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On a Parametric Problem of the Calculus of Variations without Convexity Assumptions

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The parametric integral

$$I(C)=\int_a^b f(x'(t))\,dt$$

attains the minimum in a class of rectifiable curves $C: x=x(t)$, $a\leqslant t\leqslant b$, under slow growth conditions and no convexity assumption on f.

INTRODUCTION

Let $f\colon \mathbb{R}^n\to\mathbb{R}$ be continuous and positive homogeneous of degree one. The primary purpose of this paper is to show that if f satisfies the growth assumption

$$\forall\xi\in\mathbb{R}^n : f(\xi)\geqslant\gamma\,|\xi|$$

then Tonelli's convexity assumption on f can be omitted for the existence of the minimum of the parametric integral

$$I(C)=\int_a^b f(x'(t))\,dt$$

on the set of rectifiable Fréchet-curves $C: x=x(t)$, $a\leqslant t\leqslant b$, with prescribed boundary conditions $(x(a), x(b))\in K\times B$, K (resp. B) being compact (resp. closed).

The main tool is an extension of Liapunov's Theorem on the range of vector measures (Theorem 1).

I thank Professor L. D. Berkovitz who carefully read the manuscript and suggested the present version of the first part of the proof of Theorem 2, which is more concise and elegant than the original one. I also thank Professor A. Cellina for the useful conversations we had during the preparation of this paper.

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Parametric Curves

A parametric curve C in $\mathbb{R}^n$ is a suitable equivalence class of n-vector continuous maps

$$x = x(t),\ a \leqslant t \leqslant b; \qquad y = y(s),\ c \leqslant s \leqslant d$$

leaving unchanged the sense in which the curve is travelled.

Usually, two continuous maps x and y are said to be equivalent if there is a strictly increasing continuous map

$$s = h(t),\ a \leqslant t \leqslant b, \qquad h(a) = c,\ h(b) = d$$

such that

$$y(h(t)) = x(t), \qquad a \leqslant t \leqslant b.$$

For technical reasons a weaker equivalence relation is needed.

Definition 1 [3, 14.1.A]. Two continuous maps x and y as above are said to be Fréchet equivalent if for every $\varepsilon \geqslant 0$ there is some homeomorphism

$$h\colon s = h(t),\ a \leqslant t \leqslant b, \qquad h(a) = c,\ h(b) = d$$

such that

$$|y(h(t)) - x(t)| \leqslant \varepsilon, \qquad a \leqslant t \leqslant b.$$

A class of F-equivalent maps is called a parametric curve or F(réchet)-curve.

It is easily seen that for any given F-curve $C\colon x = x(t)$, $a \leqslant t \leqslant b$, the subsets

$$[C] = [x] = \{x(t)\colon a \leqslant t \leqslant b\} \qquad \text{and} \qquad \{x(a)\},\ \{x(b)\}$$

of $\mathbb{R}^n$ are F-invariant. The same holds for the *Jordan length* $L(C)$ of a Fréchet curve C, which is defined as a total variation,

$$L(C) = \sup \sum_{i=1}^{N} |x(t_i) - x(t_{i-1})|, \tag{1}$$

where sup is taken with respect to all subdivisions

$$a = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_N = b \qquad \text{of } [a, b].$$

A F-curve is said to be *rectifiable* if $L(C) < +\infty$. The following proposition justifies the definition of F-curve.

Proposition 1 [3, 14.1.I]. *A rectifiable curve C possesses A.C. representations. In particular, the arc-length parameter s yields a unique A.C. representation*

$$x = x(s),\ 0 \leqslant s \leqslant L(C), \qquad |x'(s)| = 1 \ \text{a.e. in } [0, L].$$

If $x(t)$, $a \leqslant t \leqslant b$, is an A.C. representation of C, the Jordan length $L(C)$ is given by

$$L(C) = \int_a^b |x'(t)|\, dt. \tag{2}$$

Let $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and C be a rectifiable F-curve, $x(t)$, $a \leqslant t \leqslant b$, be any of its A.C. representations. Then the integral

$$I[x] = \int_a^b f(x(t), x'(t))\, dt \tag{3}$$

is independent of the chosen A.C. representation if and only if f is a *parametric integrand*, i.e., f does not depend on t and is positive homogeneous of degree one in x', that is, $\forall k \geqslant 0$: $f(x, kx') = kf(x, x')$ [3, 14.1.B]. In this situation (3) defines the *parametric integral* $I(C)$ for any F-curve C and for any of its A.C. representations.

Preliminary Results

Let $f\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a function and let, for $p \geqslant 1$, (h_p) be the following growth condition on f:

(h_p) there exist $\gamma > 0$ and a function $\delta \in L^1([0, T])$ such that

$$\forall (t, x) \in [0, T] \times \mathbb{R}^n\colon f(t, x) \geqslant \gamma |x|^p + \delta(t). \tag{4p}$$

The following theorem is an extension of Liapunov's Theorem on the range of a vector measure [3, Chap. 16]. Its proof, given here for the convenience of the reader, is based on an argument of A. Cellina and G. Colombo [2]. Let us indicate by χ_E the characteristic function of a set E.

Theorem 1. *Let Ω be a measurable bounded subset of $\mathbb{R}^n$, $f_1, \ldots, f_n$ (resp. $u_1, \ldots, u_m$) be a vector-valued measurable functions with values in $\mathbb{R}^l$ (resp. $\mathbb{R}^k$). Let $p_1, \ldots, p_m$ be real valued, measurable and such that:*

(i) $p_i(\omega) \geqslant 0$, $\sum_i p_i = 1$;

(ii) $\sum_i p_i f_i \in L^1(\Omega)$;

(iii) *there exist an l-valued* L^1 *function* δ, *a positive vector* γ *such that*

$$f_j(x) \geqslant \delta(x) + \gamma |u_j(x)|^p \qquad (x \in \Omega,\ 1 \leqslant p < \infty).$$

Then there exists a measurable partition $E_1, ..., E_m$ *of* Ω *with the property that* $\sum_i f_i \chi_{E_i} \in L^1(\Omega)$, $\sum_i u_i \chi_{E_i} \in L^p(\Omega)$, *and the following equalities hold*:

$$\int_\Omega \sum_i p_i f_i \, d\mu = \sum_i \int_{E_i} f_i \, d\mu, \tag{5}$$

$$\int_\Omega \sum_i p_i u_i \, d\mu = \sum_i \int_{E_i} u_i \, d\mu. \tag{6}$$

With the above notations, let us remark that if the functions u_i are chosen to be zero, then Theorem 1 yields the following Corollary:

COROLLARY. *Let* $f_1, ..., f_m$ *be measurable, bounded below be an integrable function, and such that* $\sum_{i=1}^m p_i f_i \in L^1$. *Then there exists a measurable partition* $E_1, ..., E_m$ *of* Ω *such that* (5) *holds.*

Remark. The above Corollary is a slightly different version of [4, Proposition 4.1] and takes into account the fact that the growth condition (iii) is necessary for (5) to hold. In fact, let us consider for instance $\Omega =]0, 1]$, $u_1 = u_2 = 0$, $f_1(t) = 1/t$, $f_2 = -f_1$, $p_1 = p_2 = 1/2$. Then the function $p_1 f_1 + p_2 f_2 = 0 \in L^1$ but for each measurable partition E_1, E_2 of $]0, 1]$ the function $f = f_1 \chi_{E_1} + f_2 \chi_{E_2}$ is not an element of $L^1(|f(t)| = 1/t$ a.e.).

Proof of Theorem 1. Let us suppose that $l = k = 1$, the general case being similar. By Lusin's Theorem there exists a sequence $(K_j)_{j \in \mathbb{N}}$ of disjoint compact subsets of Ω and a null set N such that $\Omega = N \cup (\bigcup_j K_j)$ and the restriction of each of the maps f_i to any K_j is continuous. In this situation, the growth assumption (iii) implies that the functions u_i restricted to K_j belong to $L^p(K_j) \subset L^1(K_j)$ $(j \in \mathbb{N})$. For any j fixed in $\mathbb{N}$, Liapunov's Theorem on the range of vector measures [3, Chap. 16] provides the existence of a measurable partition $(E_i^j)_{i=1,...,m}$ of K_j with the property that

$$\int_{K_j} \sum_i p_i f_i \, d\mu = \sum_i \int_{E_i^j} f_i \, d\mu, \tag{7}$$

$$\int_{K_j} \sum_i p_i u_i \, d\mu = \sum_i \int_{E_i^j} u_i \, d\mu. \tag{8}$$

Set, for any $v \in \mathbb{N}$, the function s_v to be

$$s_v = \sum_{j \leqslant v} \sum_{i=1}^m (f_i - \delta) \chi_{E_i^j}.$$

By (iii), each term of the right-hand side of the above equality is a sum of non-negative terms, hence the sequence s_v is monotone non-decreasing. Furthermore, by (7) we have

$$\begin{aligned} \int_\Omega s_v \, d\mu &= \sum_{j \leqslant v} \sum_{i=1}^m \int_{E_i^j} (f_i - \delta) \, d\mu \\ &= \sum_{j \leqslant v} \int_{K_j} \sum_{i=1}^m p_i (f_i - \delta) \, d\mu \\ &\leqslant \int_\Omega \left(\sum_i p_i f_i - \delta \right) du \end{aligned}$$

which, by (ii), is finite. Moreover, if we set $E_i = \bigcup_{j \in \mathbb{N}} (E_i^j)$, we have

$$\lim_v s_v = \sum_i f_i \chi_{E_i} - \delta \qquad \text{a.e.}$$

Then Beppo Levi's convergence theorem implies that

$$\sum_i f_i \chi_{E_i} \in L^1(\Omega)$$

and

$$\begin{aligned} \int_\Omega \sum_i f_i \chi_{E_i} \, d\mu &= \int_\Omega \lim_v s_v \, d\mu + \int_\Omega \delta \, d\mu \\ &= \lim_v \int_\Omega s_v \, d\mu + \int_\Omega \delta \, d\mu \\ &= \int_\Omega \sum_i p_i (f_i - \delta) \, d\mu + \int_\Omega \delta \, du \\ &= \int_\Omega \sum_i p_i f_i \, d\mu, \end{aligned}$$

which proves (5). In this situation assumption (iii) implies that $\sum_i u_i \chi_{E_i} \in L^p(\Omega)$. Hence, if we set s_v' to be

$$s_v' = \sum_{j \leqslant v} \sum_{i=1}^m u_i \chi_{E_i}$$

we have

$$s'_\nu \leqslant \sum_{i=1}^{m} |u_i| \chi_{E_i} \in L^p(\Omega) \qquad \text{and} \qquad s'_\nu \to \sum_i u_i \chi_{E_i} \text{ a.e.}$$

Lebesgue's dominated convergence theorem and equality (8) yield the conclusion.

Main Result

THEOREM 2. *Let K (resp. B) be a compact (resp. closed) subset of $\mathbb{R}^n$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous, positive homogeneous of degree one. Furthermore, suppose that f satisfies the following growth assumption (h_1):*

(h_1) *there exists $\gamma > 0$ such that, for every $x' \in \mathbb{R}^n$*

$$f(x') \geqslant \gamma |x'|.$$

Then the parametric integral

$$I(C) = \int_a^b f(x'(t))\, dt$$

has an absolute minimum in the class Δ of all rectifiable F-curves C: $x = x(t)$, $a \leqslant t \leqslant b$, satisfying the boundary conditions $x(a) \in K$, $x(b) \in B$.

Proof. Let us consider the following equivalent control problem:

$$\min \int_0^{s_1} f(u)\, ds, \qquad \text{subject to}$$

$$\frac{dx}{ds} = u(s), \qquad x(0) \in K, \qquad x(s_1) \in B. \tag{P}$$

We are considering the A.C. representation with arc length as parameter, hence s_1 is not fixed. The relaxed version of this problem is

$$\min \int_0^{s_1} \sum_{i=1}^{i=n+1} p_i(s) f(u_i(s))\, ds, \qquad \text{subject to}$$

$$\frac{dx}{ds} = \sum_{i=1}^{n+1} p_i(s)\, u_i(s), \tag{PR}$$

$$p_i(s) \geqslant 0, \qquad \sum_{i=1}^{i=n+1} p_i(s) = 1.$$

The relaxed control vector is $(p_1, ..., p_{n+1}, u_1, ..., u_{n+1})$.

The growth assumption (h_1) implies that there exists an $M > 0$ such that the length s_1 of any relaxed curve is $\leqslant M$. Condition (h_1) and $p_i \geqslant 0$, $\sum_i p_i = 1$, imply that all controls in a minimizing sequence all lie in a given ball in L^1. This fact and the form of the state equations imply that all curves in a minimizing sequence are equi-absolutely continuous. It then follows from [1, Theorem 8.5, Chap. III] that the relaxed problem has a solution

$$(x(s), p_1(s), ..., p_{n+1}(s), u_1(s), ..., u_{n+1}(s)).$$

Thus, if we set $\Omega = [0, s_1]$ and $f_i(t) = f(u_i(t))$ then Theorem 1 can be applied. Let $E_1, ..., E_m$ be the measurable partition of $[0, s_1]$ such that (5) and (6) of Theorem 1 hold. We claim that the parametric curve $\tilde{C}$ represented by $\tilde{x} = \tilde{x}(t)$, $0 \leqslant t \leqslant s_1$, defined as

$$\tilde{x}'(t) = \sum_{i=1}^{n+1} u_i(t)\, \chi_{E_i}(t), \qquad \tilde{x}(0) = x(0)$$

is a minimum of I in the class Δ.

Clearly $\tilde{x}$ is A.C. and, by (6) we have $\tilde{x}(s_1) = x(s_1) \in B$, hence $\tilde{C} \in \Delta$. Furthermore, by (5) we have

$$\begin{aligned} I(\tilde{C}) &= \int_0^{s_1} f\left(\sum_i u_i(t)\, \chi_{E_i}(t)\right) dt \\ &= \sum_i \int_{E_i} f(u_i(t))\, dt \\ &= \int_0^{s_1} \sum_i p_i(t) f(u_i(t))\, dt \\ &= \min(\mathrm{PR}). \end{aligned}$$

It follows that

$$\inf(\mathrm{P}) = \inf(I) \leqslant I(\tilde{C}) = \min(\mathrm{PR}) \leqslant \inf(\mathrm{P}),$$

hence the above equalities are in fact equalities.

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