Équations différentielles/*Ordinary Differential Equations*

Oriented measures and the bang-bang principle

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Abstract – We introduce the notion of an oriented measure. The interior of the range $\mathcal{R}$ of a non-atomic oriented measure μ is $\{\mu(\mathrm{E}) : \chi_\mathrm{E}$ has n discontinuity points$\}$ and moreover $\mu(\mathrm{E}) \in \partial\mathcal{R} \Leftrightarrow \chi_\mathrm{E}$ has less than $n-1$ discontinuity points.

Given a solution x to $x^{(n)} + a_{n-1}(t)\, x^{(n-1)} + \ldots + a_1(t)\, x' + a_0(t)\, x \in [\phi_1, \phi_2]$ on $[a, b]$ there exist two bang-bang solutions y and z having a contact of order n with x at a and b such that $y \leqq x \leqq z$. An application to the calculus of variations yields a density result.

Mesures orientées et principe du boum-boum

Résumé – *Nous introduisons la notion de mesure orientée. L'intérieur de l'image $\mathcal{R}$ d'une mesure orientée non-atomique μ est l'ensemble $\{\mu(\mathrm{E}) : \chi_\mathrm{E}$ a n points de discontinuité$\}$ et $\mu(\mathrm{E}) \in \partial\mathcal{R} \Leftrightarrow \chi_\mathrm{E}$ a au plus $n-1$ points de discontinuité.*

Si x est une solution de $x^{(n)} + a_{n-1}(t)\, x^{(n-1)} + \ldots + a_1(t)\, x' + a_0(t)\, x \in [\phi_1, \phi_2]$ sur $[a, b]$ il existe deux solutions boum-boum y et z ayant un contact d'ordre n avec x en a et b et telles que $y \leqq x \leqq z$. Une application au calcul des variations donne un résultat de densité.

Version française abrégée – Nous introduisons la notion de mesure orientée ([3], [4]).

Définition 1. – Une mesure vectorielle $\mu = (\mu_1, \ldots, \mu_n)$ sur $[a, b]$ est dite orientée si pour tout entier k dans $\{1, \ldots, n\}$ et pour tout k-uplet $\mathrm{I}_1, \ldots, \mathrm{I}_k$ d'intervalles non-triviaux de $[a, b]$ deux à deux disjoints, le déterminant

$$\begin{vmatrix} \mu_1(\mathrm{I}_1) & \cdots & \mu_1(\mathrm{I}_k) \\ \vdots & \ddots & \vdots \\ \mu_k(\mathrm{I}_1) & \cdots & \mu_k(\mathrm{I}_k) \end{vmatrix}$$

ne s'annule pas.

Posons $\Gamma_k = \{(\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k : a \leqq \gamma_1 \leqq \ldots \leqq \gamma_k \leqq b\}$.

A tout k-uplet $\gamma = (\gamma_1, \ldots, \gamma_k)$ de Γ_k nous associons les deux ensembles

$$\mathrm{E}_\gamma^- = \bigcup_{\substack{0 \leqq i \leqq k \\ i \text{ impair}}} [\gamma_i, \gamma_{i+1}], \qquad \mathrm{E}_\gamma^+ = \bigcup_{\substack{0 \leqq i \leqq k \\ i \text{ pair}}} [\gamma_i, \gamma_{i+1}].$$

Théorème 1. – *Soit μ une mesure non-atomique orientée sur $[a, b]$ et $\rho \in \mathrm{L}^1([a, b])$, $0 \leqq \rho \leqq 1$. Il existe deux n-uplets $\alpha = (\alpha_1, \ldots, \alpha_n)$ et $\beta = (\beta_1, \ldots, \beta_n)$ dans Γ_n tels que*

$$(\star) \qquad \mu(\mathrm{E}_\alpha^-) = \int_a^b \rho\, d\mu = \mu(\mathrm{E}_\beta^+).$$

Si de plus $0 < \rho < 1$ alors α et β sont les seuls éléments de Γ_n vérifiant $(\star)$ et en outre

$$a < \alpha_1 < \ldots < \alpha_n < b, \qquad a < \beta_1 < \ldots < \beta_n < b.$$

Nous formulons un nouveau résultat sur les solutions boum-boum d'un problème de contrôle décrit par une équation différentielle linéaire. Considérons le problème de contrôle

$$\text{(P)} \quad \mathrm{L}(\mathrm{D})\, x = x^{(n)} + a_{n-1}(t)\, x^{(n-1)} + \ldots + a_1(t)\, x' + a_0(t)\, x \in [\phi_1, \phi_2] \quad p.p. \text{ sur } [a, b]$$

avec $a_0, \ldots, a_{n-1}$ dans $\mathcal{C}^{n-2}([a, b])$ et ϕ_1, ϕ_2 dans $\mathrm{L}^1([a, b])$ telles que $\phi_1 \leqq \phi_2$.

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La fonction y est une solution boum-boum de (P) si $L(D)\,y \in \{\phi_1, \phi_2\}$.

THÉORÈME 2. – *Soit* $x \in W^{n,1}([a, b])$ *une solution du problème de contrôle* (P). *Alors il existe deux solutions boum-boum* y *et* z *vérifiant les conditions de tangence*

$$\forall k \in \{0, \ldots, n-1\}, \quad y^{(k)}(a) = x^{(k)}(a) = z^{(k)}(a), \qquad y^{(k)}(b) = x^{(k)}(b) = z^{(k)}(b)$$

et les inégalités $y \leqq x \leqq z$. *De plus* $L(D)\,y$ *et* $L(D)\,z$ *sont de la forme* $\chi_E \phi_1 + (1 - \chi_E)\,\phi_2$ *où l'ensemble* E *est une réunion finie d'intervalles.*

Nous appliquons ce résultat au calcul des variations.

A theorem of Lyapunov states that the range $\mathcal{R}$ of a non-atomic vector measure μ on $[a, b]$

$$\mathcal{R} = \{\mu(A) : \text{A measurable subset of } [a, b]\}$$

coincides with the convex set

$$\left\{\int_a^b \rho\, d\mu : 0 \leqq \rho \leqq 1\right\}.$$

However for a given ρ, $0 \leqq \rho \leqq 1$, the usual proofs based on convexity-extreme points arguments [5] are not constructive and do not give any information about the existence of a "nice" set E (for instance a finite union of intervals) such that

$$\mu(E) = \int_a^b \rho\, d\mu.$$

We introduce ([3], [4]) the notion of oriented measures.

DEFINITION 1. – A vector measure $\mu = (\mu_1, \ldots, \mu_n)$ on $[a, b]$ is said to be oriented on $[a, b]$ if for each k in $\{1, \ldots, n\}$ and for each k-tuple of non-trivial disjoint intervals $I_1, \ldots, I_k$ of $[a, b]$ the determinant

$$\begin{vmatrix} \mu_1(I_1) & \cdots & \mu_1(I_k) \\ \vdots & \ddots & \vdots \\ \mu_k(I_1) & \cdots & \mu_k(I_k) \end{vmatrix}$$

does not vanish.

We shall use the following notations: if $I_1, \ldots, I_n$ are n intervals, by $I_1 < \ldots < I_n$ we mean that $I_1, \ldots, I_n$ are non-trivial and for almost all n-tuple $(x_1, \ldots, x_n)$ of $I_1 \times \ldots \times I_n$ we have $x_1 < \ldots < x_n$. If $\mu = (\mu_1, \ldots, \mu_n)$ is a vector measure on $[a, b]$ and ρ belongs to $L^1([a, b])$, we note

$$\int_a^b \rho\, d\mu = \left(\int_a^b \rho\, d\mu_1, \ldots, \int_a^b \rho\, d\mu_n\right).$$

We shall denote by Γ_k the set

$$\Gamma_k = \{(\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k : a \leqq \gamma_1 \leqq \cdots \leqq \gamma_k \leqq b\}.$$

To each k-tuple $\gamma = (\gamma_1, \ldots, \gamma_k)$ belonging to Γ_k we associate the two sets

$$E_\gamma^- = \bigcup_{\substack{0 \leqq i \leqq k \\ i \text{ odd}}} [\gamma_i, \gamma_{i+1}], \qquad E_\gamma^+ = \bigcup_{\substack{0 \leqq i \leqq k \\ i \text{ even}}} [\gamma_i, \gamma_{i+1}]$$

where by convention $\gamma_0 = a$, $\gamma_{k+1} = b$.

An oriented measure has the following remarkable properties:

THEOREM 3. – *Let μ be a non-atomic oriented measure on* $[a, b]$ *and let* ρ *be a measurable function defined on* $[a, b]$ *with values in* $[0, 1]$.

There exist two n-uples $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and* $\beta = (\beta_1, \ldots, \beta_n)$ *in* Γ_n *such that*

$$(\star) \qquad \mu\,(\mathrm{E}_\alpha^-) = \int_a^b \rho\, d\mu = \mu\,(\mathrm{E}_\beta^+).$$

If in addition $0 < \rho < 1$ *then* α *and* β *in* Γ_n *satisfying* $(\star)$ *are unique and verify*

$$a < \alpha_1 < \ldots < \alpha_n < b, \qquad a < \beta_1 < \ldots < \beta_n < b.$$

Let $\theta : \Gamma_n \to \mathcal{R}$ be the function defined by $\theta\,(\gamma) = \mu\,(\mathrm{E}_\gamma^-)$.

PROPOSITION 1. – *The interior* $\mathring{\mathcal{R}}$ *of the range* $\mathcal{R}$ *coincides with* $\left\{ \int_a^b \rho d\mu : 0 < \rho < 1 \right\}$.

THEOREM 4. – *The range of* θ *coincides with* $\mathcal{R}$; *the map* θ *induces an homeomorphism from* $\mathring{\Gamma}_n$ *onto* $\mathring{\mathcal{R}}$ *and maps* $\partial\Gamma_n$ *onto* $\partial\mathcal{R}$.

COROLLARY 1. – *If* $\mu\,(\mathrm{E})$ *belongs to* $\partial\mathcal{R}$ *and* $\mu\,(\mathrm{E}) = \mu\,(\mathrm{F})$ *then* $\mathrm{E}\Delta\mathrm{F}$ *is negligible.*

Our approach discloses the recursive structure of the boundary of the range of an oriented measure. For k belonging to $\{0, \ldots, n\}$ let

$$\mathcal{R}_k^- = \{\mu\,(\mathrm{E}_\gamma^-) : \gamma \in \Gamma_k\}, \qquad \mathcal{R}_k^+ = \{\mu\,(\mathrm{E}_\gamma^+) : \gamma \in \Gamma_k\}.$$

PROPOSITION 2. – *The function* $\gamma \in \mathring{\Gamma}_k \mapsto \mu\,(\mathrm{E}_\gamma^-) \in \mathcal{R}_k^-$ (resp. $\gamma \in \mathring{\Gamma}_k \mapsto \mu\,(\mathrm{E}_\gamma^+) \in \mathcal{R}_k^+$) *is an homeomorphism from* $\mathring{\Gamma}_k$ *onto its range which coincides with* $\mathring{\mathcal{R}}_k^-$ (resp. $\mathring{\mathcal{R}}_k^+$).

PROPOSITION 3. –*The boundary of the range* $\mathcal{R}$ *of an oriented non-atomic n-dimensional measure is partitioned into*

$$\partial\mathcal{R} = \mathring{\mathcal{R}}_{n-1}^- \cup \ldots \cup \mathring{\mathcal{R}}_1^- \cup \{0\} \cup \{\mu\,(a, b)\} \cup \mathring{\mathcal{R}}_1^+ \cup \ldots \cup \mathring{\mathcal{R}}_{n-1}^+.$$

Let T *be the symmetry with respect to* $\mu(a, b)/2$. *Then for each k belonging to* $\{0, \ldots, n\}$ *we have* $\mathrm{T}\,(\mathring{\mathcal{R}}_k^-) = \mathring{\mathcal{R}}_k^+$, $\mathrm{T}\,(\mathcal{R}_k) = \mathcal{R}_k$.

One of the essential tools for the proofs is the following lemma which allows to perturb a function leaving its integral unchanged.

LEMMA 1. – *Let* $\mu = (\mu_1, \ldots, \mu_n)$ *be a non-atomic oriented measure on the interval* $[a, b]$ *and* $\mathrm{I}_0 < \mathrm{I}_1 < \ldots < \mathrm{I}_n$ *be* $n+1$ *subintervals of* $[a, b]$. *Then, given a positive* ϵ, *there exist* $n+1$ *positive real numbers* $\lambda_0, \ldots, \lambda_n$ *such that*

$$\forall l \in \{0, \ldots, n\}, \quad 0 < \lambda_l < \epsilon \qquad \text{and} \qquad \sum_{l=0}^{n} (-1)^l\, \lambda_l\, \mu\,(\mathrm{I}_l) = 0.$$

Let μ be an absolutely continuous measure on $[a, b]$ whose density functions $f_1, \ldots, f_n$ satisfy the

ORIENTATION CONDITION Δ. – We say that n real functions $f_1, \ldots, f_n$ verify condition Δ on an interval $[a, b]$ if for each k in $\{1, \ldots, n\}$, the determinant

$$\begin{vmatrix} f_1\,(x_1) & f_1\,(x_2) & \ldots & f_1\,(x_k) \\ f_2\,(x_1) & f_2\,(x_2) & \ldots & f_2\,(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_k\,(x_1) & f_k\,(x_2) & \ldots & f_k\,(x_k) \end{vmatrix}$$

is not equal to zero whenever the $x_i \in [a, b]$ are distinct and its sign is constant on the k-tuples $(x_1, \dots, x_k)$ such that $a \leqq x_1 < x_2 < \dots < x_k \leqq b$.

Then the following nice formula

$$\begin{vmatrix} \int_{I_1} f_1 & \cdots & \int_{I_k} f_1 \\ \vdots & \ddots & \vdots \\ \int_{I_1} f_k & \cdots & \int_{I_k} f_k \end{vmatrix} = \int \cdots \int_{I_1 \times \dots \times I_k} \begin{vmatrix} f_1(s_1) & \cdots & f_1(s_k) \\ f_2(s_1) & \cdots & f_2(s_k) \\ \vdots & \ddots & \vdots \\ f_k(s_1) & \cdots & f_k(s_k) \end{vmatrix} ds_1 \dots ds_k$$

shows that the measure μ is oriented.

We also give an operational criterion for the fulfillment of the condition Δ.

If $f_1, \dots, f_{k+1}$ are of class C^k on $[a, b]$ we denote their Wronskian by

$$W(f_1, \dots, f_{k+1})(t) = \begin{vmatrix} f_1(t) & \cdots & f_{k+1}(t) \\ \vdots & \ddots & \vdots \\ f_1^{(k)}(t) & \cdots & f_{k+1}^{(k)}(t) \end{vmatrix}.$$

PROPOSITION 4. – *Let $f_1, \dots, f_n \in C^{n-1}([a, b])$ be such that*

$$\forall t \in [a, b], \quad W(f_1)(t) \neq 0, \dots, W(f_1, \dots, f_n)(t) \neq 0.$$

Then $f_1, \dots, f_n$ satisfy the orientation condition Δ.

As a consequence whenever a function $x \in C^{2n-2}([a, b])$ satisfies

$$x(0) = \dots = x^{(n-2)}(0) = 0 \quad \text{and} \quad x^{(n-1)}(0) = 1$$

then the n functions $x^{(n-1)}, \dots, x', x$ verify Δ on a neighbourhood of 0. This allows us to formulate a new result concerning bang-bang solutions to linear control systems described by a generic linear differential equation. We consider the n-dimensional linear control system

$$\text{(P)} \quad L(D)x = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x \in [\phi_1, \phi_2] \ a.e. \text{ on } [a, b]$$

where the n functions $a_0, \dots, a_{n-1}$ belong to $C^{n-2}([a, b])$ and ϕ_1, ϕ_2 in $L^1([a, b])$ verify $\phi_1 \leqq \phi_2$. The function y is said to be a bang-bang solution to (P) if $L(D)y \in \{\phi_1, \phi_2\}$.

THEOREM 5. – *Let x in $W^{n,1}([a, b])$ be a solution to the control problem (P). Then there exist two bang-bang solutions y and z satisfying the tangency conditions*

$$\forall k \in \{0, \dots, n-1\}, \qquad y^{(k)}(a) = x^{(k)}(a) = z^{(k)}(a), \qquad y^{(k)}(b) = x^{(k)}(b) = z^{(k)}(b)$$

and the inequalities $y \leqq x \leqq z$. Moreover the functions $L(D)y$ and $L(D)z$ are of the form $\chi_E\phi_1 + (1 - \chi_E)\phi_2$ where the set E is a finite union of intervals.

Finally, we give an application to the calculus of variations. We consider the problem of minimizing the integral functionals

$$I(x, u) = \int_a^b f(t, x(t), u(t))\, dt$$

where $x : [a, b] \to \mathbb{R}^n$ is such that $x^{(k)}(a), x^{(k)}(b)$ $(0 \leqq k \leqq n-1)$ are fixed and u is a control belonging to $U(t, x) \subset \mathbb{R}^n$. If $u \mapsto f(t, x, u)$ is not convex the problem does not always admit a solution. Our results yield a straightforward generalization of [1].

THEOREM 6. – *There exists a dense subset $\mathcal{D}$ of $C(\mathbb{R})$ for the uniform convergence such that for g in $\mathcal{D}$ the problem*

$$\min\left\{\int_a^b g(x(t))\,dt + \int_a^b h(\mathrm{L}(\mathrm{D})\,x(t))\,dt : x \in \mathrm{W}^{n,1}([a, b])\right\}$$

with prescribed initial and final conditions admits at least one solution for every lower semicontinuous function h satisfying the growth condition $h(u) \geqq c\psi(|u|)$, ψ *being l.s.c. and convex,* $\lim_{r \to +\infty} \psi(r)/r = +\infty$.

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