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A Generalization of the Cellina–Colombo Theorem for a Class of Non-convex Variational Problems

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We state a condition under which the integral functional $I(x) = \int_0^T L(t, x(t), x'(t))\, dt$ attains a minimum under the assumption that $x \mapsto L(t, x, x')$ is concave.

INTRODUCTION

This paper concerns the problem (P) of the existence of a minimum for the integral functional I defined by

$$I(x) = \int_0^T L(t, x(t), x'(t))\, dt$$

on the set of functions $x(\cdot)$ belonging to $W^{1,p}([0, T], \mathbb{R}^n)$ $(p \geqslant 1)$ such that $x(0) = a$, $x(T) = b$, in the case where L does not necessarily satisfy Tonelli's classical assumption of convexity with respect to x'.

In this situation, the most general result is the Cellina–Colombo theorem [2] stating that if $L(t, x, x') = g(t, x) + h(t, x')$ and $x \mapsto g(t, x)$ is concave for a.e. t then Problem (P) admits at least one solution. For the case where the integrand is not the sum of two functions whose arguments are t, x and t, x' separately, it is not known whether the concavity assumption on the map $x \mapsto L(t, x, x')$ is sufficient for the existence of a solution to Problem (P). The purpose of this note is to consider this problem.

In Theorem 3 we prove that the functional I (under the concavity condition) attains a minimum if we assume further the existence of a solution

$$(\tilde{x}, p_1, ..., p_{n+1}, v_1, ..., v_{n+1})$$

to the associated relaxed problem (PR') satisfying

$$\bigcap_{i=1}^{n+1} \partial_x(-L(t, \tilde{x}(t), v_i(t))) \neq \varnothing \qquad \text{a.e.} \tag{C}$$

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Obviously, each solution to (P) is a solution to (PR′) satisfying (C); the cases for which our theorem can be usefully applied are those where the converse does not hold. For instance, condition (C) is automatically satisfied (for each $\tilde{x}$, $v_1, ..., v_{n+1}$) when the integrand L is the sum of two functions whose arguments are t, x and t, x' separately. In this situation, Theorem 3 yields Cellina and Colombo's existence result; however, it is well known that a solution to the associated relaxed problem is not, in general, a solution to the original one.

As a further application of our condition we show that Problem (P) attains a minimum if $L(t, x, x') = l(t, x) + f(t, x)\, g(t, x')$ and its bipolar $L^{**}(t, x, \cdot)$ is locally constant on $A(t, x) = \{\xi : L(t, x, \xi) > L^{**}(t, x, \xi)\}$.

The main tools are basically the arguments of [2]: an extension of Liapunov's theorem on the range of a vector measure and a selection theorem.

Assumptions and Preliminary Results

The following hypothesis is considered:

Hypothesis (H). *The set-valued map $\Phi: [0, T] \to 2^{\mathbb{R}^n}$ is measurable [1, Def. III.1.1] with non-empty closed values. In addition we assume that there exists at least one $v \in L^p([0, T], \mathbb{R}^n)$ such that $v(t) \in \Phi(t)$ a.e. and $\int_0^T v(t)\, dt = b - a$. The Caratheodory function $L: [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following growth assumption: if $p = 1$, there exist a convex l.s.c. monotonic function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$, a constant β_1, and a function $\alpha_1(\cdot)$ in L^1 such that*

$$L(t, x, \xi) \geqslant \alpha_1(t) - \beta_1 |x| + \psi(|\xi|)$$

$$\text{for each } x, \xi \text{ and for a.e. } t \left(\lim_{r \to \infty} \frac{\psi(r)}{r} = +\infty \right);$$

if $p > 1$, there exist a positive constant γ_p, a constant β_p (β_p / γ_p) is strictly smaller than the best Sobolev constant in $W_0^{1,p}([0, T], \mathbb{R}^n)$), a function $\alpha_p(\cdot)$ in L^1 such that

$$L(t, x, \xi) \geqslant \alpha_p(t) - \beta_p |x|^p + \gamma_p |\xi|^p \qquad \text{for each } x, \xi \text{ and for a.e. } t.$$

For each t, x, let us denote by $L^{**}(t, x, \cdot)$ the bipolar of the map $\xi \mapsto L(t, x, \xi)$ [4, Sect. I.4.2]. For each function L satisfying Hypothesis (H), its bipolar fulfills Tonelli's classical assumptions for the existence of a solution to the relaxed problem (PR) associated to (P),

$$\text{minimize} \int_0^T L^{**}(t, x(t), x'(t))\, dt \qquad \text{(PR)}$$

on the subset of $W^{1,p}([0, T], \mathbb{R}^n)$ of those functions x satisfying $x(0) = a$, $x(T) = b$, $x'(t) \in \Phi(t)$ a.e. Now, consider the problem

$$\begin{aligned} &\text{minimize} \int_0^T \sum_{i=1}^{n+1} p_i(t)\, L(t, x(t), v_i(t))\, dt \\ &p_i: [0, T] \to \mathbb{R}, \qquad v_i: [0, T] \to \mathbb{R}^n \text{ measurable} \\ &\sum_i p_i(t) = 1, \qquad p_i \geqslant 0, \qquad v_i(t) \in \Phi(t) \text{ a.e.} \qquad \text{(PR}') \\ &x'(t) = \sum_i p_i(t)\, v_i(t) \in L^p \\ &x(0) = a, \qquad x(T) = b. \end{aligned}$$

Clearly,

$$\min \text{PR} \leqslant \inf \text{PR}' \leqslant \inf \text{P}.$$

Moreover, we have the following:

Theorem 1 [4, Th. IX.4.1, Sect. IX.4.5]. *Let L satisfy Hypothesis (H). Then* $\min \text{PR}' = \min \text{PR} = \inf \text{P}$.

Let us denote by $\chi_E(\cdot)$ the characteristic function of a set E. Theorem 2 is an extension of Liapunov's theorem on the range of a vector measure [3, Chap. 16].

Theorem 2 [2, 6, 9]. *Let $p_1, ..., p_m: [0, T] \to [0, 1], f_1, ..., f_m: [0, T] \to \mathbb{R}^l$ ($l \geqslant 1$) be measurable ($\sum_i p_i = 1$) and bounded below by an integrable function. Let us further assume that $\sum_i p_i f_i \in L^1$. Then there exists a measurable partition $E_1, ..., E_m$ of $[0, T]$ with the property that $\sum_i f_i \chi_{E_i} \in L^1$ and the following equality holds:*

$$\int_0^T \sum_i p_i f_i\, dt = \int_0^T \sum_i f_i \chi_{E_i}\, dt.$$

Lemma 1 below concerns a property of the subdifferential of a convex function [4, Sect. I.5.1]; its proof follows directly from [2, Lemma 1].

Let us denote by $\partial_x(f(t, x, \xi))$ the subdifferential of the function $x \mapsto f(t, x, \xi)$. Also, for a subset Q of $\mathbb{R}^n$, we write $\|Q\|$ for the set $\{|q| : q \in Q\}$.

Lemma 1. *Let $f: [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Caratheodory function satisfying:*

(i) $f(t, x, \xi) \leqslant \alpha(t) + \beta |x|^p$ ($\beta > 0$, $\alpha \in L^1$);

(ii) $x \mapsto f(t, x, \xi)$ *is convex for a.e. t and for each* ξ.

Let $\tilde{x}$ *be continuous,* $v_1, ..., v_{n+1}$ *be measurable and such that*

$$\Psi(t) = \bigcap_i \partial_x f(t, \tilde{x}(t), v_i(t)) \neq \varnothing \qquad a.e.$$

Then, the set-valued map Ψ *admits an integrable selection.*

Remark. The proof of [2, Lemma 1] points out the fact that an integrable selection of Ψ exists if, instead of (i), we assume that there exists a function $\alpha(\cdot)$ in L^1 and a function $c\colon \mathbb{R}^+ \to \mathbb{R}$ such that

$$\|\partial_x(f(t, x, \xi))\| \leqslant \alpha(t) + c(\Delta) \qquad \text{for each} \quad t, \xi, |x| \leqslant \Delta.$$

LEMMA 2. *Let* $f, g\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *and set* $h(t, x, \xi) = f(t, x)\, g(t, \xi)$. *Then for each* t, x, ξ,

$$h^{**}(t, x, \xi) = f(t, x)\, G(t, \xi),$$

where

$$G(t, \xi) = \begin{cases} g^{**}(t, \xi) & \text{if } f(t, x) \geqslant 0; \\ -(-g)^{**}(t, \xi) & \text{if } f(t, x) < 0. \end{cases}$$

Proof. Let us suppose $f(t, x) < 0$, the other case ($f(t, x) \geqslant 0$) being similar. In this situation, the inequality

$$(-g)^{**}(t, \xi) \leqslant -g(t, \xi)$$

implies

$$-f(t, x)((-g)^{**}(t, \xi)) \leqslant f(t, x)\, g(t, \xi),$$

whence

$$f(t, x)\, G(t, \xi) \leqslant h^{**}(t, x, \xi). \tag{1}$$

Conversely, let t, x be fixed and ψ be any convex function satisfying $\psi(\xi) \leqslant f(t, x)\, g(t, \xi)$ for each ξ. Then

$$\frac{-1}{f(t, x)} \psi(\xi) \leqslant -g(t, \xi) \qquad \text{for each } \xi,$$

whence

$$\frac{-1}{f(t, x)} \psi(\xi) \leqslant (-g)^{**}(t, \xi).$$

In particular, for $\psi(\xi) = h^{**}(t, x, \xi)$,

$$h^{**}(t, x, \xi) \leqslant f(t, x)\, G(t, \xi). \tag{2}$$

The conclusion follows from (1) and (2).

MAIN RESULT

THEOREM 3. *Let L satisfy Hypothesis* (H). *Let us further suppose that the function* $x \mapsto L(t, x, \xi)$ *is concave for each* t, ξ. *Then, the problem*

$$\textit{minimize} \int_0^T L(t, x(t), x'(t))\, dt \tag{P}$$

on the subset of $W^{1,p}$ *of those functions satisfying* $x(0) = a$, $x(T) = b$, $x'(t) \in \Phi(t)$ *a.e. in* $[0, T]$ *admits a solution if and only if there exists a solution* $(\tilde{x}, p_1, ..., p_{n+1}, v_1, ..., v_{n+1})$ *to the associated relaxed problem* (PR′) *satisfying*

$$\bigcap_{i=1}^{n+1} \partial_x(-L(t, \tilde{x}(t), v_i(t))) \neq \varnothing \qquad a.e. \tag{C}$$

Note that, when $L(t, x, \xi)$ is differentiable in x, condition (C) reduces to

$$\frac{\partial L}{\partial x}(t, \tilde{x}(t), v_i(t)) = \frac{\partial L}{\partial x}(t, \tilde{x}(t), v_j(t)) \qquad \text{for each } i, j.$$

Proof of Theorem 3. The necessity is due to the fact that each solution to (P) satisfies (C).

Conversely, let $(\tilde{x}, p_i, ..., p_{n+1}, v_1, ..., v_{n+1})$ be a solution to (PR′) satisfying condition (C). By Lemma 1, let $\delta(\cdot) \in L^1$ be a selection to

$$t \mapsto \bigcap_{i=1}^{n+1} \partial_x(-L(t, \tilde{x}(t), v_i(t))).$$

Then, for each $y \in \mathbb{R}^n$ and $i \in \{1, ..., n+1\}$, we have

$$L(t, \tilde{x}(t), v_i(t)) \geqslant L(t, y, v_i(t)) + \langle \delta(t), y - \tilde{x}(t) \rangle, \tag{3}$$

($\langle \cdot, \cdot \rangle$) being the usual scalar product in $\mathbb{R}^n$. Set

$$B(t) = \int_0^t \delta(s)\, ds, \qquad f_i(t) = (v_i(t), L(t, \tilde{x}(t), v_i(t)), \langle v_i(t), B(t) \rangle).$$

The growth assumptions on L (Hypothesis (H)) imply that the conditions concerning the functions f_i stated in Theorem 2 are satisfied: let $E_1, ..., E_{n+1}$ be a measurable partition of $[0, T]$ such that

$$\sum_i v_i \chi_{E_i} \in L^p, \qquad \int_0^T \sum_i p_i(t)\, v_i(t)\, dt = \int_0^T \sum_i v_i(t)\, \chi_{E_i}(t)\, dt,$$

$$\int_0^T \sum_i p_i(t) \langle v_i(t), B(t)\rangle\, dt = \int_0^T \sum_i \langle v_i(t), B(t)\rangle\, \chi_{E_i}(t)\, dt, \tag{4}$$

$$\int_0^T \sum_i p_i(t)\, L(t, \bar{x}(t), v_i(t))\, dt = \int_0^T \sum_i L(t, \bar{x}(t), v_i(t))\, \chi_{E_i}(t)\, dt,$$

and set $\tilde{x}(t) = a + \int_0^t \sum_i v_i(s)\, \chi_{E_i}(s)\, ds$.

We show that $\tilde{x}$ is a solution to (P). Clearly, by (4), $\tilde{x}(T) = \bar{x}(T) = b$ and $\tilde{x} \in W^{1,p}$. Furthermore, by (3),

$$\sum_i L(t, \tilde{x}(t), v_i(t))\, \chi_{E_i}(t) \geqslant \sum_i L(t, \bar{x}(t), v_i(t))\, \chi_{E_i}(t) + \langle \delta(t), \tilde{x}(t) - \bar{x}(t)\rangle.$$

The integration of the above inequality and (4) yield

$$\begin{aligned} \min(\mathrm{PR}') &= \int_0^T \sum_i L(t, \bar{x}(t), v_i(t))\, \chi_{E_i}(t)\, dt \\ &\geqslant \int_0^T \sum_i L(t, \tilde{x}(t), v_i(t))\, \chi_{E_i}(t)\, dt + \int_0^T \langle \delta(t), \bar{x}(t) - \tilde{x}(t)\rangle\, dt. \end{aligned} \tag{5}$$

Let us remark that

$$\sum_i L(t, \tilde{x}(t), v_i(t))\, \chi_{E_i}(t) = L(t, \tilde{x}(t), \tilde{x}'(t))$$

and that the Tonelli–Fubini theorem and integration by parts give

$$\int_0^T \langle \delta(t), \tilde{x}(t)\rangle\, dt = \int_0^T \sum_i (\chi_{E_i}(t) - p_i(t)) \langle v_i(t), B(T) - B(t)\rangle\, dt.$$

Then, (4) and (5) together yield

$$\min(\mathrm{P}) \geqslant \min(\mathrm{PR}') \geqslant \int_0^T L(t, \tilde{x}(t), \tilde{x}'(t))\, dt \geqslant \min(\mathrm{P}):$$

the conclusion follows.

Some Applications

THEOREM 4 (Cellina and Colombo [2]). *Let $L(t, x, x') = g(t, x) + h(t, x')$ satisfy Hypothesis* (H) *and $x \mapsto g(t, x)$ be concave for a.e. t. Then Problem* (P) *admits at least one solution.*

Proof. Since we have $\partial_x(-L(t, x, \xi)) = \partial_x(-g(t, x))$ for each t, x, ξ then condition (C) trivially holds: Theorem 3 yields the conclusion.

We shall assume the following hypothesis:

HYPOTHESIS ($\tilde{\mathrm{H}}$). *Set $\Lambda(t) = a + \mathrm{co}\{\int_0^T \Phi(s)\, ds\}$ (see [7]). We assume that:*

($\tilde{\mathrm{H}}_1$) *the functions l, f, g: $[0, T] \times \mathbb{R}^n \to \mathbb{R}$ are such that $L(t, x, \xi) = l(t, x) + f(t, x)\, g(t, \xi)$ satisfies Hypothesis* (H) *for a.e. t, for each ξ, and for each $x \in \Lambda(t)$;*

($\tilde{\mathrm{H}}_2$) *either*

$$\text{for a.e. } t\text{: } f(t, x) > 0 \qquad \text{for each} \quad x \in \Lambda(t)$$

or

$$\text{for a.e. } t\text{: } f(t, x) < 0 \qquad \text{for each} \quad x \in \Lambda(t);$$

($\tilde{\mathrm{H}}_3$) *for a.e. t and $x \in \Lambda(t)$, the set $A(t, x) = \{\xi \in \Phi(t) : L^{**}(t, x, \xi) < L(t, x, \xi)\}$ is open and, on it, the function $\xi \mapsto L^{**}(t, x, \xi)$ is locally constant;*

($\tilde{\mathrm{H}}_4$) *there exist a function $\alpha(\cdot)$ in L^1 and a function c: $\mathbb{R}^+ \to \mathbb{R}$ such that*

$$\begin{aligned} &\text{for a.e. } t\text{: } \|\partial_x(f(t, x, \xi))\| \\ &\qquad \leqslant \alpha(t) + c(\Delta) \qquad \text{for each} \quad \xi \in \Phi(t),\ x \in \Lambda(t),\ |x| \leqslant \Delta. \end{aligned}$$

Let us remark that the class of non-trivial functions satisfying the hypothesis quoted above is non-empty.

EXAMPLE. $\Phi(t) = \mathbb{R}^+$, $a = 0$, $L(t, x, \xi) = -\gamma x^2 + (1 + x)\, |\xi - \phi(t)|\, |\xi - \psi(t)|$ ($\phi, \psi \in L^\infty$, $\phi, \psi \geqslant 0$, γ being strictly smaller than the best Sobolev constant) satisfies Hypothesis ($\tilde{\mathrm{H}}$).

As a further application of Theorem 3, we have the following

THEOREM 5. *Let l, f, g, Φ satisfy Hypothesis* ($\tilde{\mathrm{H}}$). *Then the problem*

$$\text{minimize } I(x) = \int_0^T l(t, x(t))\, dt + \int_0^T f(t, x(t))\, g(t, x'(t))\, dt \tag{P}$$

on the subset of $W^{1,p}$ of those $x(\cdot)$ satisfying $x(0)=a$, $x(T)=b$, $x'(t)\in\Phi(t)$ a.e. in $[0, T]$ admits at least one solution.

Proof. Clearly, in view of Theorem 3, it is enough to prove the existence of a solution $(\tilde{x}, p_1, ..., p_{n+1}, v_1, ..., v_{n+1})$ to (PR′) satisfying

$$L(t, x, v_i(t)) = L(t, x, v_j(t))$$

for each t, x and $i, j \in \{1, ..., n+1\}$. For this purpose, let $(\tilde{x}, p_1, ..., p_{n+1}, w_1, ..., w_{n+1})$ be an arbitrary solution to (PR′). Then, by Theorem 1

$$L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) = \sum_i p_i(t)\, L(t, \tilde{x}(t), w_i(t)). \tag{6}$$

The map $\xi \mapsto L^{**}(t, \tilde{x}(t), \xi)$ being convex, we can assume

$$L^{**}(t, \tilde{x}(t), w_i(t)) = L(t, \tilde{x}(t), w_i(t)) \qquad \text{a.e.} \tag{7}$$

Set $A = \{t : L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) < L(t, \tilde{x}(t), \tilde{x}'(t))\} = \{t : \tilde{x}'(t) \in A(t, \tilde{x}(t))\}$. By $(\tilde{\mathrm{H}}_3)$, for a.e. $t\in A$, the convex function $L^{**}(t, \tilde{x}(t), \cdot)$ is constant in a neighbourhood of $\tilde{x}'(t)$. As a consequence

$$L^{**}(t, \tilde{x}(t), \xi) \geqslant L^{**}(t, \tilde{x}(t), \tilde{x}'(t))$$

for a.e. $t\in A$ and each $\xi \in \mathbb{R}^n$. In particular

$$L^{**}(t, \tilde{x}(t), w_i(t)) \geqslant L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \qquad \text{for a.e.} \quad t \in A;$$

hence, by (6), we can assume

$$L^{**}(t, \tilde{x}(t), w_i(t)) = L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \qquad \text{for a.e.} \quad t\in A. \tag{8}$$

Equalities (7) and (8) prove that, if we set

$$v_i = w_i \chi_A + \tilde{x}' \chi_{[0,T]\setminus A}$$

then $(\tilde{x}, p_1, ..., p_{n+1}, v_1, ..., v_{n+1})$ is a solution to (PR′) satisfying

$$L(t, \tilde{x}(t), v_i(t)) = L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) \qquad \text{a.e.} \tag{9}$$

By Lemma 2, there exists a function G: $[0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L^{**}(t, x, \xi) = l(t, x) + f(t, x)\, G(t, \xi).$$

Thus (9) yields

$$g(t, v_i(t)) = G(t, \tilde{x}'(t)) \qquad \text{a.e.}$$

The claim is proved.

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