CONTINUITY PROPERTIES OF SOLUTIONS TO SOME DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We consider a nonlinear (possibly) degenerate elliptic operator $Lv = -\operatorname{div} a(\nabla v) + b(x, v)$ where the functions a and b are, unnecessarly strictly, monotonic. For a given boundary datum ϕ we prove the existence of the maximum and the minimum of the solutions and formulate a Haar-Rado type result, namely a continuity property for these solutions that may follow from the continuity of ϕ. In the homogeneous case we formulate a generalization of the Bounded Slope Condition and use it to obtain the existence of solutions to $Lv = 0$ that are Lipschitz, or locally Lipschitz, or Hölder upon the behavior of ϕ.

Introduction

A famous result due to Hartman and Stampacchia in 1966 [6] shows the existence of a globally Lipschitz solution to the equation

$$-\operatorname{div} a(\nabla v) + F(u) = 0 \text{ on } \Omega, \quad u = \phi \text{ on } \partial\Omega \tag{0.1}$$

when the boundary datum ϕ satisfies the *Bounded Slope Condition* (*BSC*). It is required moreover that $a : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the ellipticity condition

$$\forall \xi, \eta \in \mathbb{R}^n \qquad (a(\xi) - a(\eta)) \cdot (\xi - \eta) \geq \mu|\xi - \eta|^2 \qquad (\mu > 0) \tag{0.2}$$

and F satisfies various technical assumptions that we omit here. In the case where $F = 0$ the result is obtained even under the weaker assumption that a is just (unnecessarily strictly) monotonic. Few years before under the same assumption on the boundary datum Stampacchia obtained in [?] the existence of a minimizer of an integral functional among Lipschitz functions. After many years this result became a source of inspiration for some new results in the Calculus of Variations concerning the regularity of the minimizers for the problem

$$\min \int_\Omega f(\nabla v(x))\, dx : \qquad u \in \phi + W_0^{1,1}(\Omega). \tag{P}$$

We mention Cellina who first revisited their paper in this framework and established in [3] the *Lipschitz continuity* of the minimizers of (P) when ϕ satisfies the (BSC); Clarke in [5] introduced the new one sided *Lower/Upper* (BSC) and obtained under this condition the *local Lipschitz* continuity of the solutions to (P) by assuming moreover that Ω is convex. In both cases the lagrangian f was supposed to be strictly convex due mostly to the lack of the validity of the Comparison Principles, a key tool, when the epigraph of f has some non trivial flat faces. The methods developed by these authors allowed Bousquet to prove in [2] the continuity of the minimizers for a continuous boundary datum, and us to establish in [10] the *global Hölder continuity* of the minimizers of (P) once ϕ is Lipschitz and f is coercive; there we were also able to drop the usual strict convexity assumption on the lagrangian.

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Following the same path of Hartman and Stampacchia these latter results obtained in the framework of the Calculus of Variations are now giving some new existence theorems in the framework of the Partial Differential Equations (PDE's in the rest of the paper). Bousquet considered the very same operator studied in [6] and obtained in [1] the existence of a solution to (0.1) among locally Lipschitz functions if ϕ satisfies Clarke's unilateral (BSC) and a is uniformly strictly monotonic.

Our purpose is to study the existence of regular solutions to 0.1 when a is not uniformly strictly monotonic. As in [6] and [1] we obtain the results by using the regularity of the boundary datum instead of the classical upper bounds on the growth of a. More precisely we are concerned with the problem

$$Lv \doteq -\operatorname{div} a(\nabla v) + b(x, v) = 0 \text{ on } \Omega \subset \mathbb{R}^n, \quad u = \phi \text{ on } \partial\Omega$$

where a, differently from [1], is *not* supposed to be a strictly monotone operator, namely we require that

$$\forall \xi, \eta \in \mathbb{R}^n \qquad (a(\eta) - a(\xi)) \cdot (\eta - \xi) \geq 0. \tag{0.3}$$

We assume moreover that $u \mapsto b(x, u)$ is monotonic and that either the equality (0.3) implies that $a(\xi) = a(\eta)$ or that $u \mapsto b(x, u)$ is strictly monotonic. This condition is fulfilled for instance if a is the gradient of a convex, $\mathcal{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$ such that the non trivial maximal faces of the epigraph of f have dimension n, a common fact for the problems of the Calculus of Variations arising from the convexification of a non convex lagrangian. The main difficulties here are the non uniqueness of the solutions to the Dirichlet problem and the fact that affine functions do no more satisfy, in general, the Comparison Principles from above or from below.

In the first parts of the paper we thoroughly study the set of the solutions, no more unique, to the Dirichlet problem associated to $Lv = 0$ and show in particular under some natural growth condition on a and b that, given a boundary datum ϕ, there is a maximal and a minimal solution to the Dirichlet problem $Lv = 0, v = \phi$ on $\partial\Omega$. We formulate some Comparison Principles that, as well as we know, are new in the case where a is not strictly monotonic and we exhibit, in the homogeneous case, a new class of solutions to $Lv = 0$, depending on the level sets of a, that satisfy the Comparison Principle.

We then establish the fact that if ω is any modulus of continuity and u is the maximum or the minimum of the solutions to $Lv = 0$ such that

$$\forall \gamma \in \partial\Omega \qquad |u(x) - \phi(\gamma)| \leq \omega(|x - \gamma|) \text{ a.e. } x$$

then $|u(y) - u(x)| \leq \omega(|y - x|)$ for a.e. x, y. More precisely the monotonicity of b is enough if b does not depend on x, otherwise we require a more general monotonic assumption on the two variables x, u. This result is well known when u belongs to the class of a-harmonic functions, the solutions to the equations that are narrowly similar to the p-Laplace equation, where the function a needs to satisfy a homogeneity assumption that we do not make here. The result is also the PDE's counterpart of the so called Haar-Rado theorem for the Lipschitz minimizers of (P) that we recently extended in [12].

The results of these parts, though similar to the corresponding ones in the Calculus of Variations, require some arguments that are new and strictly related to the structure of the partial differential operator.

In the last part of the paper we extend the (BSC) by using the new class of functions described above instead of the affine ones: it turns out that in the case where a is not strictly monotonic, the class of the boundary data that satisfy this new *Generalized* (BSC) is wider than the class of functions that satisfy the

(BSC). To clarify this statement we just mention some facts that differentiate them. The (BSC) is a quite restrictive condition: among other properties that forces the domain to be convex. On the other hand once a level set of a contains a ball centered in the origin it turns out that every Lipschitz function of a suitable rank satisfies the Generalized (BSC) with no convexity requirement on the domain Ω.

We then apply the Haar-Rado type theorem in the homogeneous case i.e. $Lv = -\operatorname{div} a(\nabla v)$. Again, without assuming the strict monotonicity of a we prove the analogue of the results of the Calculus of Variations mentioned above: existence of a Lipschitz solution when ϕ satisfies the Generalized (BSC) and, if Ω is convex, of a locally Lipschitz solution when ϕ satisfies a unilateral Generalized (BSC), and of a Hölder solution when ϕ is Lipschitz. The techniques are here similar to those of the Calculus of Variations since most of them rely on the Comparison Principles.

1. Notation and setting

If v and w are functions then $v \wedge w$ (resp. $v \vee w$) stands for the pointwise minimum (resp. maximum) of v and w. The scalar product in $\mathbb{R}^n$ is denoted by "$\cdot$".

Definition 1.1 (Modulus of continuity)**.** A modulus of continuity is a positive continuous function $\omega : [0, +\infty[$ such that $\omega(0) = 0$. A real valued function ϕ on a set X is ω–continuous if $|\phi(y) - \phi(x)| \leq \omega(|y - x|)$ for all $x, y \in X$.

Definition 1.2 (Inequalities in the trace sense)**.** Let $u, v \in W^{1,1}(D)$. We say that $u \leq v$ in ∂D *in the trace sense* if $u \wedge v \in u + W_0^{1,1}(D)$ or, equivalently, if $u \vee v$ is in $v + W_0^{1,1}(D)$.

Some basic facts about inequalities in the trace sense can be found in [12].

We consider here the following operator in divergence form

$$Lv = -\operatorname{div} a(\nabla v) + b(x, v).$$

Throughout the paper we will make use of the following assumptions.

Basic Assumptions. We assume that $Lv = -\operatorname{div} a(\nabla v) + b(x, v)$ is such that

A_1) the function $a : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotonic, i.e.

$$\forall \xi, \eta \in \mathbb{R}^n \qquad (a(\eta) - a(\xi)) \cdot (\eta - \xi) \geq 0. \tag{1.1}$$

Moreover

$$\forall \xi, \eta \in \mathbb{R}^n \quad (a(\eta) - a(\xi)) \cdot (\eta - \xi) = 0 \iff a(\xi) = a(\eta); \tag{1.2}$$

B_1) the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function that is continuous in the second variable and

$$\forall x \in \Omega \quad \forall u, v \in \mathbb{R} \qquad (b(x, v) - b(x, u))(v - u) \geq 0. \tag{1.3}$$

Remark 1.1. Under just A_1) and B_1) the solutions to $Lv = 0$ with a prescribed boundary datum may not be unique. Such solutions turn out to be unique if either (1.1) or (1.3) is strict, i.e. either L is elliptic or $u \mapsto b(x, u)$ is strictly monotonic for each x.

Remark 1.2. Condition A_1) is fulfilled if, for instance, a is the gradient of a convex, $\mathcal{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$ such that the non trivial maximal faces of its epigraph have dimension n. Indeed if $(\nabla f(\eta) - \nabla f(\xi)) \cdot (\eta - \xi) = 0$ then f is affine on the segment $[\xi, \eta]$ so that the graph of f contains the segment joining $(\xi, f(\xi))$ to $(\eta, f(\eta))$. The assumption on the non trivial faces of the graph of f implies that the segment $[\xi, \eta]$ belongs to the closure $\overline{E}$ of an open set E where f is affine, say $f(x) = k \cdot x + d$ for some $k \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Thus $\nabla f = k$ on $\overline{E}$ and in particular $\nabla f(\xi) = \nabla f(\eta) = k$.

2. Basic tools

In this section we present the basic tools to prove our regularity results and a Haar-Rado type theorem. Most of them are a reformulation in the PDE's framework of some analogous result that we proved for minimizers of integral functionals in [12]. However their proofs are not a straightforward modification of the variational ones and need some peculiar techniques that we develope here.

2.1. **Comparison Principles.** We recall the notion of sub/supersolution to the PDE $Lv = 0$. Here $p \geq 1$ and q is the conjugate exponent of p.

Definition 2.1. Let $u \in W^{1,p}(\Omega)$ be such that $a(\nabla u), b(x, u(x)) \in L^q(\Omega)$. We say that u is a *subsolution* to $Lv = 0$ (we write that $Lu \leq 0$) in Ω if

$$\forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geq 0 \text{ a.e.} \qquad \int_\Omega a(\nabla u(x))\nabla\varphi(x) + b(x, u(x))\varphi(x)\, dx \leq 0\,; \quad (2.1)$$

u is a *supersolution* to $Lv = 0$ (we write that $Lu \geq 0$) in Ω if

$$\forall \varphi \in W_0^{1,p}(\Omega),\ \varphi \geq 0 \text{ a.e.} \qquad \int_\Omega a(\nabla u(x))\nabla\varphi(x) + b(x, u(x))\varphi(x)\, dx \geq 0. \quad (2.2)$$

Finally, u is a *solution* to $Lv = 0$ (i.e. $Lu = 0$) in Ω if

$$\forall \varphi \in W_0^{1,p}(\Omega), \qquad \int_\Omega a(\nabla u(x))\nabla\varphi(x) + b(x, u(x))\varphi(x)\, dx = 0. \quad (2.3)$$

A subsolution (resp. supersolution) to $Lv = 0$ is said to be *strict* if the inequality (2.1) (resp. (2.2)) is strict whenever φ is non zero.

The next Lemma is a key tool in the proof of the subsequent Comparison Principle.

Lemma 2.1. *Assume that L satisfies A_1) and B_1). Let u, w be such that $Lu \leq 0$, $Lw \geq 0$ and $u \leq w$ on $\partial\Omega$. The the following statements hold*

a) $a(\nabla u) = a(\nabla w)$ *and* $b(x, u(x)) = b(x, w(x))$ *a.e. on the set*

$$\Sigma = \{x \in \Omega : u(x) > w(x)\};$$

b) $L(u \wedge w) \leq 0$ *and* $L(u \vee w) \geq 0$*;*

c) $L(u \wedge w) = 0$ *if* $Lu = 0$ *and* $L(u \vee w) = 0$ *if* $Lw = 0$.

Proof. a) Let $\Sigma = \{x \in \Omega : u(x) > w(x)\}$. Since, by taking $(u - w)^+$ as a test function

$$\int_\Omega a(\nabla u) \cdot \nabla(u - w)^+ + b(x, u)(u - w)^+\, dx \leq 0$$

and

$$\int_\Omega a(\nabla w) \cdot \nabla(u - w)^+ + b(x, w)(u - w)^+\, dx \geq 0$$

then

$$\int_\Sigma (a(\nabla u) - a(\nabla w)) \cdot (\nabla u - \nabla w) + (b(x, u) - b(x, w)(u - w)\, dx \leq 0$$

so that A_1) and B_1) imply that

$$(a(\nabla u) - a(\nabla w)) \cdot (\nabla u - \nabla w) = 0, \quad (b(x, u) - b(x, w))(u - w) = 0 \text{ a.e. on } \Sigma :$$

again A_1) and B_1) yield $a(\nabla u) = a(\nabla w)$ and $b(x, u) = b(x, w)$ a.e. on Σ.

We show now that $u \vee w$ is a supersolution to $Lv = 0$. If $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$ a.e. then by a) we have

$$\begin{aligned}
&\int_\Omega a(\nabla(u \vee w)) \cdot \nabla\varphi + b(x, u \vee w)\varphi\, dx \\
&\quad = \int_{u \leq w} a(\nabla w) \cdot \nabla\varphi + b(x, w)\varphi\, dx + \int_\Sigma a(\nabla u) \cdot \nabla\varphi + b(x, u)\varphi\, dx \\
&\quad = \int_{u \leq w} a(\nabla w) \cdot \nabla\varphi + b(x, w)\varphi\, dx + \int_\Sigma a(\nabla w) \cdot \nabla\varphi + b(x, w)\varphi\, dx \\
&\quad = \int_\Omega a(\nabla w) \cdot \nabla\varphi + b(x, w)\varphi\, dx \geq 0.
\end{aligned}$$

Notice that the last inequality is actually a equality if $Lw = 0$, proving the parts of claims b) and c) concerning $u \vee w$. The statements concerning $u \wedge w$ follow similarly. □

In the next Comparison Principle we give some conditions the ensure that $u \leq w$ a.e. on Ω once $Lu \leq 0$, $Lw \geq 0$ and $u \leq w$ on $\partial\Omega$. The first of these conditions is the strict monotonicity of a or b: in case *i*) the conclusion is well known and we write it here just for the convenience of the reader. We underline that in this case the basic point is the uniqueness of the solutions to $Lw = 0$ for a prescribed boundary datum. In the general case, without assuming *i*), the solutions to $Lv = 0$ may not be unique and it may happen that the Comparison Principle does not hold for arbitrary solutions to $Lv = 0$. An example of this situation can be found in [4] in a variational setting. The new fact here is that we take into account the case where solutions to $Lv = 0$ are not unique. Namely in the case *iii*) of Theorem 2.1the function that are involved in the Comparison Principle are some special solutions to $Lv = 0$: the maximum and/or the minimum one for a prescribed boundary datum.

Definition 2.2. We say that $u \in W^{1,p}(\Omega)$ is the *maximum* (resp. *minimum*) of the solutions to $Lv = 0$ if $Lu = 0$ and $v \leq u$ (resp. $v \geq u$) a.e. for every $v \in u + W_0^{1,p}(\Omega)$ satisfying $Lv = 0$.

We underline that these solutions do both trivially exist in the case where the solutions to $Lv = 0$ are unique. We will show in § 3 that they still exist if L satisfies some suitable growth conditions.

Theorem 2.1 (Comparison Principle for extremal solutions)**.** *Assume that L satisfies* A_1) and B_1). Let u, w be such that $Lu \leq 0$, $Lw \geq 0$ and $u \leq w$ on $\partial\Omega$. Assume moreover that one of the following assumptions holds:

i) Either the function a is strictly monotonic or the function $u \mapsto b(x, u)$ is strictly monotonic for a. e. x;
ii) u is a strict subsolution or w is a strict supersolution to $Lv = 0$;
iii) u is the minimum of the solutions or w is the maximum of the solutions to $Lv = 0$.

Then $u \leq w$ a.e. on Ω.

Proof. i) Claim a) of Lemma 2.1 implies that $a(\nabla u) = a(\nabla w)$ and $b(x, u(x)) = b(x, w(x))$ a.e. on $\Sigma = \{x \in \Omega : u(x) > w(x)\}$. The strict monotonicity of a (resp. of $b(x, \cdot)$) implies that $\nabla u = \nabla w$ (resp. $u = w$) a.e. on Σ: in both cases we obtain that $(u - w)^+ = 0$ a.e. on Ω.

ii) Assume that $Lu < 0$ and that by contradiction that $u > w$ on a non negligible set Σ. Then by taking $(u - w)^+$ as a test function we get

$$\int_\Sigma a(\nabla u) \cdot \nabla(u - w)^+ + b(x, u)(u - w)^+\, dx < 0.$$

Now Lemma 2.1 implies that $a(\nabla u) = a(\nabla w)$ and $b(x, u(x)) = b(x, w(x))$ a.e. on $\Sigma = \{x \in \Omega : \, u(x) > w(x)\}$ and thus

$$\int_\Omega a(\nabla w) \cdot \nabla (u-w)^+ + b(x,w)(u-w)^+ \, dx < 0,$$

contradicting the fact that $Lw \geq 0$; it follows that $u \leq w$ a.e.. The case where w is a strict supersolution follows similarly.

iii) Assume that u is the minimum of the solutions to $Lv = 0$. Since by c) of Lemma 2.1 $u \wedge w$ is still a solution to $Lu = 0$ then the minimality of u yields $u \leq u \wedge w$ so that $u \leq w$ a.e.. The case where w is the maximum of the solutions to $Lu = 0$ follows similarly. □

2.2. **Behavior of solutions with respect to translations.** Let now ω be any positive modulus of continuity. We consider the following monotonicity assumption on b, more general then B_1).
Assumption B_2**)**.

$$\forall x, y \in \mathbb{R}^n, \, \forall u, v \in \mathbb{R} \quad v \geq u + \omega(|y-x|) \Rightarrow b(y,v) \geq b(x,u).$$

Remark 2.1. Notice that B_2) implies B_1): indeed if $x \in \Omega$ then $\omega(0) = 0 = x - x$ so that if $v \geq u$ then $v \geq u + \omega(0)$ and the validity of B_2) yields $b(x,v) \geq b(x,u)$. Moreover if $b(x,u) = b(u)$ does not depend on x then B_2) is fulfilled if and only if b is increasing.

The following theorem states that, in the case we are considering, the property of being a subsolution or supersolutions is preserved under suitable translations.

Theorem 2.2. *Let ω be a modulus of continuity, $h \in \mathbb{R}^n$ and assume that L satisfies* A_1) *and* B_1)*. Let u be a subsolution of $Lv = 0$. Then $u(y-h) - \omega(|h|)$ is a subsolution of $Lv = 0$ on $h + \Omega$. Analogously, if u is a supersolution of $Lv = 0$ on Ω then $u(y+h) + \omega(|h|)$ is a supersolution of $Lv = 0$ on $h + \Omega$.*

Proof. Let u be a subsolution of $Lv = 0$ and set $c = \omega(|h|)$, $w(y) = u(y-h) - c$. Let $\varphi \in W_0^{1,2}(h + \Omega)$ be positive a.e.; the change of variables $y = x + h$ yields

$$I \doteq \int_{h+\Omega} a(\nabla w(y)) \cdot \nabla \varphi(y) + b(y, w(y))\varphi(y) \, dy =$$
$$= \int_\Omega a(\nabla u(x)) \cdot \nabla \psi(x) + b(x+h, u(x) - c)\psi(x) \, dx$$

where we set $\psi(x) = \varphi(x+h)$, a function of $W_0^{1,p}(\Omega)$. Therefore $I = \Xi + \Pi$ with

$$\Xi = \int_\Omega a(\nabla u(x)) \cdot \nabla \psi(x) + b(x, u(x))\psi(x) \, dx$$
$$\Pi = \int_\Omega (b(x+h, u(x) - c) - b(x, u(x)))\psi(x) \, dx.$$

Now $\Xi \leq 0$ since u is a subsolution of $Lv = 0$; moreover since $u(x) - (u(x) - c) = c \geq c$ then B_2) implies that $b(x, u(x)) \geq b(x+h, u(x) - c)$ so that $\Pi \leq 0$: it follows that $I \leq 0$, i.e. $w(y) = u(y-h) - \omega(|h|)$ is a subsolution. The part of the claim concerning supersolutions follows similarly. □

2.3. **A Haar-Rado type theorem.** The next result is in the flavor of well know properties that hold both in the Calculus of Variations and in the PDE's setting. In the first case it is known as Haar-Rado theorem and it holds for Lipschitz minimizers of strictly convex functionals of the gradient, whereas for differential equation can be found in [7, Lemma 6.47] for a-harmonic functions. The proof there is based on the particular structure of the operator (strict monotonicity and homogeneity in the gradient variable) that allows the use of Harnack inequality in the interior of the domain. Our approach is based on the validity of the Comparison Principles stated before. Our proof is directly inspired by our recent generalization of Haar-Rado theorem in the Calculus of Variations [12]; we give it here for the sake of completeness.

Theorem 2.3 (Haar-Rado type). *Let $\omega : [0,+\infty[\to [0,+\infty[$ be a modulus of continuity, ϕ be a function in $W^{1,p}(\Omega)$ that is ω-continuous on $\overline{\Omega}$. Let L satisfy* A$_1$) *and* B$_1$) *and u be the maximum or the minimum of the solutions to $Lv = 0$ on $\phi + W_0^{1,p}(\Omega)$. Assume moreover that one the following assumptions holds*

H$_1$) *$u, \phi \in \mathcal{C}(\overline{\Omega})$ and*

$$\forall \gamma \in \partial\Omega, \quad \forall x \in \Omega \qquad |u(x) - \phi(\gamma)| \le \omega(|x-\gamma|); \tag{2.4}$$

H$_2$) *$\Omega \cap (h + \Omega)$ is regular for all $h \in \mathbb{R}^n$; moreover*

$$\forall \gamma \in \partial\Omega \qquad |u(x) - \phi(\gamma)| \le \omega(|x-\gamma|) \quad \text{a.e. } x; \tag{2.5}$$

H$_3$) *there exist $\ell_1, \ell_2 \in \phi + W_0^{1,1}(\Omega)$ that are ω-continuous on $\overline{\Omega}$ and such that*

$$\ell_1(x) \le u(x) \le \ell_2(x) \text{ a.e. on } \Omega. \tag{2.6}$$

Then $|u(y) - u(x)| \le \omega(|y - x|)$ for every Lebesgue points x and y of u.

Proof. Assume that u is the maximum of the solutions to $Lv = 0$. Fix h in $\mathbb{R}^n$: we know from [12, Lemma 4.1] that $u_h - \omega(|h|) \le u$ on $\partial(\Omega \cap \Omega_h)$ in the trace sense. By Theorem 2.2 $u_h - \omega(|h|)$ is a subsolution of $Lv = 0$ on Ω_h and thus on $\Omega \cap \Omega_h$ whereas u is still the maximum of the solutions to $Lv = 0$ on $\Omega \cap \Omega_h$. The Comparison Principle (Theorem 2.1) implies that $u_h - \omega(|h|) \le u$ a.e. on $\Omega \cap \Omega_h$. Now let x, y be two Lebesgue points of u and let $r > 0$ be such that $B_r(x)$ and $B_r(y)$ are contained in Ω. Let $h = y - x$; since $u(z + h) \le u(z) + \omega(|h|)$ for a.e. $z \in B_r(x)$, it turns out by integration on balls of radius r and then passing to the limit as r tends to 0 that $u(y) - u(x) \le \omega(|y - x|)$; proving the claim. The case where u is the minimum of the solutions follows similarly. □

3. The set of solutions to $Lv = 0$

In this section we assume $p > 1$ and we posit that L satisfies the Basic Assumptions A$_1$) and B$_1$) and moreover, the following *growth condition*.

Definition 3.1. For $a : \mathbb{R}^n \to \mathbb{R}^n$ a *level set for a* is a set of the form

$$F_\xi = \{\eta \in \mathbb{R}^n : a(\eta) = a(\xi)\}$$

for some $\xi \in \mathbb{R}^n$.

Growth Assumption G). The functions a, b satisfy one of the following conditions. Either

G$_1$) The level sets of a are uniformly bounded, i.e. there is $K > 0$ such that $\operatorname{diam} F_\xi \le K$ for all $\xi \in \mathbb{R}^n$ and there exists a modulus of continuity τ such that $|b(x,v) - b(x,u)| \le \tau(|v-u|)$ for a.e. x and every $u, v \in \mathbb{R}$; or

G$_2$)

$$\forall \xi \in \mathbb{R}^n \qquad a(\xi)\cdot\xi \ge \alpha|\xi|^p, \quad |a(\xi)| \le \beta|\xi|^{p-1} + r$$
$$\forall u \in \mathbb{R} \qquad |b(x,u)| \le \gamma|u|^{p-1} + s(x) \text{ a.e. in } \Omega$$

for some $\alpha, \beta, \gamma > 0$, $r \ge 0$ and $s \in L^q(\Omega)$.

Remark 3.1. Conditions G$_1$) and G$_2$) are independent.??

Lemma 3.1. *Let F be a* level set *for a. F is closed and moreover*

a) *if a satisfies* A$_1$) *then F is convex;*

b) *if a Assumption* G_2) *then*

$$\forall \eta, \xi \in F \qquad \alpha|\eta|^{p-1} \leq \beta|\xi|^{p-1} + r.$$

Proof. a) Let $\eta, \xi \in F$ and $\zeta = \lambda\xi + (1-\lambda)\eta$ for some $\lambda \in]0,1[$. Now

$$0 \leq (a(\zeta) - a(\xi)) \cdot (\zeta - \xi) = (1-\lambda)(a(\zeta) - a(\xi)) \cdot (\eta - \xi)$$

$$0 \leq (a(\zeta) - a(\eta)) \cdot (\zeta - \eta) = \lambda(a(\zeta) - a(\xi)) \cdot (\xi - \eta)$$

Since $a(\xi) = a(\eta)$ it follows that $(a(\zeta) - a(\xi)) \cdot (\zeta - \xi) = 0$: the monotonicity assumption A_1) yields the conclusion. b) The claim follows directly from the inequalities

$$\alpha|\eta|^p \leq a(\eta) \cdot \eta = a(\xi) \cdot \eta \leq (\beta|\xi|^{p-1} + r)|\eta|.$$

□

Theorem 3.1. *Assume that* L *satisfies* A_1), B_1) *and* G). *Let* ϕ *in* $W^{1,p}(\Omega)$ *and assume that the set of solutions* X *to* $Lv = 0$ *in* $\phi + W_0^{1,p}(\Omega)$ *is non empty. The following assumptions hold:*

a) *Let* $u \in X$ *and* $v \in \phi + W_0^{1,p}(\Omega)$. *Then* $v \in X$ *if and only if* $a(\nabla v) = a(\nabla u)$ *and* $b(x,u) = b(x,v)$ *a.e. in* Ω*;*
b) *if* $u, v \in X$ *then* $u \vee v, u \wedge v \in X$*;*
c) X *is convex;*
d) X *is weakly compact.*

Proof. a) It is obvious that if $a(\nabla v) = a(\nabla u)$ and $b(x,u) = b(x,v)$ a.e. in Ω then $Lv = Lu = 0$. Conversely assume that $v \in X$; by taking $\varphi = (v-u)^+$ as a test function we obtain that

$$\int_{\{v \geq u\}} (a(\nabla v) - a(\nabla u)) \cdot (\nabla v - \nabla u) + (b(x,v) - b(x,u))(v-u)\, dx = 0$$

whereas, by taking $\varphi = (u-v)^+$ as a test function we obtain that

$$\int_{\{v \leq u\}} (a(\nabla v) - a(\nabla u)) \cdot (\nabla v - \nabla u) + (b(x,v) - b(x,u))(v-u)\, dx = 0$$

and thus

$$\int_{\Omega} (a(\nabla v) - a(\nabla u)) \cdot (\nabla v - \nabla u) + (b(x,v) - b(x,u))(v-u)\, dx = 0.$$

Since from A_1) and B_1) the integrand is positive then

$$(a(\nabla v) - a(\nabla u)) \cdot (\nabla v - \nabla u) = 0 \quad (b(x,v) - b(x,u))(v-u) = 0 \text{ a.e.}$$

it follows that $a(\nabla u) = a(\nabla v)$ and $b(x,v) = b(x,u)$ a.e.. b) If $u, v \in X$ then $a(\nabla u) = a(\nabla v)$ and $b(x,u) = b(x,v)$ a.e. so that $a(\nabla(u \wedge v)) = a(\nabla(u \vee v)) = a(\nabla u)$ and analogously $b(x, u \wedge v) = b(x, u \vee v) = b(x,u)$ a.e.: a) implies that both $u \wedge v$, $u \vee v \in X$. c) Let $u, v \in X$. Then by a) we have $a(\nabla u) = a(\nabla v)$ and $b(x,u) = b(x,v)$ a.e.. It follows from Lemma 3.1 a) that if $w = \lambda u + (1-\lambda)v$ for some $\lambda \in [0,1]$ then $a(\nabla w) = a(\nabla u)$ a.e.; from the monotonicity of $b(x,\cdot)$ we infer that $b(x, \lambda u + (1-\lambda)v) = b(x,u)$: a) yields the claim. d) Assume first that G_1) holds. Let $(u_k)_k$ be a sequence in X. Then by a) $a(\nabla u_k) = a(\nabla u_1)$. Since the level sets are uniformly bounded there is $K > 0$ such that $|\nabla(u_k(x) - u_1(x))| \leq K$ a.e. for every k. It follows that the sequence $(u_k - u_1)_k$ is weakly* precompact in $W^{1,\infty}(\Omega)$ so that there exists a subsequence weakly converging to a function v in $W^{1,1}(\Omega)$: set $u = u_1 + v$. Since $\nabla u_k \rightharpoonup \nabla u$ in $L^1(\Omega)$, by Mazur's Lemma there is a sequence $(\nabla w_k)_k$ of convex combination of ∇u_k that converges strongly to ∇u in $L^1(\Omega)$. Notice that since the level sets of a are convex and $\nabla u_k(x) \in F_{\nabla u_1(x)}$ a.e. then $\nabla w_k(x) \in F_{\nabla u_1(x)}$ a.e. for every k. Modulo a subsequence we may assume that ∇w_k converges a.e. to ∇u. At every point x of convergence we thus have that

$\nabla u(x) \in F_{\nabla u_1(x)}$ a.e. or equivalently $a(\nabla u(x)) = a(\nabla u_1(x))$ a.e.. Moreover the uniform continuity of $b(x,\cdot)$ implies that $b(x,u_k(x)) \to b(x,u(x))$ a.e.. It follows then by a) that $u \in X$ proving that the sequence $(u_k)_k$ has a subsequence weakly converging in X.

Assume now that G_2) holds. The set X is closed in $W^{1,p}(\Omega)$: indeed if $u_k \in X$ and $u_k \to u$ in $W^{1,p}(\Omega)$ then, passing to a convenient subsequence, we may assume that $u_k \to u$, $\nabla u_k \to \nabla u$ a.e. and $|u_k| \leq h$, $|\nabla u_k| \leq h$, for some $h \in L^p(\Omega)$. The continuity of a and $b(x,\cdot)$ thus implies that $a(\nabla u_k) \to a(\nabla u)$, $b(x,u_k) \to b(x,u)$ a.e. and the estimates from above in G) give both $|a(\nabla u_k)| \leq \beta |\nabla u_k|^{p-1} \leq \beta h^{p-1}$ and $|b(x,u_k)| \leq \gamma |u_k|^{p-1} + |s| \leq \gamma h^{p-1} + |s|$, with $h^{p-1}, s \in L^q(\Omega)$. Thus, if $\varphi \in W_0^{1,p}(\Omega)$ the Lebesgue dominated convergence theorem yields

$$\int_\Omega a(\nabla u) \cdot \nabla \varphi + b(x,u)\varphi \, dx = \lim_{k\to\infty} \int_\Omega a(\nabla u_k) \cdot \nabla \varphi + b(x,u_k)\varphi \, dx = 0,$$

proving that $u \in X$; the convexity of X then yields the closure of X in the weak topology. Moreover X is bounded in $W^{1,p}(\Omega)$, indeed fix $u \in X$: if $v \in X$ then from a) we deduce that $a(\nabla v) = a(\nabla u)$; Lemma 3.1 b) then implies that

$$\alpha |\nabla v|^{p-1} \leq \beta |\nabla u|^{p-1} + r \text{ a.e.}$$

so that $\|\nabla v\|_p$ is bounded by a constant (depending on u). Since $v \in u + W_0^{1,p}(\Omega)$, Poincaré inequality yields the conclusion. □

Remark 3.2. The existence of a solution to $Lv = 0$ follows the Browder-Minty Theorem [14] if the constants in the growth conditions are suitably chosen in such a way that the operator L is coercive.

We are now in the position to prove the existence of the minimum and the maximum of the solutions to $Lv = 0$ with a prescribed boundary datum.

Theorem 3.2 (Existence of extremal solutions)**.** *Assume that L satisfies* A_1)*,* B_1) *and G). Given $\phi \in W^{1,p}(\Omega)$ there are $u^-, u^+ \in \phi + W_0^{1,p}(\Omega)$ solutions to $Lv = 0$ satisfying $u^- \leq u \leq u^+$ a.e. for all $u \in \phi + W_0^{1,p}(\Omega)$ with $Lu = 0$.*

Proof. Let X be the set of solutions to $Lv = 0$ in $\phi + W_0^{1,p}(\Omega)$. From the closure and the convexity of X together with the separability of $W^{1,p}(\Omega)$ there is a dense sequence $(u_k)_k$ in X. For every $k \in \mathbb{N}$ set $v_k = u_1 \vee \cdots \vee u_k$ and let u^+ be the pointwise limit of v_k. From the weak compactness of X there is $w \in X$ such that v_k converges weakly to w; thus v_k converges strongly to w in $L^p(\Omega)$ so that $u^+ = w \in X$. Clearly $u^+ \geq u$ for every $u \in X$. The existence of u^- follows similarly. □

4. The generalized (BSC)

From now on we consider the homogeneous case

$$Lv = -\operatorname{div} a(\nabla v)$$

and we assume that the operator satisfies the Basic Assumption A_1).

4.1. A class of functions that satisfies the Comparison Principle.

We consider the translates of the support functions of a compact and convex set, first introduced by Cellina [4] in the framework of the Calculus of Variations to deal with non strictly convex problems.

Definition 4.1 (A class of functions)**.** Whenever F is a compact and convex subset of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ we consider the functions

$$h^+_{F,x_0}(x) = \max\{\xi \cdot (x - x_0) : \xi \in F\}$$

$$h^-_{F,x_0}(x) = \min\{\xi \cdot (x - x_0) : \xi \in F\} = -h^+_{-F,x_0}(x).$$

Example 1. Let F be the unit ball. Then $h^+_{F,x_0}(x) = |x - x_0|$ and $h^-_{F,x_0}(x) = -|x - x_0|$ for all x_0.

It is worth mentioning that the functions just defined are Lipschitz, that $\nabla h^\pm_{F,x_0} \in F$ and that $h^\pm_{F,x_0}(x) = \nabla h^\pm_{F,x_0}(x) \cdot (x - x_0)$ a.e.: this follows easily from the properties of the support function to a set [15] or see [4] for a direct proof; they are nothing more than affine when F is reduced to a single point. We show now that these functions satisfy the Comparison Principle with respect to any other minimizer (not just the minimum or the maximum ones). The proposition is the reformulation in this PDE's setting of a result by Cellina in [4] in the more general form as in [9].

Proposition 4.1. *Assume that a satisfies* A_1) *and let F be a compact level set for a. For every $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ the functions $c + h^\pm_{F,x_0}$ are solutions to $-\operatorname{div} a(\nabla v) = 0$. Moreover if $x_0 \notin \Omega$ they satisfy the Comparison Principle:*

$$u \in (c+h^\pm_{F,x_0})+W^{1,1}_0(\Omega) \quad Lu \le 0,\ u \le c+h^\pm_{F,x_0} \text{ on } \partial\Omega \Rightarrow u \le c+h^\pm_{F,x_0} \text{ a.e. on } \Omega,$$

$$u \in (c+h^\pm_{F,x_0})+W^{1,1}_0(\Omega) \quad Lu \ge 0,\ u \ge c+h^\pm_{F,x_0} \text{ on } \partial\Omega \Rightarrow u \ge c+h^\pm_{F,x_0} \text{ a.e. on } \Omega.$$

Proof. Let $F = \{\eta \in \mathbb{R}^n : a(\eta) = a(\xi)\}$ for some ξ. The fact that $c + h^\pm_{F,x_0}$ are solutions follows immediately since

$$a(\nabla(c + h^\pm_{F,x_0})) = a(\xi)$$

is a constant. Assume now without restriction that $c = 0$. Set $h = h^+_{F,x_0}$ and let $u \in h + W^{1,1}_0(\Omega)$ be a solution to $Lv = 0$. By Theorem 3.1 we obtain that $a(\nabla u) = a(\nabla h)$ a.e. so that $\nabla u \in F$ a.e.; by the very definition of h we thus obtain that

$$\nabla u(x) \cdot (x - x_0) \le \nabla h^+_{F,x_0}(x) \cdot (x - x_0) = h(x) \text{ a.e.}$$

and therefore, if we set $\psi = h - u$, we have

$$\psi \in W^{1,1}_0(\Omega), \quad \nabla\psi(x) \cdot (x - x_0) \ge 0 \text{ a.e. on } \Omega.$$

We resume here the same reasoning that was carried on in [8]: there is a representative ψ^* of ψ that is zero on $\partial\Omega$ and such that ψ^* is absolutely continuous on a.e. line through x_0 and such that, for a.e. $x \in \Omega$,

$$\forall t \quad \frac{d}{dt}\psi^*(x_0 + t(x - x_0)) = \nabla\psi(x_0 + t(x - x_0)) \cdot (x - x_0) \ge 0$$

so that ψ^* increases along a.e. line from x_0. Since $\psi^* = 0$ on $\partial\Omega$ it follows that ψ^* does actually vanish along these lines, so that $\psi^* = 0$ a.e. on Ω. Thus $\psi = 0$ a.e. on Ω and $u = h$, so that $h = h^+_{F,x_0}$ is the only solution with such a boundary datum. The same reasoning applies to h^-_{F,x_0}. Remark **??** yields the conclusion. □

4.2. **Bounded Slopes Conditions.** We first recall the Bounded Slope Condition introduced by Hartmann and Stampacchia in [6] .

Definition 4.2 (BSC)**.** The function ϕ satisfies the *Bounded Slope Condition* of rank $M \ge 0$ if for every $\gamma \in \partial\Omega$

$$\exists |z^-_\gamma| \le M \qquad \forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z^-_\gamma \cdot (\gamma' - \gamma) \le \phi(\gamma') \tag{4.1}$$

$$\exists |z^+_\gamma| \le M \qquad \forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z^+_\gamma \cdot (\gamma' - \gamma) \ge \phi(\gamma') \tag{4.2}$$

Remark 4.1. We remind that ϕ satisfies the (BSC) if and only if it is the restriction of a convex function and of a concave function, both defined on $\mathbb{R}^n$ and globally Lipschitz. Under a uniform convexity assumption on the domain any $\mathcal{C}^2$ function satisfies the (BSC) [13]. The (BSC) is a quite restrictive condition: it forces for instance the function ϕ to be affine on the flat parts of $\partial\Omega$ and Ω to be convex.

Recently, some new conditions that are less restrictive than the (BSC) appeared in the literature for problems of the Calculus of Variations depending on the gradient. The Lower (resp. Upper) (BSC) was introduced by Clarke: it requires the validity of just (4.1) (resp. (4.2)), which turns out in [5] to be sufficient to obtain the local Lipschitz continuity of the minimizers of strictly convex functionals. A generalized (BSC) was introduced by Cellina, where the functions $c+h^{\pm}_{F,x_0}$ defined above replace affine functions in the (BSC): when the sets F are the projections onto $\mathbb{R}^n$ of the faces of the epigraph of the lagrangian the condition turns out to be sufficient in [11] to obtain the Lipschitz continuity of the minimizers. The condition is particularly suitable and interesting when the lagrangian is not strictly convex, since in this case some of the faces of its epigraph are not reduced to a point, so that the functions $h^{\pm}_{F,x_0}$ are not affine.

In what follows we convert the results quoted above in the framework of elliptic PDE's. Let us first formulate the definition of the Generalized (BSC) in this context.

We recall that we consider the operator $Lv = -\operatorname{div} a(\nabla v)$, where $a : \mathbb{R}^n \to \mathbb{R}^n$ satisfies Assumption A_1).

Definition 4.3 (Generalized (BSC) or (GBSC))**.** The pair (ϕ, a) satisfies the *Generalized* (*BSC*) of rank $M \geq 0$ if for every $\gamma \in \partial\Omega$:

i) there exists a level set F^- for a, contained in a ball of radius M, such that

$$\forall \gamma' \in \partial\Omega \qquad \phi(\gamma) + h^-_{F^-_\gamma,\gamma}(\gamma') \leq \phi(\gamma'); \tag{4.3}$$

ii) there exists a level set F^+ for a, contained in a ball of radius M, such that

$$\forall \gamma' \in \partial\Omega \qquad \phi(\gamma) + h^+_{F^+_\gamma,\gamma}(\gamma') \geq \phi(\gamma'). \tag{4.4}$$

The pair (ϕ, a) is said to satisfy the *Generalized Lower* (resp. *Upper*) (*BSC*) if just (4.3) (resp. (4.4)) holds.

Remark 4.2. Opposite to the (BSC) the definition of the Generalized (BSC) involves the operator a. This is why it appears slightly different to the analogous condition formulated in the context of the Calculus of Variations [11] where it depends on the lagrangian. This also explains why the proof of the subsequent results differ from their analogous versions that have been established in the frameworks of the Calculus of variations.

The term "Generalized" in the new (BSC) is motivated by the following result.

Proposition 4.2. *Assume that the function a satisfies* A_1)*,G) and that (ϕ, a) satisfies the Lower* (*resp. Upper*) (*BSC*) *of rank M. Then ϕ satisfies the Lower* (*resp. Upper*) *Generalized* (*BSC*) *of a rank depending only on a and M.*

Proof. Assume that for some γ and $z \in \mathbb{R}^n$ with $|z| \leq M$ we have

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z \cdot (\gamma' - \gamma) \leq \phi(\gamma').$$

Let F_z be the level set for a defined by

$$F_z = \{\eta \in \mathbb{R}^n : a(\eta) = a(z)\}.$$

Since, clearly, for all $\gamma' \in \partial\Omega$

$$h^-_{F_z,\gamma}(\gamma') = \min\{\xi \cdot (\gamma' - \gamma) : \xi \in F_z\} \leq z \cdot (\gamma' - \gamma)$$

then $\phi(\gamma) + h^-_{F_z,\gamma}(\gamma') \leq \phi(\gamma')$. If G_1) holds then F_z is contained in a ball of center 0 and radius $M + K$. If g_2) then by Lemma 3.1 b) we have

$$|\alpha\eta|^{p-1} \leq \beta |z|^{p-1} + r \leq \beta M^{p-1} + r$$

so that F is contained in a ball of center 0 and radius depending on a and M, thus proving that the validity of the Lower (BSC) implies that of the Generalized Lower

(BSC) in both the cases. The version of the result for the Upper (BSC) follows similarly. □

Example 2. The Generalized (BSC) is strictly more general than the (BSC). For instance if

$$a(\xi) = \nabla f(\xi) \qquad f(\xi) = \begin{cases} (|\xi|^2 - 1)^2 & \text{if } |\xi| \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

then the level set F of a containing the origin is the closed unit ball. It follows from Example 1 that $h^+_{F,x_0}(x) = |x - x_0|$ and $h^-_{F,x_0}(x) = -|x - x_0|$. Therefore any Lipschitz function ϕ of rank less or equal than 1 is such that (ϕ, a) satisfies the Generalized (BSC); note that the domain may be not convex.

5. Regularity results for the solutions to $-\operatorname{div} a(\nabla v) = 0$

In this section we assume that the operator $Lv = -\operatorname{div} a(\nabla v)$ satisfies the Basic Assumption A_1) and the Growth Assumption G). In this situation where $b = 0$ the Browder-Minty Theorem ensures the existence of a solution to the Dirichlet problem associated to L.

5.1. **Lipschitz continuity.** We first formulate a variant of the celebrated result on the existence of a Lipschitz solution to $Lu = 0$ [6, Theorem 13.1]. There it is required that a satisfies the monotonicity assumption (1.1) and that ϕ satisfies the (BSC). We allow here ϕ to belong to a wider class, whereas a counterpart due to the nonlinarity, we assume moreover that a satisfies the slightly stronger condition A_1).

Theorem 5.1 (Lipschitz continuity with the Generalized (BSC))**.** *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Assume that the function a satisfies* A_1), *G) and that (ϕ, a) satisfies the* Generalized (BSC) *of rank M. Let u be the maximum (resp. minimum) of the solutions to $Lv = 0$ on $\phi + W^{1,1}_0(\Omega)$. Then u is Lipschitz of rank depending on a and M.*

Proof. Let u be any solution to $Lv = 0$ on $\phi + W^{1,1}_0(\Omega)$. It follows from Proposition 4.1 that the inequalities (4.3) and (4.4) hold true for every $\gamma' \in \Omega$ (instead of just for $\gamma' \in \partial\Omega$). For $x \in \overline{\Omega}$ set

$$\ell^-(x) = \sup\{\phi(\gamma) + h^-_{F^-_\gamma,\gamma}(x) : \gamma \in \partial\Omega\}$$

and

$$\ell^+(x) = \inf\{\phi(\gamma) + h^+_{F^+_\gamma,\gamma}(x) : \gamma \in \partial\Omega\}.$$

Then $\ell^\pm$ are Lipschitz, both belong to $\phi + W^{1,1}_0(\Omega)$ and we have

$$\ell^- \leq u \leq \ell^+ \quad \text{a.e. on } \Omega$$

so that Assumption H_3) of Theorem 2.3 is satisfied: its application yields the Lipschitz continuity of u whenever u is the maximum or the minimum of the minimizers. □

It follows from Proposition 4.2 that the conclusion of Theorem 5.1 does hold if one assumes the more restrictive assumption that ϕ satisfies the (BSC) instead of the Generalized (BSC), thus obtaining the conclusion of [6, Theorem 13.1] as a corollary. We have to say however that in [6] it is not required that the function a satisfies the assumptions A_1) and G). The fact that G) can be omitted there is due to the fact that the functions involved are just Lipschitz of a prescribed rank; here we need G) first to define the functions h_{F,x_0} defined upon some level sets that turn out to be bounded under G).

5.2. **Local Lipschitz and Hölder continuity.** In [5] Clarke introduced the unilateral (BSC) to obtain the local Lipschitz regularity of the minimum of a variational problem of the gradient. We generalized it to the case of non strictly convex lagrangian in [11]. The Comparison Principles established here allow us to convert the result in the framework of PDE's. We underline that, beside the interest of this results in itself, it is also a basic tool in the subsequent proof of the Hölder continuty of the solutions to $Lv = 0$ (Theorem 5.3).

Theorem 5.2 (Local Lipschitz continuity)**.** *Let Ω be an open, convex and bounded subset of $\mathbb{R}^n$. Assume that the function a satisfies* A_1)*, G) and that (ϕ, a) fulfills the Lower or the Upper (GBSC). Let u be the maximum (resp. minimum) of the solutions to $Lv = 0$. Then u is locally Lipschitz.*

Proof. The proof is similar to those of [5, Theorems 2.1] and [11, Theorem 4.1]. We just point out the new fact that we are dealing with solutions to a PDE instead of minimizers of an integral functional, by showing that the main arguments of these proofs do still work in this setting. Let u be the maximum of the solutions to $Lu = 0$ on $\phi + W_0^{1,p}(\Omega)$ and (ϕ, a) satisfies the Lower (GBSC).
i) For $\lambda \in]0, 1]$ and $\gamma \in \partial\Omega$ the function $u_\lambda(x) = \lambda u\left(\dfrac{x-\gamma}{\lambda} + \gamma\right)$ is still a solution to $Lv = 0$ on $\Omega_\lambda = \lambda(\Omega - \gamma) + \gamma$: it is the greater one among those that share the same boundary datum.
ii) By Proposition 4.1 the Lower (GBSC) implies that $u(x) \geq \phi(\gamma) + h^-_{F^-_\gamma,\gamma}(x)$ for a.e. $x \in \Omega$.
iii) The solutions to $Lv = 0$ with $v \in \phi + W_0^{1,p}(\Omega)$ are bounded. Fix $\gamma \in \partial\Omega$: since $u(x) \geq \phi(\gamma) + h^-_{F^-_\gamma,\gamma}(x)$ then u is bounded from below. Moreover let $C = \|\phi\|_\infty$ and u_C be the greatest solution to $Lv = 0$ with $v = C$ on $\partial\Omega$. By Theorem 3.1 we have $a(0) = a(\nabla C) = a(\nabla u_C)$ so that ∇u_C belongs a.e. to the level set $\{\eta \in \mathbb{R}^n : a(\eta) = a(0)\}$ which is bounded thanks to assumption G) and Lemma 3.1 b). Thus u_C is Lipschitz and bounded by a constant depending only on C, a and $\operatorname{diam}\Omega$.
The proof then proceeds as in [5, Theorem 2.1] □

Finally we have the analogue of [10, Theorem 4.5] yield the following result.

Theorem 5.3 (Continuity and Hölder continuity)**.** *Let Ω be an open, convex and bounded subset of $\mathbb{R}^n$. Assume that the function a satisfies* A_1) *and G). Assume moreover, under G_1, that $a(\xi) \cdot \xi \geq \alpha|\xi|^p$. Let u be the maximum or the minimum of the solutions to $Lv = 0$ on $\phi + W_0^{1,p}(\Omega)$. Then*

a) *If ϕ is Lipschitz then u is Hölder continuous in $\overline{\Omega}$ of order $\alpha = \frac{p-1}{n+p-1}$;*
b) *if ϕ is continuous then u is continuous on $\overline{\Omega}$.*

Proof. Again we just show that the main arguments of the proof of [10, Theorem 4.5] continue to hold true in this PDE's setting. Assume first that ϕ is Lipschitz.
i) The analogue of [5, Lemma 2.11] is still valid. More precisely: let u be the maximum (resp. minimum) of the solutions to $Lv = 0$ when the boundary datum ϕ satisfies the Lower (resp. Upper) BSC and the domain is a polyhedron Q. Then there exists a constant C depending only on the diameter of Q, $\|\phi\|_\infty$, $\|\nabla\phi\|_\infty$ such that

$$\forall\gamma \in \partial\Omega,\ \forall x \in \Omega \quad u(x) - \phi(\gamma) \leq C|x-\gamma|^\alpha \text{ (resp. } \phi(\gamma) - u(x) \leq C|x-\gamma|^\alpha).$$

Indeed the result is a consequence of the local Lipschitz continuity of u, that we established in Theorem 5.2, and of a uniform bound of $\|\nabla u\|_{L^p(Q)}$ that follows there from the coercivity of the functional and the fact that u is a minimizer. Here such an estimate follows from Lemma 3.1 b) with $\eta = \nabla u$ and $\xi = \nabla\phi$.

ii) Let now u be any solution to $Lv = 0$ with $v = \phi$ on $\partial\Omega$, where ϕ is Lipschitz of rank M. Following the steps of the proof of Theorem 4.5 of [10], for $\gamma \in \partial\Omega$, we consider the convex function $\phi_\gamma(x) = \phi(\gamma) + M|x - \gamma|$ and a cube Q_γ that is tangent to Ω at γ, contains Ω, and is isometric to a cube Q that does not depend on γ. Let u_γ be the maximum of the solutions on Q_γ to $Lv = 0$, $v = \phi_\gamma$ on ∂Q_γ. Since Q_γ is polyhedron we know that

$$\forall x \in Q_\gamma \qquad u_\gamma(x) - \phi(\gamma) \leq C|x - \gamma|^\alpha \tag{5.1}$$

where C depends only on $\operatorname{diam} Q_\gamma = \operatorname{diam} Q$ and on $\|\nabla \phi_\gamma\|_\infty = M$, so not on γ itself. Since ϕ_γ is convex, then the Comparison Principle 2.1 implies that $u_\gamma \geq \phi_\gamma$ on Q_γ so that $u_\gamma \geq \phi_\gamma \geq \phi$ on Ω. Now u_γ is still the maximum of the solutions to $Lv = 0$ on Ω among the functions that share the same boundary datum. Again, Theorem 2.1 shows that $u_\gamma \geq u$ a.e. on Ω. It follows from (5.1) that

$$u(x) - \phi(\gamma) \leq u_\gamma(x) - \phi(\gamma) \leq C|x - \gamma|^\alpha \text{ a.e. on } \Omega.$$

Analogously one obtains that $\phi(\gamma) - u(x) \leq C|x - \gamma|^\alpha$ a.e. on Ω.
iii) If u is the maximum or the minimum of the solutions to $Lv = 0$ with $v = \phi$ on $\partial\Omega$ then a) follows from the Haar-Rado type Theorem 2.3.
iv) Claim b) follows as in the last lines of the proof of [2, Lemma 7] by approximating ϕ by means of Lipschitz functions and of the Comparison Principle. □

We have proved in Lemma 3.1 that if a satisfies assumptions A_1) and G) then for every $\xi \in \mathbb{R}^n$ the level set $F_\xi = \{\eta \in \mathbb{R}^n : a(\eta) = a(\xi)\}$ is compact. If this requirement is slightly strengthened the previous results hold for *every* solution to $Lv = 0$, not just for the maximum and the minimum ones.

Corollary 5.1. *Under the above assumptions assume moreover that the diameters of the level sets of a are bounded by a constant. The conclusions of Theorem 5.1, Theorem 5.2 and of Theorem 5.3 hold for every solution to $Lv = 0$, $v = \phi$ on $\partial\Omega$.*

Proof. It is enough to note that if u and w are solutions to $Lv = 0$ with the same boundary datum then, by Proposition 3.1, their gradients belong to a same level set. Our assumption that $w = u + \ell$ where ℓ is a Lipschitz function whose rank depends only on a: the Hölder or Lipschitz regularity of w is then inherited by that of u. □

Similarly to [10, Corollary 4.10] it can be easily shown that if the diameters of the sets F_ξ are uniformly bounded then the conclusions of .

Remark 5.1. The conclusion of Theorem 5.3 is, from one hand, an extension of a well known result among a-harmonic function, i.e. solutions to $-\operatorname{div} a(\nabla v) = 0$ where a is strictly monotonic and satisfies the further assumption that $a(\xi) \sim c|\xi|^{p-1}$ in the sense that $a(\lambda\xi) = \lambda|\lambda|^{p-2}a(\xi)$ whenever $\lambda \in \mathbb{R}$ is non zero [7, Theorem 6.44]. We note however that this classical results holds even when the boundary datum ϕ is Hölder, the domain is regular and a has a suitable dependence on x. The extension of the validity of our result to Hölder boundary data or regular, though non convex domains, remains open.

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