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COMPARISON RESULTS WITHOUT STRICT CONVEXITY

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To Paolo Marcellini on the occasion of his 60-th birthday

Abstract. In this paper we establish a comparison result for solutions to the problem

$$\text{minimize} \int_\Omega l(\|\nabla u(x)\|)\, dx$$

or to the problem

$$\text{minimize} \int_\Omega l(\gamma_C(\nabla u(x))\, dx,$$

for a special class of solutions, without assuming neither smoothness nor strict convexity of l.

1. Introduction.

This paper is partially a sequel to [1], to [2] and to [4], where special classes of solutions were presented, for variational problems involving general convex Lagrangeans L, with the purpose of proving comparison theorems without assumptions of smoothness and without the assumption of strict convexity on L. Comparison theorems are the basis for several powerful tools used in the Calculus of Variations or in Partial Differential Equations, including the Strong Maximum Principle and the Moving Plane Method. These results are obtained exploiting the property of uniqueness of solutions; in turn, this property depends on the strict convexity of the Lagrangean. The assumption of strict convexity, however, forbids considering important classes of Lagrangeans, in particular those Lagrangeans that are generated from the convexification of functionals that, originally, were not convex. Although, without this assumption of strict convexity, one cannot hope to prove a comparison theorem that would be true for *any* two solutions, in the applications of this principle, in general, one of the solutions belongs to a special class of solutions and one aims at results for this more restricted class of solutions.

In the present paper we consider the problem of minimizing

$$\int_\Omega l(\|\nabla u(x)\|)\, dx \tag{1}$$

with $\|\cdot\|$ the Euclidean norm, or, more generally, of minimizing

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$$\int_\Omega l(\gamma_C(\nabla u(x)))\, dx \tag{2}$$

where γ_C is the *gauge* of the convex set C (see Section 3). In either case, the function l is convex, lower semicontinuous and even. In particular, the problems we consider contain the minimization of

$$\int_\Omega l(\langle \nabla u(x), Q\nabla u(x)\rangle^{\frac{1}{2}})\, dx \tag{3}$$

with Q a symmetric, positive definite $N \times N$ matrix.

We show that, without any regularity assumption and without the assumption of strict convexity, a comparison result holds when one of the two solutions involved in the comparison belongs to special classes of solutions, that we will denote $w_c^{x_0,k}$. These solutions are defined through the polar of the Lagrangean L, without exploiting, in the definition, a differential equation, a procedure that would require the regularity of L, that we do not assume. Our definition of the class $w_c^{x_0,k}$, when applied to the classical case of the Lagrangean $L(\xi) = \frac{1}{2}\|\xi\|^2$, gives back the solutions (when x_0 does not belong to the closure of Ω) given by $w_c^{x_0,k}(x) = c\log\|x - x_0\| + k$, for $N = 2$, and $w_c^{x_0,k}(x) = -\frac{c}{N-2}\frac{1}{\|x-x_0\|^{N-2}} + k$, for $N > 2$. As it is well known, these are the solutions that are used to establish the validity of the Strong Maximum Principle and of Hopf's Lemma for harmonic functions.

As an application of our comparison Theorem, we will provide a Maximum Principle for a class of extended valued Lagrangeans.

2. **Comparison Theorems. The rotationally symmetric case.** In this paper, Ω is a bounded, open set. By a *solution* we mean a function providing a *finite* minimum of (1) among those functions assuming the same boundary values. We denote by $\mathrm{Dom}(L)$ the *effective domain* of the function L, i.e. the set of points where L is finite, and by $\partial L(p)$ the *subdifferential* of L at p. By L^* we mean the *polar* or *Legendre transform* of L, a (possibly extended valued) convex function; we refer to [5] for the details on these subjects.

In this section we consider the problem of minimizing (1). The results of this section are strictly contained in the results of Section 3.

Consider the Lagrangean $L(\xi) = l(\|\xi\|)$, where

$$l : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

is convex, even and lower semicontinuous.

We will use the fact that L^* is rotationally invariant and that

$$\partial L^*(p) = \partial l^*(\|p\|)\frac{p}{\|p\|} \text{ if } p \neq 0, \qquad \partial L^*(0) = \partial l^*(0)B$$

where B is the unit ball; moreover $\partial l^*(\cdot)$ is an upper semicontinuous, increasing map with values the closed intervals of $\mathbb{R}$. Our results will be based on the properties of the function $w_c^{x_0,k}$, defined as follows.

Let $z^*(t)$ be any measurable selection from $\partial l^*(t)$ and, for $c \neq 0$, set

$$w_c^{x_0,k}(x) = \mathrm{sign}(c)\int_1^{\|x-x_0\|} z^*\Big(\frac{|c|}{s^{N-1}}\Big)\, ds + k$$

whose gradient is $\mathrm{sgn}(c)z^*(\frac{|c|}{\|x-x_0\|^{N-1}})\frac{x-x_0}{\|x-x_0\|}$. This function need not be well defined for every choice of x_0: for instance, in the case $L(t) = |t| - \sqrt{|t|}$, whose polar is

$L^*(p) = \frac{1}{4}\frac{1}{1-|p|}$ for $p \in (-1,1)$, whenever $x_0 \in \Omega$, for any $c \neq 0$, the values of the map $\frac{c}{\|x-x_0\|^{N-1}}$ when x varies in Ω are never entirely contained in $(-1,1)$, where L^* is defined. This explains the need of some further assumptions, as the one that follows.

Assumption (A). We shall assume that either

i) $\mathrm{Dom}(l^*) \supset [-\ell, +\ell]$ and $dist(x_0, \Omega) > (\frac{|c|}{\ell})^{\frac{1}{N-1}}$ or
ii) $\mathrm{Dom}(l^*) = (-\infty, +\infty)$ and $dist(x_0, \Omega) > 0$, or
iii) $\mathrm{Dom}(l) \subset [-\ell, +\ell]$ and $x_0 \notin \Omega$.

Proposition 1. *Assume (A). Then $w_c^{x_0,k}$ is well defined; for every admissible variation η,*

$$\int_\Omega \langle \frac{x-x_0}{\|x-x_0\|^N}, \nabla\eta(x)\rangle \, dx = 0, \tag{4}$$

and $w_c^{x_0,k}(x)$ is a solution to the problem of minimizing (1).

Proof. For $x_0 \notin \Omega$, $\frac{x-x_0}{\|x-x_0\|^N}$ is the gradient of the harmonic function $\frac{\|x-x_0\|^{2-N}}{2-N}$, or of the harmonic function $\log\|x-x_0\|$ for $N=2$, so (4) is true for $\eta \in C_c^\infty(\Omega)$.

In case iii) of Assumption (A), solutions are Lipschitzian and so are variations; the map $\frac{x-x_0}{\|x-x_0\|^N}$ is in $L^1(\Omega)$; hence, by approximation, (4) holds for every admissible, i.e. lipschitzian, variation η. In case i) we have that $\frac{|c|}{\|x-x_0\|^{N-1}} \leq |\ell|$ and in ii), $\frac{x-x_0}{\|x-x_0\|^N}$ is in $L^\infty(\Omega)$, so that, again by approximation, the validity of (4) holds for η in $W_0^{1,1}(\Omega)$. So (4) holds for every admissible variation.

From the definition of $w_c^{x_0,k}$, we have that $\frac{\nabla w_c^{x_0,k}(x)}{\|\nabla w_c^{x_0,k}(x)\|} = \mathrm{sgn}(c)\frac{x-x_0}{\|x-x_0\|}$ a.e. and

$$\|\nabla w_c^{x_0,k}(x)\| \frac{\nabla w_c^{x_0,k}(x)}{\|\nabla w_c^{x_0,k}(x)\|} = \nabla w_c^{x_0,k}(x) \in \mathrm{sgn}(c)\partial l^*(\frac{|c|}{\|x-x_0\|^{N-1}})\frac{x-x_0}{\|x-x_0\|}$$

so that $\|\nabla w_c^{x_0,k}(x)\| \in \partial l^*(\frac{|c|}{\|x-x_0\|^{N-1}})$ and $\frac{|c|}{\|x-x_0\|^{N-1}} \in \partial l(\|\nabla w_c^{x_0,k}(x)\|)$ a.e.. Hence, (4) is the Euler Lagrange equation of problem (1) computed along the function $w_c^{x_0,k}$.

Since $w_c^{x_0,k}$ satisfies the Euler Lagrange equation, the convexity of the functional implies that it is a solution to the problem of minimizing (1). □

The following is our main result.

Theorem 2.1 (Comparison Theorem)**.** *Assume (A).*

i) Let w be a solution to the problem of minimizing (1) such that, on $\partial\Omega$, we have $w \leq w_c^{x_0,k}$ in the sense of $W^{1,1}(\Omega)$. Then, a.e. in Ω, $w(x) \leq w_c^{x_0,k}(x)$.
ii) Assume that, instead, on $\partial\Omega$ we have $w \geq w_c^{x_0,k}$. Then $w(x) \geq w_c^{x_0,k}(x)$ a.e. in Ω.

Proof. Ad i). Set $\eta^+(x) = \max\{0, w(x) - w_c^{x_0,k}(x)\}$ and $E^+ = \{x \in \Omega : \eta^+(x) > 0\}$. The assumption on the boundary data implies that $\eta^+ \in W_0^{1,1}(\Omega)$: we wish to prove that E^+ has measure zero.

a) We have that

$$\nabla(w_c^{x_0,k} + \eta^+)(x) = \begin{cases} \nabla w_c^{x_0,k}(x) & \text{if } w(x) \leq w_c^{x_0,k}(x) \\ \nabla w_c^{x_0,k}(x) + \nabla\eta^+(x) & \text{otherwise} \end{cases}$$

so that

$$\begin{aligned}&\int_\Omega [L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x)) - L(\nabla w_c^{x_0,k}(x))]\,dx\\ &= \int_{E^+}[L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x)) - L(\nabla w_c^{x_0,k}(x))]\,dx.\end{aligned}$$

We claim that

$$\int_{E^+} L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x))\,dx = \int_{E^+} L(\nabla w_c^{x_0,k}(x))\,dx. \tag{5}$$

In fact, it cannot be that

$$\int_{E^+} L(\nabla w(x))\,dx = \int_{E^+} L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x))\,dx < \int_{E^+} L(\nabla w_c^{x_0,k}(x))\,dx$$

since, otherwise,

$$\int_\Omega L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x))\,dx < \int_\Omega L(\nabla w_c^{x_0,k}(x))\,dx$$

and $w_c^{x_0,k}$ would not be a solution. It cannot be, either, that

$$\int_{E^+} L(\nabla w(x))\,dx > \int_{E^+} L(\nabla w_c^{x_0,k}(x))\,dx = \int_{E^+} L(\nabla w(x)-\nabla\eta^+(x))\,dx,$$

since w would not be a solution. Then,

$$\int_{E^+} L(\nabla w(x))\,dx = \int_{E^+} L(\nabla w_c^{x_0,k}(x))\,dx.$$

This proves the claim.

Set $u(x) = \frac{x-x_0}{\|x-x_0\|^N}$: from Proposition 1 we have that $\int_{E^+}\langle u(x),\nabla\eta^+(x)\rangle\,dx = 0$. Thus

$$\int_{E^+} L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x)) - [L(\nabla w_c^{x_0,k}(x)) + \langle cu(x),\nabla\eta^+(x)\rangle]\,dx = 0. \tag{6}$$

On the other hand, since $cu(x)\in\partial L(\nabla w_c^{x_0,k}(x))$ a.e., then

$$\psi(x) \doteq L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x)) - [L(\nabla w_c^{x_0,k}(x)) + \langle cu(x),\nabla\eta^+(x)\rangle] \geq 0 \text{ a.e.}$$

Now ψ is measurable, a.e. non negative and its integral vanishes: it follows that $\psi = 0$ a.e., or equivalently,

$$\begin{cases} L(\nabla w_c^{x_0,k}(x)+\nabla\eta^+(x)) = L(\nabla w_c^{x_0,k}(x)) + \langle cu(x),\nabla\eta^+(x)\rangle \text{ a.e.}\\ cu(x)\in\partial L(\nabla w_c^{x_0,k}(x)) \text{ a.e.}\end{cases} \tag{7}$$

Let x be such that (7) holds: it follows, by convexity that, for every $\lambda\in[0,1]$,

$$L(\nabla w_c^{x_0,k}(x)+\lambda\nabla\eta^+(x)) = L(\nabla w_c^{x_0,k}(x)) + \lambda\langle cu(x),\nabla\eta^+(x)\rangle$$

Hence, with the exception of a subset of E^+ of measure zero, this set independent of λ, the map $\lambda\mapsto L(\nabla w_c^{x_0,k}(x)+\lambda\nabla\eta^+(x))$ is affine on the interval $[0,1]$.

b) Whenever $\nabla\eta^+(x)\neq 0$, we have that

$$\xi_1 \doteq \nabla w_c^{x_0,k}(x) \neq \nabla w_c^{x_0,k}(x)+\nabla\eta^+(x) \doteq \xi_2.$$

From the previous result, on the non-trivial segment $(\xi_1,\xi_2)\subset\mathbb{R}^N$ the map L is affine, i.e., in the $N+1$ dimensional space, the segment joining $(\xi_1, L(\xi_1))$ to $(\xi_2, L(\xi_2))$ is contained in a proper face F of the epigraph of L. Since $L(\xi) = l(\|\xi\|)$ is rotationally symmetric, its epigraph has at most one face of dimension strictly greater than 1: if it exists, it contains $(0, L(0))$ and its slope is 0. Here the slope of

F is $cu(x) \neq 0$, therefore we must have that ξ_2 is parallel to ξ_1: we have obtained that $\nabla\eta^+(x)$ and $\nabla w_c^{x_0,k}(x)$ are parallel.

c) We wish to show that this conclusion implies that $\nabla\eta^+(x) = 0$, except possibly on a set of measure zero.

Consider the one-dimensional faces of the epigraph of the convex function l, corresponding to countably many non trivial intervals $[\alpha_i, \beta_i]$ such that the restriction of l to $[\alpha_i, \beta_i]$ is affine, and, for $t \in (\alpha_i, \beta_i)$, $\partial l(t)$ equals the constant singleton $\{l'_i\}$. Whenever $\nabla\eta^+(x)$ is parallel to $\nabla w_c^{x_0,k}(x)$, we have that

$$\|\nabla w_c^{x_0,k}(x) + \lambda\nabla\eta^+(x)\| - \|\nabla w_c^{x_0,k}(x)\| = \lambda\|\nabla\eta^+(x)\| \operatorname{sign}\langle c(x-x_0), \nabla\eta^+(x)\rangle,$$

so they are distinct if $\nabla\eta^+(x) \neq 0$; hence, in this case, from (7), these values must belong, for some i, to the same interval $[\alpha_i, \beta_i]$, so that

$$\{l'_i\} \in \partial l(\|\nabla w_c^{x_0,k}(x) + \lambda\nabla\eta^+(x)\|) \cap \partial l(\|\nabla w_c^{x_0,k}(x)\|),$$

and we have

$$\begin{aligned} l(\|\nabla w_c^{x_0,k}(x) &+ \lambda\nabla\eta^+(x)\|) \\ &= l(\|\nabla w_c^{x_0,k}(x)\|) + \lambda l'_i\|\nabla\eta^+(x)\| \ \operatorname{sign}\langle c(x-x_0), \nabla\eta^+(x)\rangle. \end{aligned} \tag{8}$$

Since

$$\langle c\frac{x-x_0}{\|x-x_0\|^N}, \nabla\eta^+(x)\rangle = \frac{|c|}{\|x-x_0\|^{N-1}}\|\nabla\eta^+(x)\| \operatorname{sign}\langle c(x-x_0), \nabla\eta^+(x)\rangle, \tag{9}$$

comparing (8) with (7), we obtain

$$\frac{|c|}{\|x-x_0\|^{N-1}} = l'_i.$$

The set of $x \in \mathbb{R}^N$ such that $\frac{|c|}{\|x-x_0\|^{N-1}} = l'_i$, for some $i \in \mathbb{N}$, is of N dimensional measure zero. This shows that, outside a set of measure zero, $\nabla\eta^+(x)$ is null. Now $\eta^+ \in W_0^{1,1}(\Omega)$: it follows that $\eta^+ = 0$ a.e., so that E^+ has measure zero.

Ad ii). Set $\eta^-(x) = \min\{0, w(x) - w_c^{x_0,k}(x)\}$ and $E^- = \{x \in \Omega : \eta^-(x) < 0\}$. Steps a) to c) above do not depend on the boundary values of $w - w_c^{x_0,k}$, so we reach the conclusion that $\eta^- = 0$ on Ω and thus E^- is negligible. □

Examples. 1. When $N = 2$ and $L(\xi) = \frac{1}{2}\|\xi\|^2$, $\partial l^*(p) = p$ and

$$w_c^{x_0,k}(x) = \operatorname{sign}(c)\int_1^{\|x-x_0\|} z^*\Big(\frac{|c|}{s}\Big)\, ds + k = c\log\|x-x_0\| + k.$$

If there exist $r_0 > 0$, $r_1 > 0$, such that for $\|x - x_0\| = r_0$ we have $w(x) \geq c\log(r_0) + k$ and for $\|x-x_0\| = r_1$, $w(x) \geq c\log(r_1) + k$, and w is continuous then, on the annulus centered at x_0 with radii between r_0 and r_1, $w(x) \geq c\log\|x-x_0\| + k$; this is the argument used in the proof of the Strong Maximum Principle.

2. For the Lagrangean

$$L(\xi) = \begin{cases} 0 & |\xi| \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

we obtain $L^*(p) = \|p\|$; $\nabla w_c^{x_0,k}(x) = \pm\frac{x-x_0}{\|x-x_0\|}$ and $w_c^{x_0,k}(x) = \pm\|x-x_0\|$.

3. For
$$L(t) = \begin{cases} 0 & |t| \le 1, \\ \frac{1}{2}(|t|-1)^2 & |t| > 1 \end{cases}$$
whose polar is $L^*(p) = \frac{1}{2}|p|^2 + |p|$ we have
$$\partial L^*(p) = \begin{cases} B(0,1) & p = 0, \\ p + \operatorname{sgn}(p) & p \neq 0. \end{cases} \tag{10}$$
and, for $c > 0$, $N = 2$, we deduce that
$$w_c^{x_0,k}(x) = \int_1^{\|x-x_0\|} z^*\Big(\frac{c}{s}\Big)\, ds = \int_1^{\|x-x_0\|} \Big(\frac{c}{s} + 1\Big)\, ds = c \log \|x - x_0\| + \|x - x_0\|.$$
4. For $L(\xi) = \|\xi\|$, we have
$$l^*(t) = \begin{cases} 0 & |t| \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$
and $w_c^{x_0,k}(x) \equiv k$.

As an application of Theorem 1, consider the case $\mathrm{Dom}(l) = (-\ell, +\ell)$; any function w that makes (1) finite, has to be Lipschitzian with Lipschitz constant ℓ. It is not clear whether there can be pairs of points (x^*, y^*) in the closure of Ω, such that $|w(x^*) - w(y^*)| = \ell \|x^* - y^*\|$. The following result, that was proved in [3] with the additional assumptions of smoothness and of strict convexity of l, shows that this is not possible for w a solution and for special pairs of points.

Theorem 2.2 (A Maximum Principle for extended valued Lagrangeans)**.** *Assume that* $\mathrm{Dom}(l) = (-\ell, +\ell)$*. Let* w *be a solution to the problem of minimizing (1) on* $\phi + W^{1,1}(\Omega)$*, where* ϕ *is Lipschitzian of constant* $\lambda < \ell$*. Then, for no* $x^* \in \Omega$ *there exists* $y^* \in \partial\Omega$ *such that* $|w(x^*) - \phi(y^*)| = \ell \|x^* - y^*\|$*.*

Proof. a) Since $\lim_{t \to \ell} l(t) = +\infty$ then, by convexity, for any selection $z(\cdot)$ from $\partial l(\cdot)$, we have $\lim_{t\to\ell} z(t) = +\infty$. Let z^* be any selection from ∂l^*: we have that $z^*(p)$ tends to ℓ as $p \to +\infty$. Hence, for every x_0 and $c \neq 0$,
$$\lim_{x \to x_0} z^*\Big(\frac{|c|}{\|x - x_0\|^{N-1}}\Big) = \ell.$$

b) Assume that there exist x^* and $y^* \in \partial\Omega$ satisfying $w(x^*) - \phi(y^*) = \ell\|x^* - y^*\|$. Fix $\lambda < \Lambda < \ell$. Let M be such that $z^*(p) > \Lambda$ whenever $p > M$. Let D be the diameter of Ω; choose c so large that $\frac{c}{D^{N-1}} > M$; a fortiori, for $t \le D$, we have
$$z^*\Big(\frac{c}{t^{N-1}}\Big) > \Lambda;$$
notice that $z^* \le \ell$, so that $\int_0^1 z^*\Big(\frac{c}{s^{N-1}}\Big)\, ds \le \ell$, and consider the solution $w_c^{y^*,k}$, as provided by Proposition 1, centered at y^*, with $k = \int_0^1 z^*\Big(\frac{c}{s^{N-1}}\Big)\, ds + \phi(y^*)$, i.e.,
$$w_c^{y^*,k}(x) = \int_1^{\|x-y^*\|} z^*\Big(\frac{c}{s^{N-1}}\Big)\, ds + \int_0^1 z^*\Big(\frac{c}{s^{N-1}}\Big)\, ds + \phi(y^*)$$

$$= \int_0^{\|x-y^*\|} z^*\Big(\frac{c}{s^{N-1}}\Big)\, ds + \phi(y^*).$$

Notice that since $z^* \le \ell$ then $w_c^{y^*,k}$ is defined on $\mathbb{R}^N$; moreover for a.e. y on the ball of center y^* and radius D we have $\dfrac{c}{\|y-y^*\|^{N-1}} \ge \dfrac{c}{D^{N-1}} > M$ and thus

$$\|\nabla w_c^{y^*,k}(x)\| = z^*\Big(\frac{c}{\|y-y^*\|^{N-1}}\Big) \ge \Lambda.$$

Let now $y \in \partial\Omega$: the latter inequality yields that

$$w_c^{y^*,k}(y) - \phi(y^*) = w_c^{y^*,k}(y) - w_c^{y^*,k}(y^*) \ge \Lambda\|y^* - y\|.$$

Since $w(y) - \phi(y^*) = \phi(y) - \phi(y^*) \le \lambda\|y^* - y\|$ we have obtained that, on $\partial\Omega$, one has $w_c^{y^*,k}(y) \ge w(y)$.

On the other hand, at the point x^* one has $w(x^*) - \phi(y^*) = \ell\|y^* - x^*\|$, while, from $w_c^{y^*,k}(x^*) - \phi(y^*) = \int_0^{\|x^*-y^*\|} z^*\Big(\frac{c}{s^{N-1}}\Big)\, ds$ and the inequality $z^* < \ell$, one obtains $w_c^{y^*,k}(x^*) - \phi(y^*) < \ell\|y^* - x^*\|$ or, equivalently,

$$w_c^{y^*,k}(x^*) \le \phi(y^*) + \ell\|y^* - x^*\| = w(x^*).$$

Hence the open set $\{x \in \Omega : w(x) > w_c^{y^*,k}(x)\}$ contains x^*, a contradiction to Theorem 1.

c) Since L is radial then $-w$ is a solution to the problem of minimizing (1) on $-\phi + W_0^{1,1}(\Omega)$. It follows from b) that there are no $x^* \in \Omega$ and $y^* \in \partial\Omega$ such that $(-w)(x^*) - (-\phi)(y^*) = \ell\|x^* - y^*\|$, proving that for such pair of points the equality $\phi(y^*) - w(x^*) = \ell\|x^* - y^*\|$ does not hold. □

3. **A less symmetric case.** In this section we consider an extension of the minimization problem (3) to cover a more general, non rotationally symmeytric, case.

Let C be a closed, bounded and convex subset of $\mathbb{R}^N$ containing the origin in its interior and let C^* be its *polar*, defined by

$$C^* = \{x^* \in \mathbb{R}^N : \forall x \in C,\ \langle x, x^*\rangle \le 1\}.$$

We denote by γ_C the *gauge* of C defined on $\mathbb{R}^N$ by

$$\gamma_C(x) = \inf\{\lambda > 0 : x \in \lambda C\}.$$

We have that the polar $(I_C)^*(x^*) = \inf_{\{x\in C\}}\langle x, x^*\rangle$ of the *indicator function* of C is the gauge of the polar C^* of C, and that (see for instance [5], Corollary 15.1.2) γ_{C^*} is also the *support function* of C, i.e.

$$\gamma_{C^*}(x^*) = \sup\{\langle x, x^*\rangle : x \in C\}. \tag{11}$$

Example 1. Let Q be a symmetric, positive definite $N \times N$ matrix, and let

$$C = \{x \in \mathbb{R}^N : \langle x, Qx\rangle \le 1\}.$$

Then $\gamma_C(\xi) = \langle \xi, Q\xi\rangle^{\frac{1}{2}}$ and $\gamma_{C^*}(x^*) = \langle x^*, Q^{-1}x^*\rangle^{1/2}$.

Let $l : [0, +\infty[\to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous function that attains its minimum at 0 and consider the minimization problem (2). In particular, in the case where Q is the identity matrix, we have that $l(\gamma_C(\xi)) = l(\|\xi\|)$, and problem (2) reduces to problem (1).

Let $z^*(t)$ be any measurable selection from $\partial l^*(t)$. For $x_0 \notin \Omega$, $c \neq 0$ and $k \in \mathbb{R}$, set

$$w_c^{x_0,k}(x) = \operatorname{sgn}(c) \int_1^{\gamma_{C^*}(x-x_0)} z^*\Big(\frac{|c|}{s^{N-1}}\Big)\, ds + k. \tag{12}$$

In the case where Q is the identity matrix, the function $w_c^{x_0,k}$ defined in (12) coincides with the function defined in Section 2. We will show that the maps $w_c^{x_0,k}$ are minimizers of (2) that satisfy the Comparison Principle stated in Theorem 1. Hence, the results of this section will contain the results of Section 2.

The following is the analog of Proposition 1 in this more general setting.

Proposition 2. *Let $x_0 \notin \Omega$ and C be strictly convex. The field*

$$u(x) = \frac{x - x_0}{(\gamma_{C^*}(x-x_0))^N}$$

is divergence free.

Proof. The assumption that C is strictly convex implies that $x \mapsto \gamma_{C^*}(x-x_0)$ does not vanish and that is differentiable for every $x \neq x_0$. We have

$$\frac{\partial u_i}{\partial x_i} = \frac{(\gamma_{C^*}(x-x_0))^N - N(\gamma_{C^*}(x-x_0))^{N-1}(x_i - (x_0)_i)\frac{\partial}{\partial x_i}\gamma_{C^*}(x-x_0)}{(\gamma_{C^*}(x-x_0))^{2N}}$$

so that

$$\operatorname{div}(u(x)) = N(\gamma_{C^*}(x-x_0))^{N-1}\frac{\gamma_{C^*}(x-x_0) - \sum_{i=1}^N (x_i - (x_0)_i)\frac{\partial}{\partial x_i}\gamma_{C^*}(x-x_0)}{(\gamma_{C^*}(x-x_0))^{2N}} = 0$$

since γ_{C^*} is positively homogeneous of degree 1. □

Proposition 2 and the arguments of the previous section yield the following analog of Theorem 1.

Theorem 3.1. *Assume (A) and let C be strictly convex. Then $w_c^{x_0,k}$ is a well defined Lipschitz solution to the problem of minimizing (1). Moreover $w_c^{x_0,k}$ satisfies the Comparison Principle stated in Theorem 1.*

Proof. We sketch the proof, which is similar to that of Proposition 1 and Theorem 1, by just mentioning the technical arguments that are new. The chain rule gives, for a.e. x,

$$\nabla w_c^{x_0,k}(x) = \operatorname{sgn}(c)\, z^*\Big(\frac{|c|}{(\gamma_{C^*}(x-x_0))^{N-1}}\Big)\nabla\gamma_{C^*}(x-x_0)$$

Since γ_{C^*} is positively homogeneous of degree 1, we have

$$\frac{|c|}{(\gamma_{C^*}(x-x_0))^{N-1}} = \gamma_{C^*}(x-x_0)\frac{|c|}{(\gamma_{C^*}(x-x_0))^N} = \gamma_{C^*}\Big(|c|\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big)$$

and

$$\nabla\gamma_{C^*}(x-x_0) = \nabla\gamma_{C^*}\Big(|c|\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big)$$

so that

$$\nabla w_c^{x_0,k}(x) = \operatorname{sgn}(c)\, z^*\Big(\gamma_{C^*}\Big(|c|\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big)\Big)\,\nabla\gamma_{C^*}\Big(|c|\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big).$$

The differentiability of γ_{C^*} and [5, Theorem 15.3] give, for $p \in \mathbb{R}^N \setminus \{0\}$,

$$\partial L^*(p) = \partial l^*(\gamma_{C^*}(p))\nabla\gamma_{C^*}(p) \tag{13}$$

proving that

$$\nabla w_c^{x_0,k}(x) \in \partial L^*\Big(c\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big) \text{ a.e.}$$

By Lemma 1, the arguments of Proposition 1 show then that $w_c^{x_0,k}$ is a minimizer of (1).

By (13) the faces of the epigraph of L with slope different from 0 are at most one–dimensional. To conclude that $w_c^{x_0,k}$ satisfies the Comparison Principle, the proof of Theorem 1 shows that it is enough to show that

$$\partial l^*\Big(\gamma_{C^*}\Big(c\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big)\Big)$$

is a.e. reduced to a point, or equivalently that the sets

$$\Big\{x \in \mathbb{R}^N : \qquad \gamma_{C^*}\Big(c\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big) = l_i'\Big\}$$

are negligible for every i. This occurs since, again by homogeneity,

$$\gamma_{C^*}\Big(c\frac{x-x_0}{(\gamma_{C^*}(x-x_0))^N}\Big) = |c|\frac{1}{(\gamma_{C^*}(x-x_0))^{N-1}}$$

and moreover, for every $\lambda > 0$,

$$\gamma_{C^*}(x-x_0) = \lambda \iff x \in x_0 + \lambda\partial C^*$$

where ∂C^* is the boundary of C^*, a negligible set. □

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