(\overline{\Omega})$. In view of Theorem 2.1, this is due to the fact that in this case h^-_{F,x^0} (respectively h^+_{F,x^0}) is not the minimum (respectively the maximum) of the minimizers of I.

Lemma 3.1. *Let F be the projection onto $\mathbb{R}^n$ of a bounded exposed face of the epigraph of f. Then, for every x in $\mathbb{R}^n$, the map $y \mapsto h^+_{F,y}(x)$ is Lipschitz of constant*

$$K_F = \max\{|\eta| : \eta \in F\}.$$

Proof. Let $y_1, y_2 \in \mathbb{R}^n$; then

$$\begin{aligned} h^+_{F,y_2}(x) &= \sup_{k \in F} k \cdot (x - y_2) \\ &\leq \sup_{k \in F} k \cdot (x - y_1) + \sup_{k \in F} k \cdot (y_1 - y_2) \\ &\leq h^+_{F,y_1}(x) + K_F|y_2 - y_1| \end{aligned}$$

proving the claim. □

Proof of Theorem 3.2. Assume that $h = h^+_{F,x^0} \geq u_0$ on $\partial\Omega$ and that $w > h$ on a non negligible set E. Since $w \leq h$ on $\partial\Omega$ then $w \wedge h \in w + W^{1,1}_0(\Omega)$ and thus $I(w \wedge h) \geq I(w)$ or, equivalently,

$$0 \geq \int_E f(\nabla w(x)) - f(\nabla h(x))\,dx.$$

Assume that $\{(x,z) : z = p \cdot x + b\}$ is a supporting hyperplane to the face $\widehat{F}$ of the epigraph of f whose projection is F; by Corollary 3.1, we have $p \in \partial f(\nabla h(x))$ a.e. on Ω so that

$$f(\nabla w(x)) - f(\nabla h(x)) \geq p \cdot (\nabla w(x) - \nabla h_F(x)) \quad \text{a.e. on } E.$$

Since $w \wedge h = w$ out of E the latter inequalities yield

$$0 \geq \int_E f(\nabla w(x)) - f(\nabla h(x))\,dx \geq \int_E p \cdot (\nabla w - \nabla h)(x)\,dx$$
$$= \int_\Omega p \cdot \nabla(w - w \wedge h)(x)\,dx = 0$$

since $w - w \wedge h \in W_0^{1,1}(\Omega)$. Therefore, a.e. on E we have

$$f(\nabla w(x)) - f(\nabla h(x)) = p \cdot (\nabla w - \nabla h)(x)$$

so that the points $(\nabla w(x), f(\nabla w(x)))$ and $(h(x), f(h(x)))$ belong a.e. on E to $\widehat{F}$. Thus $\nabla w(x) \in F$ a.e. and in particular, by Proposition 3.1,

$$\nabla w(x) \cdot (x - x^0) \leq \max\{k \cdot (x - x^0) : k \in F\} = \nabla h_F(x) \cdot (x - x^0) \quad \text{a.e. on } E.$$

Therefore, if we set $\eta = w - w \wedge h$ we have

$$\nabla \eta(x) \cdot (x - x^0) \leq 0 \quad \text{a.e.,} \quad \eta \in W_0^{1,1}(\Omega), \quad \eta > 0 \text{ on a non negligible set} \tag{3.1}$$

(we will still call η on $\mathbb{R}^n$ its trivial extension out of Ω). There exists $\delta > 0$ such that $\eta \geq \delta$ on a non negligible subset E_δ of Ω.

If $x^0 \notin \overline{\Omega}$ there is a closed ball $B(x^0, r]$ centered in x^0 of radius $r > 0$ that is contained in $\mathbb{R}^n \backslash \overline{\Omega}$. There is a cone C with apex x^0 and aperture less than (for instance) $\pi/2$ that intersects E_δ on a non negligible set E_δ^C. A slight modification of the proof of a classical result on the existence of a representative of a given Sobolev function that is absolutely continuous on almost every line (see, for instance, [8, Theorem 3.35]) yields a representative η^* of η such that $\eta^* = 0$ on $\partial B(x^0, r] \cap C$, for almost every direction ν of the cone the map $t \mapsto \eta^*(x^0 + t\nu)$ is absolutely continuous on $[r, +\infty[$ and

$$\frac{d}{dt}\eta^*(x^0 + t(x - x^0)) = \nabla \eta(x^0 + t(x - x^0)) \cdot (x - x^0) \quad \text{for a.e. } t \geq r.$$

In particular, for a.e. x in C, we have

$$\frac{d}{dt}\eta^*(x^0 + t(x - x^0)) = \nabla \eta(x^0 + t(x - x^0)) \cdot (x - x^0) \leq 0 \quad \text{for a.e. } t \geq r$$

so that η^* is decreasing on a.e. direction of the cone. In particular $\delta \leq \eta = \eta^* \leq 0$ a.e. on E_δ^C, a contradiction.

Let $x^0 \in \partial\overline{\Omega}$ and (x_k^0) be a sequence in $\mathbb{R}^n \backslash \overline{\Omega}$ such that $|x_k^0 - x^0| \leq 1/k$. By Lemma 3.1 for every x in $\mathbb{R}^n$ and $k = 1, 2, \ldots$ we have

$$|h^+_{F,x^0}(x) - h^+_{F,x_k^0}(x)| \leq \frac{K_F}{k} \tag{3.2}$$

so that, for every k,

$$w(x) - \frac{K_F}{k} \leq h^+_{F,x_k^0}(x) \quad \text{on } \partial\Omega.$$

Since $w - \frac{K_F}{k}$ is still a minimizer of I and $x_k^0 \notin \overline{\Omega}$, the first part of the proof shows that

$$w(x) - \frac{K_F}{k} \leq h^+_{F,x_k^0}(x) \quad \text{a.e. on } \Omega$$

and therefore, using again (3.2), we obtain that, for all $k = 1, 2, \ldots$,

$$w(x) \leq h_F(x) + 2\frac{K_F}{k} \quad \text{a.e. on } \Omega.$$

By letting $k \to +\infty$ we get $\eta = w - h_F \leq 0$ a.e. on Ω, contradicting (3.1). □

4. Some Applications to the Regularity of the Minimizers

We consider here the functional

$$I(v) = \int_\Omega f(\nabla v(x))\,dx$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex. In this section, we will often assume that the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are uniformly bounded or that f is superlinear. Any strictly convex function satisfies this condition. This assumption implies that the faces of the epigraph of f are bounded; this weaker condition was called (CGA) or *Conical Growth Condition* in [7, 9] to obtain some global Lipschitz regularity results. The next example shows that there exist superlinear functions such that the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are not uniformly bounded.

Example 4.1 (The Projections of Faces of a Superlinear Function May Not be Uniformly Bounded). Let $k_i = \frac{i(i+1)}{2}$ $(i \in \mathbb{N})$ so that $k_{i+1} - k_i = k_i - k_{i-1} + 1$; let g be defined by

$$g(\xi) = \begin{cases} k_i^2 & \text{if } |\xi| = k_i \\ +\infty & \text{otherwise} \end{cases} \quad \xi \in \mathbb{R}^n$$

and set $f(\xi) = g^{**}(\xi)$ for any $\xi \in \mathbb{R}^n$. Then, for every ξ, $f(\xi) \geq |\xi|^2$ and the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are the sets

$$F_i^u = \{\lambda u : \lambda \in [k_i, k_{i+1}]\}, \quad |u| = 1, \quad i \in \mathbb{N}.$$

Moreover $\operatorname{diam}(F_i^u) = k_{i+1} - k_i = k_i - k_{i-1} + 1 = \operatorname{diam}(F_{i-1}^u) + 1 \to +\infty$ for $i \to +\infty$.

We will apply here the *Comparison Principle* for extreme minima formulated in Theorem 2.1; we first give a condition that ensures the existence of the lowest and the greatest of the minimizers of I among the functions that share the same boundary value.

Proposition 4.1 (Existence of the Minimum and the Maximum of the Minimizers). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Assume that either f is superlinear or that the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are uniformly bounded.*

Let $u_0 \in W^{1,1}(\Omega)$ and assume that the functional

$$I(v) = \int_\Omega f(\nabla v(x))\,dx$$

has a minimizer in $u_0 + W_0^{1,1}(\Omega)$. Then the sets of the minimizers of I in the same space has a minimum and a maximum element.

Proof. Let us first point out that, in this situation, the set X of the minima of I is a weakly compact subset of $W^{1,1}(\Omega)$: the proof of this claim is straightforward if f is superlinear; otherwise, let u_k, $k \in \mathbb{N}$, be a sequence of minima of I in $u_0 + W_0^{1,1}(\Omega)$. By [7, Lemma 4.9], for every $k = 1, 2, \ldots$, the points $(u_k(x), f(\nabla u_k(x)))$ and $(u_1(x), f(\nabla u_1(x)))$ belong a.e. to the same face of the epigraph of f. If the diameter of the projection onto $\mathbb{R}^n$ of the faces of the epigraph of f is bounded by D then

$$|\nabla u_k(x) - \nabla u_1(x)| \le D \quad \text{a.e.}$$

Therefore the functions $u_k - u_1$ ($k = 1, 2, \ldots$) of $W_0^{1,1}(\Omega)$ are equi-Lipschitz and thus converge, up to a subsequence, weakly and a.e. to a function w in $u_0 + W_0^{1,1}(\Omega)$; the weak lower semicontinuity of I then shows that w is a minimum too, so that X is weakly compact. We now prove that X has a maximum. The space $W^{1,1}(\Omega)$ being separable, there exists a countable dense subset $\{w_1, \ldots, w_k, \ldots\}$ of X. Set $v_k = w_1 \vee \cdots \vee w_k$; Theorem 2.1 shows that $v_k \in X$; let v be the pointwise limit of the monotonic sequence v_k. By the weak compactness of X, we may assume that v_k weakly converges to a function in X which, by Mazur's Lemma, actually turns out to be equal to v. Therefore $v \in X$ and, since $v \ge w_k$ for every k then $v \ge u$ for any $u \in X$. The existence of a minimum follows similarly. □

Remark 4.1. The proof of Proposition 4.1 shows that the conclusion holds true if, instead of assuming that f is superlinear or that the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are uniformly bounded, the function f is such that the *faces of its epigraph are bounded* and I has a *Lipschitz minimizer.*

A first consequence of Theorem 4.1 is a refinement of Theorem 2.2. We denote by R_0 the radius of the smallest ball centered in the origin which enclose the projection onto $\mathbb{R}^n$ of the exposed face of the epigraph of f containing $(0, f(0))$.

Theorem 4.1 (The Minimizers are Bounded). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Assume that either f is superlinear or that the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are uniformly bounded. If $u_0 \in W^{1,1}(\Omega)$ is essentially bounded, then every minimizer u of*

$$I(v) = \int_\Omega f(\nabla v(x))\,dx$$

in $u_0 + W_0^{1,1}(\Omega)$ is essentially bounded; moreover, if $m \le u_0 \le M$ on $\partial\Omega$, then

$$m - R_0 \operatorname{diam}(\Omega) \le u \le M + R_0 \operatorname{diam}(\Omega) \quad \textit{a.e. in } \Omega.$$

In particular, $\|u\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$ if $(0, f(0))$ is an extreme point of the epigraph of f.

Proof. Let m and M be such that $m \le u_0 \le M$ a.e. The constant functions are minimizers of I among the functions that have the same boundary datum: Proposition 4.1 yields a maximum w^+ among the minima of I that are equal to M on $\partial\Omega$. By [7, Lemma 4.9] the points $(\nabla w^+(x), f(\nabla w^+(x)))$ and $(0, f(0))$ belong to a same exposed face of the epigraph of f. Moreover, since $w^+ - M \in W_0^{1,1}(\Omega)$, then w^+ is Lipschitz of constant less than R_0 so that, in particular, for every $x^0 \in \partial\Omega$,

$$w^+(x) \le w^+(x^0) + R_0|x - x^0| \le M + R_0 \operatorname{diam}(\Omega) \quad \text{a.e. in } \Omega.$$

Analogously, the minimum of the minima of I in $m + W_0^{1,1}(\Omega)$ is a function w^- that is Lipschitz of constant R_0 and therefore

$$w^-(x) \ge w^-(x^0) - R_0|x - x^0| \ge m - R_0 \operatorname{diam}(\Omega) \quad \text{a.e. in } \Omega.$$

Again, using the *Comparison Principle* (Theorem 2.1) we obtain $w^- \le u \le w^+$ a.e. in Ω and therefore

$$m - R_0 \operatorname{diam}(\Omega) \le u \le M + R_0 \operatorname{diam}(\Omega) \quad \text{a.e. in } \Omega. \qquad \square$$

The following assumptions are the one-sided versions of what we called (CBSC) or *Cellina Bounded Slope Condition* in [7], a condition introduced by Cellina in [6] to generalize the classical (BSC) or *Bounded Slope Condition* (see [2, 4]). They represent, in this non strictly convex setting, the generalization of the *Lower and Upper Bounded Slope Condition*, the one-sided versions of the (BSC) introduced by Clarke in [5]: the affine functions involved in the Lower (respectively Upper) (BSC) are replaced here by the functions of the form h^-_{F,x^0} (respectively and h^+_{F,x^0}).

Definition 4.1 (Lower (CBSC)). We say that (u_0, f) satisfies the *Lower (CBSC)* of constant $K \ge 0$ if for every $x^0 \in \partial\overline{\Omega}$ there exists an exposed face of the epigraph of f whose projection $F(x^0)$ onto $\mathbb{R}^n$ is contained in the ball centered in 0 of radius K and, moreover,

$$\forall x \in \partial\overline{\Omega}, \quad u_0(x^0) + h^-_{F(x^0),x^0}(x) \le u_0(x).$$

Definition 4.2 (Upper (CBSC)). We say that (u_0, f) satisfies the *Upper (CBSC)* of constant $K \ge 0$ if for every $x^0 \in \partial\overline{\Omega}$ there exists an exposed face of the epigraph of f whose projection $F(x^0)$ onto $\mathbb{R}^n$ is contained in the ball centered in 0 of radius K and, moreover,

$$\forall x \in \partial\overline{\Omega}, \quad u_0(x^0) + h^+_{F(x^0),x^0}(x) \ge u_0(x).$$

Let us recall the definition of (CBSC), that we formulate here in terms of Lower and Upper (CBSC).

Definition 4.3 (CBSC). We say that (u_0, f) satisfies the (CBSC) of constant $K \geq 0$ if (u_0, f) satisfies both the Lower and Upper (CBSC) of constant K.

Remark 4.2. We point out that whereas the (BSC) depends only on u_0 and its behavior on $\partial\Omega$, the validity of the (CBSC) depends also strongly on f. The original definition of (CBSC) that we gave in [7] slightly differs from the simpler one that is given here due to the fact that we do not use anymore the notion of subdifferential of the polar of f^* to define the functions $h^{\pm}_{F,x^0}$; moreover we are just concerned with the points x^0 of $\partial\overline{\Omega}$, without taking care of the other points of $\partial\Omega$ (notice that if Ω is convex, then $\partial\Omega = \partial\overline{\Omega}$).

We showed in [7] that, if f has values in $\mathbb{R}$ and the faces of the epigraph of f are bounded, then the validity of the (BSC) ensures that of the (CBSC). It follows similarly that the Lower (respectively Upper) (BSC) of Clarke implies the Lower (respectively Upper) (CBSC). We prove it here again for the convenience of the reader, since the original proof given in [7] involves the subdifferential of the polar of f and a different notation.

We underline that in the next proposition we just assume that the projections of the faces of the epigraph are bounded (and not *uniformly bounded*). We point out the fact that any superlinear function satisfies this property (see [7]).

Proposition 4.2. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and that the faces of the epigraph of f are bounded. If u_0 satisfies the Lower (respectively Upper) (BSC) of constant K then (u_0, f) fulfills the Lower (respectively Upper) (CBSC) of constant*

$$R^f_K = \max\{|\xi| : \exists |\eta| \leq K,\ (\xi, f(\xi)) \text{ and } (\eta, f(\eta)) \text{ belong to a same face of } \operatorname{epi}(f)\}.$$

Proof. Let $u_0(x^0) + a \cdot (x - x^0) \leq u_0(x)$ for all x in $\partial\Omega$. Let $p \in \partial f(a)$ and $\widehat{F}$ be the exposed face equal to the intersection of the epigraph of f with the supporting hyperplane $\{(\xi, z) : z = p \cdot (\xi - a) + f(a)\}$. Then, if F is the projection of $\widehat{F}$ on $\mathbb{R}^n$, F is bounded and $h^-_{F,x^0} = \min\{k \cdot (x - x^0) : k \in F\} \leq a \cdot (x - x^0)$ for every x. Moreover if $|a| \leq K$ then $|k| \leq R^f_K$ for every k in F. It follows that the Lower (BSC) of constant K implies the Lower (CBSC) of constant R^f_K. The "upper" version of the claim follows analogously. □

However the (CBSC) itself is much weaker than the Lower or Upper (BSC). In fact the (BSC) forces the domain Ω to be convex and u_0 to be affine on the flat parts of $\partial\Omega$ and, as a consequence of [5, Proposition 1.1], the Lower (BSC) forces the boundary datum to be convex on the flat parts of the boundary of Ω. The next result shows that the (CBSC) is much more tolerant.

Proposition 4.3. *Let Ω be an open bounded set and F be the projection of a bounded face of the epigraph of f. Assume that u_0 is Lipschitz on Ω and such that $\nabla u_0 \in F$ a.e. Then (u_0, f) satisfy the (CBSC) of constant $K_F = \max\{|\xi| : \xi \in F\}$.*

Proof. We can consider the function u_0 defined in $\mathbb{R}^n$ with $\nabla u_0 \in F$ a.e. (see, for example [10, Lemma 2.2]). Let $x^0 \in \partial\Omega$ and $x \in \partial\Omega$: by Lebourg's *Mean Value Theorem* [11, Theorem 2.4] there exists z in the segment joining x^0 to x and an element p of Clarke's generalized gradient of u_0 at z such that

$$u_0(x) - u_0(x^0) = p \cdot (x - x^0).$$

Since, by the *Generalized Gradient Formula* [11, Theorem 8.1], p is in the convex hull of the limits of sequences of gradients of u_0 (and F is convex) it then turns out that $p \in F$; therefore $h^-_{F,x^0}(x) \leq u_0(x) - u_0(x^0) \leq h^+_{F,x^0}(x)$. □

Remark 4.3. An interpretation of Proposition 4.3 is that flat parts of the epigraph of f give more chance in order that (u_0, f) satisfy the (CBSC).

Remark 4.4. We notice also the fact that, for example, the assumptions of Proposition 4.3 are fulfilled whenever the epigraph of f has a face whose projection contains a ball centered in ξ_0 of radius K and $u_0(x) - \xi_0 \cdot x$ is Lipschitz of constant K.

Example 4.2 (The Lower (CBSC) is not equivalent to the Upper (CBSC)). Let f be the function of Example 2.1. Let $\Omega = [-1, 1] \times [0, 1]$ and $u_0 : \partial\Omega \to \mathbb{R}$ be the function defined by

$$\begin{cases} u(x, 0) = 1 - |x| & -1 \leq x \leq 1, \\ u(-1, y) = u(1, y) = y^2 & y \in [0, 1], \\ u(x, 1) = 1 & x \in [0, 1]. \end{cases}$$

The projections of faces of the epigraph of f are a ball B of radius 1 and single points. Then $h^+_{B,x^0} = |x - x^0|$, $h^-_{B,x^0} = -|x - x^0|$ whereas, if F is a singleton, the functions $h^{\pm}_{F,x^0}$ are affine. Therefore (u_0, f) satisfies the Lower (CBSC). However it turns out easily that (u_0, f) does not fulfill the Upper (CBSC). Moreover u_0 is neither convex nor concave on all of the flat parts of the boundary of Ω and thus it does neither satisfy a one-sided (BSC) condition.

We recall here the Lipschitz regularity result that we established in [7, Theorem 5.12].

Theorem 4.2 (Lipschitz Continuity of Minimizers under the (CBSC)). *Let Ω be such that $\operatorname{int}(\overline{\Omega}) \setminus \Omega$ is negligible and $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Assume, moreover, that (u_0, f) satisfies the (CBSC) of constant K and that the faces of the epigraph of f containing the images of the points of radius less than K are bounded.*

Let w be a minimizer of

$$I(v) = \int_{\Omega} f(\nabla v(x))\,dx$$

in $u_0 + W_0^{1,1}(\Omega)$. Then w is Lipschitz and its Lipschitz constant is bounded by

$$R_K^f = \max\{|\xi| : \exists |\eta| \leq K \ (\xi, f(\xi)) \text{ and } (\eta, f(\eta)) \text{ belong to a same face of } \operatorname{epi}(f)\}.$$

Moreover, I has a minimizer whose Lipschitz constant is bounded by K.

Remark 4.5. [7, Theorem 5.12] requires actually that Ω is convex and that all the faces of the epigraph are bounded; its proof, in view of Theorem 3.2, shows that the results holds true under the weaker assumption of Theorem 4.2. In fact by the *Comparison Principle* (Theorem 3.2) we first obtain that the result holds true for the minima of I in $u_0 + W^{1,1}(\operatorname{int}(\overline{\Omega}))$ which coincides with $u_0 + W^{1,1}(\Omega)$ since $\operatorname{int}(\overline{\Omega})$ differs from Ω for at most a negligible set. The latter equality is satisfied for instance if Ω is convex or, more generally, if $\partial\Omega$ is negligible.

Remark 4.4 together with Theorem 4.2 yield the following result.

Corollary 4.1 (Lipschitzianity of the Minima for Every Lipschitz Boundary Datum). *Let Ω be such that $\operatorname{int}(\overline{\Omega})\setminus\Omega$ is negligible and $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Assume that the faces of the epigraph of f are bounded and that the projection F of the exposed face of the epigraph f through $(0, f(0))$ contains a ball of radius K. Then, if u_0 is any Lipschitz function of Lipschitz constant less than K, every minimizer of I in $u_0+W_0^{1,1}(\Omega)$ is Lipschitz and its Lipschitz constant is bounded by $R^f_{K_F}$.*

We are now in the position to formulate the extension of Clarke's local lipschitzianity result [5, Theorem 2.1]. We admit however that our proof is strongly inspired by the dilation method of Clarke.

We denote again by R_0 the radius of the smallest ball centered in the origin which enclose the projection onto $\mathbb{R}^n$ of the exposed face of the epigraph of f containing $(0, f(0))$.

Theorem 4.3 (Local Lipschitzianity). *Let Ω be convex. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and assume moreover that one of the following assumptions holds:*

(a) *f is superlinear, or*
(b) *the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f are uniformly bounded by a constant D*

and that the functional $I(v) = \int_\Omega f(\nabla v(x))\,dx$ admits a minimizer on $u_0+W_0^{1,1}(\Omega)$. If (u_0, f) satisfies either the Lower (CBSC) or the Upper (CBSC) of constant K, then every minimizer of I is bounded, locally Lipschitz and lower semicontinuous on the closure of Ω; moreover one of the minimizers $\overline{u}$ is such that

$$|\nabla\overline{u}(x)| \leq \frac{K'}{\operatorname{dist}(x, \partial\Omega)} \ \text{a.e.}, \quad K' = 2K\operatorname{diam}(\Omega) + 2\|u_0\|_{L^\infty(\partial\Omega)} + R_0 \operatorname{diam}(\Omega).$$

Moreover, under (b), if w is any minimizer of I, then

$$|\nabla w(x)| \leq \frac{K'}{\operatorname{dist}(x, \partial\Omega)} + D.$$

Remark 4.6. We illustrate here why Theorem 4.3 extends the aforementioned result of Clarke. We assume here the one-sided (CBSC) instead of Clarke's one-sided (BSC); moreover we allow the epigraph of f to have some non trivial faces.

The proof of Theorem 4.3 is inspired by the dilation technique of Clarke's proof described also in [12, 13].

Proof of Theorem 4.3. Let u be any minimizer of I; by Theorem 4.1, the function u is essentially bounded and

$$\|u\|_\infty \leq M = R_0 \operatorname{diam}(\Omega) + \|u_0\|_{L^\infty(\partial\Omega)}.$$

We assume that (u_0, f) satisfies the Lower (CBSC); the other case can be treated similarly. For every x^0 in $\partial\Omega$, Theorem 3.2 yields

$$u_0(x^0) + h^-_{F(x^0),x^0}(x) \leq u(x)$$

for a.e. x in Ω; we point out, in view of the subsequent Remark 4.7, that Theorem 3.2 holds without assuming that Ω is convex. Set

$$\ell(x) = \sup\{u_0(x^0) + h^-_{F(x^0),x^0}(x) : x^0 \in \partial\Omega\}. \tag{4.1}$$

Then ℓ is Lipschitz, its Lipschitz constant being bounded by K, and moreover $\ell = u_0$ on $\partial\Omega$. Let $z \in \partial\Omega$. Set, for $\lambda \in]0,1[$, $\Omega_\lambda = \lambda(\Omega - z) + z$ and, for x in Ω_λ,

$$u_\lambda(x) = \lambda u\Big(\frac{x-z}{\lambda} + z\Big).$$

Then $u_\lambda \in W_0^{1,1}(\Omega_\lambda) + \varphi_\lambda$ where $\varphi_\lambda(x) = \lambda u_0(\frac{x-z}{\lambda} + z)$ so that by [13, Lemma 1] we have

$$u_\lambda \leq u + (1-\lambda)\Big(\frac{K}{\lambda}\operatorname{diam}(\Omega) + \|u_0\|_{L^\infty(\partial\Omega)}\Big) \quad \text{on } \partial\Omega_\lambda$$

for every minimizer u of I in $u_0 + W_0^{1,1}(\Omega)$. Let $\overline{u}$ be the maximum of these minimizers; its existence follows from Proposition 4.1. Then $\overline{u}$ (restricted to Ω_λ) is also the maximum of the minimizers of $I_\lambda(v) = \int_{\Omega_\lambda} f(\nabla v(x))\,dx$ among the functions of $W^{1,1}(\Omega_\lambda)$ that coincide with $\overline{u}$ on $\partial\Omega$. Furthermore the proof of [5, Lemma 2.7] shows that $\overline{u}_\lambda$ is a minimizer relative to the domain Ω_λ. The *Comparison Principle* (Theorem 2.1) then yields

$$\overline{u}_\lambda \le \overline{u} + (1-\lambda)\Big(\frac{K}{\lambda}\operatorname{diam}(\Omega) + \|u_0\|_{L^\infty(\partial\Omega)}\Big) \quad \text{a.e. on } \Omega_\lambda$$

so that, for every $\lambda \ge \frac{1}{2}$,

$$\overline{u}_\lambda - \overline{u} \le q(1-\lambda) \quad \text{a.e. on } \Omega_\lambda \tag{4.2}$$

where

$$q = 2K\operatorname{diam}(\Omega) + \|u_0\|_{L^\infty(\partial\Omega)}.$$

For every $\lambda \ge \frac{1}{2}$ and x such that (4.2) holds we deduce that

$$\begin{aligned}\overline{u}\Big(\frac{x-z}{\lambda}+z\Big) - \overline{u}(x) &= (1-\lambda)\overline{u}\Big(\frac{x-z}{\lambda}+z\Big) + \overline{u}_\lambda(x) - \overline{u}(x)\\ &\le (1-\lambda)(\|\overline{u}\|_\infty + q) \le (1-\lambda)K'\end{aligned}$$

where $K' = M + q$. For every $k = 1, 2, \ldots$ set $\lambda_k = k/(k+1)$ so that $\lambda_k \ge \frac{1}{2}$ and $\frac{x-z}{\lambda_k} + z = x + \frac{1}{k}(x-z)$, $1-\lambda_k = 1/(k+1)$: let Z_k be a negligible set such that

$$\overline{u}\Big(x + \frac{1}{k}(x-z)\Big) - \overline{u}(x) \le \frac{K'}{k+1} \tag{4.3}$$

on $\Omega_{\lambda_k}\backslash Z_k$. The sets Ω_{λ_k} increase with k and their union is Ω. Therefore if $x \in \Omega\backslash\bigcup_{k=1}^\infty Z_k$ the inequality (4.3) holds for $k \ge k_x$, for some $k_x \in \mathbb{N}$. Now the gradient of $\overline{u}$ exists a.e. so that, passing to the limit in (4.3) for $k \to +\infty$, we obtain

$$\nabla\overline{u}(x)\cdot(x-z) \le K' \quad \text{a.e.}$$

so that, by choosing z such that

$$\frac{x-z}{|x-z|} = \frac{\nabla\overline{u}(x)}{|\nabla\overline{u}(x)|}$$

we get

$$|\nabla\overline{u}(x)| \le \frac{K'}{\operatorname{dist}(x,\partial\Omega)} \quad \text{a.e.}$$

If w is any other minimum of I, the points $(\nabla w(x), f(\nabla w(x)))$ and $(\nabla\overline{u}(x), f(\nabla\overline{u}(x)))$ belong a.e. to the same exposed face of the epigraph of f.

Therefore, under (b), $|\nabla w(x) - \nabla\overline{u}(x)| \le D$ so that

$$|\nabla w(x)| \le |\nabla\overline{u}(x)| + D \le \frac{K'}{\operatorname{dist}(x,\partial\Omega)} + D \quad \text{a.e.}$$

If (a) holds, then for x on a compact subset $\mathcal{K}$ of Ω, we have

$$|\nabla w(x)| \le |\nabla\overline{u}(x)| + D_{\mathcal{K}} \le \frac{K'}{\operatorname{dist}(x,\partial\Omega)} + D_{\mathcal{K}} \quad \text{a.e. on } \mathcal{K}$$

where $\mathcal{D}_{\mathcal{K}}$ is a bound of the diameter of the faces of the epigraph of f containing the images of the points of $\mathcal{K}$. Finally, the lower semicontinuity of w follows from the fact that $w \ge \ell$ on $\partial\Omega$ where ℓ, defined in (4.1) is Lipschitz on the closure of Ω and equal to u_0 on $\partial\Omega$. □

Remark 4.7. The proof of Theorem 4.3 shows that one may weaken the convexity assumption on Ω by assuming instead that $\operatorname{int}(\overline{\Omega}) \setminus \Omega$ is negligible: indeed, in this case, the minimizers of I turn out to be locally Lipschitz at those points x that lie in the interior of a convex combination of points of $\partial\Omega$ with respect to with Ω is star-shaped.

Remark 4.8. As Clarke points out in the paper [5], if u_0 satisfies a one-sided (BSC) and f is just convex then it is easy to see that I has at least a locally Lipschitz minimizer: it is enough to consider the perturbed problem with the lagrangian $f_k(\xi) = f(\xi) + \frac{1}{k}|\xi|^2$ instead of f and pass to the limit for $k \to +\infty$. This argument does not hold anymore if (u_0, f) satisfies a one-sided (CBSC) since it may be that (u_0, f_k) does no more satisfy a one-sided (CBSC).

5. The Nonconvex Case

In this section we will consider a Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$ that we do not assume to be convex; we assume that f is bounded from below by an affine function. We denote by f^{**} the *bipolar* of f and by I^{**} the functional defined on $W^{1,1}(\Omega)$ by

$$I^{**}(u) = \int_\Omega f^{**}(\nabla u(x))\,dx.$$

Theorem 5.1 (Relaxation). *Let Ω be open and bounded. Assume that the faces of the epigraph of f^{**} are bounded and that either*

(a) *$f(\xi) \ge a|\xi| + b$, for some $a > 0$ and $b \in \mathbb{R}$, or*
(b) *the diameters of the projections onto $\mathbb{R}^n$ of the faces of the epigraph of f^{**} are uniformly bounded.*

If I^{**} admits a locally Lipschitz minimizer, then

$$\begin{aligned}&\min\{I^{**}(u) : u \in u_0 + W_0^{1,1}(\Omega)\}\\ &\quad = \inf\{I(u) : u \in u_0 + W_0^{1,1}(\Omega),\ u \text{ locally Lipschitz}\}\end{aligned}$$

and the minimizers of I, if they exist, are locally Lipschitz.

Proof. Let A_j, $j \in \mathbb{N}$, be an increasing family of regular open sets whose union is Ω and set $\Omega_0 = A_0$ and $\Omega_j = A_j\backslash\overline{A_{j-1}}$ for $j \ge 1$. We denote by $\overline{u}$ a locally Lipschitz minimizer of I^{**}, by I_{Ω_j} and $I^{**}_{\Omega_j}$ the functionals

$$I_{\Omega_j}(u) = \int_{\Omega_j} f(\nabla u(x))\,dx, \quad I^{**}_{\Omega_j}(u) = \int_{\Omega_j} f^{**}(\nabla u(x))\,dx$$

defined for every $u \in \overline{u} + W_0^{1,1}(\Omega_j)$. The restriction of $\overline{u}$ to Ω_j is a minimizer for $I^{**}_{\Omega_j}$; let K_j be its Lipschitz constant on Ω_j. Then applying [7, Theorem 6.3] it follows that

$$I^{**}_{\Omega_j}(\overline{u}) = \inf\{I_{\Omega_j}(u),\ u \in \overline{u} + W_0^{1,1}(\Omega_j) \cap \mathrm{Lip}_{K'_j}(\Omega_j)\}$$

for some K'_j, where by $\mathrm{Lip}_{K'_j}(\Omega_j)$ is the space of Lipschitz functions on Ω_j with Lipschitz constant less than K'_j. Let $\varepsilon > 0$ and $u_j \in \overline{u} + W_0^{1,1}(\Omega_j) \cap \mathrm{Lip}_{K'_j}(\Omega_j)$ be such that

$$\int_{\Omega_j} f(\nabla u_j(x))\, dx < \int_{\Omega_j} f^{**}(\nabla \overline{u}(x))\, dx + \frac{\varepsilon}{2^j}.$$

Let v be the function defined by $v = \sum_j u_j \chi_{\Omega_j}$, where χ_{Ω_j} is the characteristic function of Ω_j; clearly v is locally Lipschitz.

If f satisfies assumption (a) then, for every j,

$$|\nabla u_j| \le \frac{1}{a} f(\nabla u_j) - \frac{b}{a} \quad \text{a.e. on } \Omega_j.$$

Therefore, for every $m \ge 1$,

$$\sum_{j=0}^{m} \int_{\Omega_j} |\nabla u_j(x)|\, dx \le \sum_{j=0}^{m} \frac{1}{a} \int_{\Omega_j} f(\nabla u_j(x))\, dx - \frac{b}{a} |\Omega_0 \cup \cdots \cup \Omega_m|$$

so that $\nabla v \in L^1(\Omega)$ and

$$\begin{aligned}\int_{\Omega} |\nabla v(x)|\, dx &\le \sum_{j=0}^{\infty} \frac{1}{a} \int_{\Omega_j} f(\nabla u_j(x))\, dx - \frac{b}{a}|\Omega| \\ &\le \frac{1}{a} \int_{\Omega} f^{**}(\nabla \overline{u}(x))\, dx - \frac{b}{a}|\Omega| + 2\varepsilon.\end{aligned}$$

It follows that the series $\sum_j u_j$ converges to v in $u_0 + W_0^{1,1}(\Omega)$.

The same conclusion holds if f satisfies assumption (b): in fact if R is a uniform bound of the diameters of the projections of the faces of the epigraph of f then, for every $j \in \mathbb{N}$, ∇u_j and $\nabla \overline{u}$ belong a.e. to the projection of the same face of the epigraph of f and thus $|\nabla u_j| \le |\nabla \overline{u}| + R$ a.e. in Ω_j, whence $\nabla v \in L^1(\Omega)$ and $|\nabla v| \le |\nabla \overline{u}| + R$ a.e. in Ω. Now, since

$$\int_{\Omega} f(\nabla v)\, dx = \sum_{j=0}^{\infty} \int_{\Omega_j} f(\nabla u_j)\, dx < \sum_{j=0}^{\infty} \int_{\Omega_j} f^{**}(\nabla \overline{u})\, dx + 2\varepsilon$$

the conclusion follows. □

The following result follows directly from Theorem 4.3.

Corollary 5.1. *Let Ω be convex. Assume that either f is superlinear or the diameters of the projection onto $\mathbb{R}^n$ of the faces of the epigraph of f^{**} are uniformly bounded. If (u_0, f^{**}) satisfies either the Lower (CBSC) or the Upper (CBSC) of constant K and the functional I^{**} admits a minimizer on $u_0 + W_0^{1,1}(\Omega)$ then*

$$\begin{aligned}&\min\{I^{**}(u) : u \in u_0 + W_0^{1,1}(\Omega)\} \\ &\quad = \inf\{I(u) : u \in u_0 + W_0^{1,1}(\Omega),\ u \text{ locally Lipschitz}\}\end{aligned}$$

and the minimizers of I, if they exist, are locally Lipschitz.

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