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# THE EXISTENCE QUESTION IN THE CALCULUS OF VARIATIONS: A DENSITY RESULT

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ABSTRACT. We show the existence of a dense subset $\mathscr{D}$ of $\mathscr{C}(\mathbb{R})$ such that, for $g$ in it, the problem

$$\text{minimum} \int_0^T g(x(t))\,dt + \int_0^T h(x'(t))\,dt\ ,\ \ x(0)=a\ ,\ x(T)=b$$

admits a solution for every lower semicontinuous $h$ satisfying growth conditions

## INTRODUCTION

The first basic aim of the Calculus of Variations is the problem of yielding sufficient conditions for the existence of solutions to the classical minimum problem

$$\text{(P)} \qquad \text{Minimize} \int_0^T f(t, x(t), x'(t))\,dt\ :\ x(0)=a\ ,\ x(T)=b$$

on a suitable space of functions $x$.The traditional answer to the sufficiency has been to impose that the map $x' \mapsto f(t, x, x')$ be convex (see [Ce]); more recently, in an effort to provide existence criteria other than convexity in $x'$, some special sufficient condition has been given. More precisely, for the class of maps of the form $f(t, x, x') = g(t, x) + h(t, x')$, existence of solutions has been obtained by requiring that the map $x \mapsto g(t, x)$ be monotonic (for $x$ in $\mathbb{R}$) [M] or, for $x$ in $\mathbb{R}^n$, that the same map be concave [C-C]. Hence, for maps of the form $g(x) + h(x')$ the property of yielding existence of solutions so far seems to belong to a very narrow and special class of functions: those that are either convex in $x'$ or concave or monotonic in $x$. Existence seems to be a property related to very special geometric behaviour in $x$ or in $x'$. The purpose of this note is to show that (for $x$ in $\mathbb{R}$) this is not so. We consider the class of functions $g(x) + h(x')$ where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $h$ is lower semicontinuous; we show that there exists a subset $\mathscr{D}$ of the space of continuous functions, dense for the topology of uniform convergence

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on compacta, such that, for $g$ in $\mathscr{D}$, problem (P) has existence of solutions for every function $h$ satisfying the usual growth conditions.

## Notation and preliminary results

Let us denote by $\partial f(x)$ the subdifferential of a function $f$ at $x$. Also, by $\operatorname{extr}(S)$ we mean the set of extreme points of $S$. The following lemma will be used later.

**Lemma 1.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be convex, l.s.c.,* $f(x') \geq \psi(|x'|) + \gamma$ $(\gamma \in \mathbb{R}$, $\psi \geq 0$ *convex, l.s.c. and* $\lim_{r\to\infty} \psi(r)/r = +\infty)$. *Then, for each measurable* $x' : I = [\alpha, \beta] \to \mathbb{R}$ *with the property that* $t \mapsto f(x'(t)) \in L^1$, *there exists an integrable selection* $\omega$ *of* $\Phi(t) : t \mapsto (\partial f)^{-1}(\partial f(x'(t)))$ *with values in* $\operatorname{extr}(\Phi(t))$ *and such that*

$$\int_I \omega(t)\,dt = \int_I x'(t)\,dt, \qquad \int_I f(\omega(t))\,dt = \int_I f(x'(t))\,dt$$

*and, for each* $t$,

$$\int_\alpha^t \omega(s)\,ds \leq \int_\alpha^t x'(s)\,ds.$$

*Proof.* Let us first remark that each inverse image under $\partial f$ of a point $c \in \mathbb{R}$ is closed, convex ($f$ being convex), and bounded, since $\lim_{x'\to\infty} f(x')/|x'| = +\infty$. It follows that each $(\partial f)^{-1}(c)$ is either empty or a closed interval. Let $(c_i)_{i\in J\subset\mathbb{N}}$ be the values of $\partial f$ whose inverse image under $\partial f$ is a nontrivial interval $[a_i, b_i]$ ($a_i \neq b_i$), and set $K_i = (x')^{-1}([a_i, b_i])$ for $i$ in $J$, $K_0 = I \setminus \bigcup_i K_i$. Then, by Lemma 3.4 in [A-C], there exists a measurable selection $\sigma_i$ of the set-valued map $\Phi(t)\chi_{K_i}(t)$ with values in $\operatorname{extr}(\Phi(t)\chi_{K_i}(t)) = \{a_i, b_i\}$ such that

$$\int_I \sigma_i(t)\chi_{K_i}(t)\,dt = \int_I x'(t)\chi_{K_i}(t)\,dt \tag{1}$$

and, for each $t$,

$$\int_\alpha^t \sigma_i(s)\chi_{K_i}(s)\,ds \leq \int_\alpha^t x'(s)\chi_{K_i}(s)\,ds. \tag{2}$$

For each $i \in J$, we have

$$\int_I f(x'(t))\chi_{K_i}(t)\,dt = \int_I f(a_i) + c_i(x'(t) - a_i)\chi_{K_i}(t)\,dt;$$

hence, by (1)

$$\begin{aligned}\int_I f(x'(t))\chi_{K_i}(t)\,dt &= \int_I f(a_i) + c_i(\sigma_i(t) - a_i)\chi_{K_i}(t)\,dt \\ &= \int_I f(\sigma_i(t))\chi_{K_i}(t)\,dt\end{aligned} \tag{3}$$

since $\sigma_i(t) \in \{a_i, b_i\}$. It follows that, for each $n \in \mathbb{N}$,

$$\int_I \psi\left(|x'|\chi_{K_0} + \sum_{i\leq n} |\sigma_i|\chi_{K_i}\right)\,dt \leq \int_I f(x'(t))\,dt - \int_I \gamma\,dt;$$

i.e., the functions $\omega_n = x'\chi_{K_0} + \sum_{i\leq n} \sigma_i \chi_{K_i}$, $n \in \mathbb{N}$, are equi-integrable. Set $\omega(t) = x'\chi_{K_0}(t) + \sum_i \sigma_i(t)\chi_{K_i}(t) = \lim_n \omega_n(t)$. Vitali's convergence theorem [E-T, VIII, Corollary 1.3] yields that $\omega$ is in $L^1$. By (1), (2), (3), $\omega$ has the required properties. □

**Definition.** $\mathscr{D}$ is the subset of $\mathscr{C}(\mathbb{R})$ of those piecewise affine functions $g$ such that: (i) any interval in which $g$ is affine with a given nonzero slope is bounded; (ii) adjacent to any such maximal interval, on both the left and the right, is one in which $g$ is constant; and (iii) any bounded set contains only finitely many such intervals.

**Proposition 1.** *$\mathscr{D}$ is dense in $\mathscr{C}(\mathbb{R})$.*

## Main result

Consider the following problem:

**Problem P.**

$$\text{Minimize} \int_0^T g(x(t))\,dt + \int_0^T h(x'(t))\,dt$$

on the subset of $W^{1,p}([0, T], \mathbb{R})$ of those functions satisfying the boundary conditions $x(0) = a$, $x(T) = b$.

We shall consider the following

**Growth condition G.** If $p = 1$, there exist a convex l.s.c. function $\psi$ such that $\lim_{r\to\infty} \psi(r)/r = +\infty$, $\gamma \in \mathbb{R}$, and $h(x') \geq \psi(|x'|) + \gamma$; if $p > 1$, there exist $\sigma > 0$ ($\beta/\sigma$ being strictly less than the best Sobolev constant), and $h(x') \geq \sigma|x'|^p + \gamma$.

We have the following

**Theorem 1.** *Let $p \geq 1$ and $g \in \mathscr{D}$ be such that $g(x) \geq \alpha - \beta|x|^p$ for every $x$ ($\alpha$, $\beta \in \mathbb{R}$). Let $h$ satisfy the growth condition* (G). *Then, Problem* (P) *admits at least a solution.*

*Proof.* Let $\tilde{x}$ be a solution to the relaxed problem associated to (P), and set

$$\Delta_1 = \min\{\tilde{x}(t) : t \in [0, T]\}, \qquad \Delta_2 = \max\{\tilde{x}(t) : t \in [0, T]\}.$$

Let $d_1$ be the greater discontinuity point of $g'$ less than or equal to $\Delta_1$, $d_2 < \cdots < d_{n-1}$ be those inside $]-\Delta_1, \Delta_2[$, $d_n$ be the next after $d_{n-1}$, and set

$$\epsilon = \tfrac{1}{5}\min\{|d_{i+1} - d_i| : i = 1, \ldots, n-1\}.$$

*Claim* (a). We claim that $[0, T]$ can be partitioned in a countable union of disjoint intervals $I_j$ ($j \in \mathbb{N}$) such that $g$ is monotonic on $\tilde{x}(I_j)$.

*Proof of the claim.* Consider the three sets $A$, $V$, $B$ defined by

$$A = \bigcup_{i=1}^{n-1} ]d_i + \epsilon, d_{i+1} - \epsilon[ , \qquad V = \{d_i - \epsilon, d_i + \epsilon : i = 1, \ldots, n\}$$

$$B = \bigcup_{i=1}^{n} ]d_i - 2\epsilon, d_i + 2\epsilon[.$$

By the continuity of $\tilde{x}$, the inverse image of $A$ under $\tilde{x}$ is a countable union of disjoint relatively open subintervals $(\sigma_i, \tau_i)$ of $[0, T]$. The image of each subinterval is contained in one of the open intervals $]d_i + \epsilon, d_{i+1} - \epsilon[$ on which

$g$ is affine. By the continuity of $\widetilde{x}$, there exists $\delta$ such that $|t-s|<\delta \Rightarrow |\widetilde{x}(t)-\widetilde{x}(s)|<\epsilon$. Consider those subintervals $(\sigma_i, \tau_i)$ whose diameter $\tau_i-\sigma_i \geq \delta$, say for $i=1, \ldots, m$. These are the first elements of our partition. Again by continuity, for each $i$, at least one between $\widetilde{x}(\sigma_i)$ and $\widetilde{x}(\tau_i)$ is in $V$ (actually both, except for the case $\sigma_i=0$ or $\tau_i=T$). Consider the finite union of closed subintervals of $[0, T]$ that is the complement of the finite union of open subintervals $(\sigma_i, \tau_i)$, $i=1, \ldots, m$; they are the intervals $[0, \sigma_1], \ldots, [\tau_{m-1}, \sigma_m], [\tau_m, T]$. For $t$ in this complement, $\widetilde{x}(t)$ is in $B$. In fact, either $t$ is in $\widetilde{x}^{-1}(A) \setminus \bigcup_{i=1}^m(\sigma_i, \tau_i)$ or $\widetilde{x}(t)$ is in $\bigcup_i[d_i-\epsilon, d_i+\epsilon]$.

In the first case, from the choice of $\delta$ and the remark on the behaviour at the extremes of the intervals, there exists $d_i$ such that $|\widetilde{x}(t)-d_i|<2\epsilon$, i.e., $\widetilde{x}(t) \in B$.

The second case holds since each $[d_i-\epsilon, d_i+\epsilon]$ is in $B$. Since $B$ is open, its counter image $\widetilde{x}^{-1}(B)$ is a countable collection of open subintervals. Consider the image of any such subinterval: it is contained in one (and only one) $]d_i-2\epsilon, d_i+2\epsilon[$. On one of $]d_i-2\epsilon, d_i[$ or $]d_i, d_i+2\epsilon[$ $g$ is constant, on the other affine; $g$ is monotonic on $]d_i-2\epsilon, d_i+2\epsilon[$. Intersect each subinterval with the finite collection of intervals $[\tau_i, \sigma_{i+1}]$ : $i=0, \ldots, m$. The union of this countable collection of intervals and of $(\sigma_i, \tau_i)$ : $i=1, \ldots, m$ is the required partition of $[0, T]$.

*Claim* (b). By [E-T, Proposition IX.3.1] in the case $p>1$ and by a slight modification of the same reasoning in the case $p=1$ under the "superlinearity" growth condition (G), there exist measurable $p_1, p_2, v_1, v_2 : [0, T] \to \mathbb{R}$ ($p_i \geq 0$, $p_1+p_2=1$) such that

$$\begin{cases} \widetilde{x}' = p_1 v_1 + p_2 v_2 \quad \text{a.e.}, \\ h^{**}(\widetilde{x}'(t)) = p_1(t) h(v_1(t)) + p_2(t) h(v_2(t)) \quad \text{a.e.}; \end{cases}$$

hence, $\operatorname{extr}(\partial h^{**})^{-1}(\partial h^{**}(\widetilde{x}'(t))) = \{v_1(t), v_2(t)\}$ a.e. Let $I_j=(\alpha_j, \beta_j)$ be one of the intervals considered in (a). Let us explicitly carry out the construction for the case $g'<0$ on $]d_{i-1}, d_i[$; the other cases are treated similarly. We claim that there exists a measurable partition $E_1^j$, $E_2^j$ of $[\alpha_j, \beta_j]$ such that:

(i) $v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j} \in L^p(\alpha_j, \beta_j)$;

(ii) $\int_{\alpha_j}^{\beta_j} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j}\, dt = \int_{\alpha_j}^{\beta_j} p_1 v_1 + p_2 v_2\, dt$ ;

(iii) $\int_{\alpha_j}^{t} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j}\, ds \geq \int_{\alpha_j}^{t} p_1 v_1 + p_2 v_2\, ds$ for each $t \in (\alpha_j, \beta_j)$;

(iv) $\int_{\alpha_j}^{\beta_j} p_1(t) h(v_1(t)) + p_2(t) h(v_2(t))\, dt$
$= \int_{\alpha_j}^{\beta_j} h(v_1(t)) \chi_{E_1^j}(t) + h(v_2(t)) \chi_{E_2^j}(t)\, dt$;

(v) $\left| \int_{\alpha_j}^{t} v_1 \chi_{E_1^j} + v_2 \chi_{E_2^j}\, ds - \int_{\alpha_j}^{t} p_1 v_1 + p_2 v_2\, ds \right| < \epsilon$ for each $t \in (\alpha_j, \beta_j)$.

*Proof of the claim.* Set $\psi(x')=|x'|^p$ if $p>1$. Let us first remark that by La Vallée-Poussin's Theorem, the set

$$\mathscr{H} = \left\{ f \in L^1(\alpha_j, \beta_j) : \int_{\alpha_j}^{\beta_j} \psi(|f|) \leq \int_{\alpha_j}^{\beta_j} h^{**}(\widetilde{x}'(t))\, dt - \int_{\alpha_j}^{\beta_j} \gamma\, dt \right\}$$

is equi-integrable. Let $\rho>0$ be such that $\int_A |f|\, dt < \epsilon/2$ for each measurable subset $A$ of $I_j$ whose measure is less than $\rho$ and for each $f \in \mathscr{H}$. Let $\alpha_j = \gamma_0 < \gamma_1 < \cdots < \gamma_m = \beta_j$ be a subdivision of $I_j$ such that $\max_i |\gamma_{i+1}-\gamma_i| < \rho$. By Lemma 1 and the above remark on the inverse image of the subdifferential of

$h^{**}$, for each interval $[\gamma_k, \gamma_{k+1}]$, there exists a measurable partition $E_{1,k}$, $E_{2,k}$ satisfying:

(i)$_k$ $v_1\chi_{E_{1,k}} + v_2\chi_{E_{2,k}} \in L^p(\gamma_k, \gamma_{k+1})$;

(ii)$_k$ $\int_{\gamma_k}^{\gamma_{k+1}} v_1\chi_{E_{1,k}} + v_2\chi_{E_{2,k}}\, dt = \int_{\gamma_k}^{\gamma_{k+1}} p_1v_1 + p_2v_2\, dt$ ;

(iii)$_k$ $\int_{\gamma_k}^{t} v_1\chi_{E_{1,k}} + v_2\chi_{E_{2,k}}\, ds \geq \int_{\gamma_k}^{t} p_1v_1 + p_2v_2\, ds$ for each $t \in (\gamma_k, \gamma_{k+1})$;

(iv)$_k$ $\int_{\gamma_k}^{\gamma_{k+1}} \sum_{i=1}^2 p_i(t)h(v_i(t))\, dt = \int_{\gamma_k}^{\gamma_{k+1}} \sum_{i=1}^2 h(v_i(t))\chi_{E_{i,k}}(t)\, dt$.

Set $E_1^j = \bigcup_{k=0}^{m-1} E_{1,k}$, $E_2^j = \bigcup_{k=0}^{m-1} E_{2,k}$. Then (i), (ii), (iii), (iv) can be trivially deduced from their corresponding (i)$_k$, (ii)$_k$, (iii)$_k$, (iv)$_k$. In order to prove (v), fix $t \in (\alpha_j, \beta_j)$ and let $k$ be such that $t \in [\gamma_k, \gamma_{k+1}[$. Let us write that

$$\int_{\alpha_j}^{t} \sum_i (p_i - \chi_{E_i^j})v_i\, ds = \int_{\alpha_j}^{\gamma_k} \sum_i (p_i - \chi_{E_i^j})v_i\, dt + \int_{\gamma_k}^{t} \sum_i (p_i - \chi_{E_i^j})v_i\, ds. \tag{4}$$

By (ii)$_k$, the first term of the right-hand side of the above equality is zero. Furthermore, we have

$$\psi(|v_1|\chi_{E_1^j} + |v_2|\chi_{E_2^j}) \leq h(v_1(t))\chi_{E_1^j}(t) + h(v_2(t))\chi_{E_2^j}(t) - \gamma$$

so that, by (iv)$_k$, $v_1\chi_{E_1^j} + v_2\chi_{E_2^j} \in \mathscr{H}$. Let us recall that $|t - \gamma_k| < \rho$; hence, by equi-integrability we have

$$\left| \int_{\alpha_j}^{t} \sum_i (p_i - \chi_{E_i^j})v_i\, ds \right| \leq \int_{\gamma_k}^{t} p_1|v_1| + p_2|v_2|\, ds + \int_{\gamma_k}^{t} |v_1|\chi_{E_1^j} + |v_2|\chi_{E_2^j}\, ds$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon ,$$

which proves the claim.

*Claim* (c). Let us denote by $\overline{x}_j : [\alpha_j, \beta_j] = I_j \to \mathbb{R}$ the function defined by

$$\overline{x}_j(t) = \widetilde{x}(\alpha_j) + \int_{\alpha_j}^{t} v_1\chi_{E_1^j} + v_2\chi_{E_2^j}\, ds.$$

Then, by (i), $\overline{x}_j' \in L^p(\alpha_j, \beta_j)$, and by (ii) we have $\overline{x}_j(\beta_j) = \widetilde{x}(\beta_j)$. Furthermore, by (iv),

$$\int_{\alpha_j}^{\beta_j} h^{**}(\widetilde{x}'(t))\, dt = \int_{\alpha_j}^{\beta_j} h(\overline{x}_j'(t))\, dt. \tag{5}$$

Since, by definition, $\overline{x}_j(\alpha_j) = \widetilde{x}(\alpha_j)$ then, by (iii), $\overline{x}_j(t) \geq \widetilde{x}(t)$ for every $t \in I_j$. Moreover, by (v), $|\overline{x}_j(t) - \widetilde{x}(t)| < \epsilon$ for every $t \in I_j$; whence, $\overline{x}_j(t) \in ]d_i - 3\epsilon, d_i + 3\epsilon[$. Then, for $g$ nonincreasing on the above interval,

$$g(\overline{x}_j(t)) \leq g(\widetilde{x}(t)) \quad \text{for every } t \in I_j. \tag{6}$$

Now, (5) and (6) together give

$$\int_{I_j} g(\overline{x}_j(t))\, dt + \int_{I_j} h(\overline{x}_j'(t))\, dt \leq \int_{I_j} g(\widetilde{x}(t))\, dt + \int_{I_j} h^{**}(\widetilde{x}'(t))\, dt. \tag{7}$$

Let $\overline{x} : [0, T] \to \mathbb{R}$ be the function whose restriction to each $I_j$ ($j \in \mathbb{N}$) is $\overline{x}_j$.

Then $\overline{x} \in W^{1,p}$ and $\overline{x}(0) = \tilde{x}(0)$, $\overline{x}(T) = \tilde{x}(T)$. Moreover, by (7)

$$\begin{aligned}\inf(\mathrm{P}) &\leq \int_0^T g(\overline{x}(t))\,dt + \int_0^T h(\overline{x}'(t))\,dt \\ &\leq \int_0^T g(\tilde{x}(t))\,dt + \int_0^T h^{**}(\tilde{x}'(t))\,dt \\ &= \min(\mathrm{PR}) \leq \inf(\mathrm{P}).\end{aligned}$$

It follows that the above inequalities are in fact equalities: $\overline{x}$ is a solution to (P). □

*Remarks.* (a) The proof of Theorem 1 holds for a class of maps larger than $\mathscr{D}$. In fact, it is enough to assume that on each interval either $g' = 0$ or $g'$ is of constant sign and that between two intervals having different signs of $g'$, there is one with $g' = 0$.

(b) The classical example of a variational problem not having a solution is given by

$$h(x') = |x' - 1||x' + 1| \qquad \text{and} \qquad g(x) = x^2, \quad x(0) = 0 = x(1).$$

By remark (a), by modifying the function $x \mapsto x^2$ in an arbitrarily small neighbourhood of zero as a constant function, one obtains that the modified problem admits solutions for every choice of the boundary data.

(c) The technical tool used, Lemma 1, is very specific to one-dimensionality; in [A-M] a counterexample to the validity of the very same lemma in higher dimension is presented. Still the authors are inclined to believe that a possibly weaker statement, e.g., density of both $g$ and $h$, should hold in general.

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