AN ALTERNATIVE PROOF OF HAN – SHAFRIR'S LOWER BOUND FOR THE ENERGY OF MAPS IN PERFORATED DOMAINS (REVISED VERSION)

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February 1996

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Introduction

Let G be an open subset of $\mathbb{R}^2$, $x_1, \dots, x_n$ be n distinct points in G, ρ, μ be positive real numbers and set $\Omega = G \setminus \cup_{i=1}^n B(x_i, \rho)$. We make the following assumptions:

$$d(x_i, \partial G) \geq 2\rho \vee \mu, \quad i = 1, \dots, n; \tag{1.1}$$

$$|x_i - x_j| \geq 4\rho \text{ for all } i \neq j \text{ in } \{1, \dots, n\}; \tag{1.2}$$

$$u \in C^1(\Omega, S^1) \cap C(\cup_{i=1}^n \partial B(x_i, \rho), S^1); \tag{1.3}$$

$$\deg(u, \partial B(x_i, \rho)) = d_i \in \mathbb{Z}, \quad i = 1, \dots, n. \tag{1.4}$$

We write $I_n = \{1, \dots, n\}$ and denote by $\mathcal{P}_n$ the set of the partitions of I_n; for each subset I of I_n and for each $\mathcal{A}$ in $\mathcal{P}_n$ we put $d_I = \sum\limits_{i \in I} d_i$, $\sigma(\mathcal{A}) = \sum\limits_{I \in \mathcal{A}} d_I^2$, $\|d\| = \sum\limits_i |d_i|$ and $F(d_1, \dots, d_n) = \min\{\sigma(\mathcal{A}) : \mathcal{A} \in \mathcal{P}_n\}$. We are concerned with the following:

Theorem 1.1. [1] *Assuming (1.1) - (1.4) there exists a constant C depending only on n and $\|d\|$ such that*

$$\int_\Omega |\nabla u|^2 \geq 2\pi F(d_1, \dots, d_n) \log(\mu/\rho) - C.$$

This lower bound is essential to obtain the main result in [1] where it has been proved in a less general form [Th. 2.1]; this optimal version is due to Z. C. Han and I. Shafrir. Here we give an alternative proof of their result: we use their "elementary" Lemma [2, Lemma 1] but instead of analysing the possibilities for the points $x_1, \dots, x_n$ to regroup together in the domain we make a simple construction of circular annuli on which the integral of the function $|\nabla u|^2$ naturally exhibits the quantity $F(d_1, \dots, d_n)$.

2. Notations and preliminary results

We denote by Γ the family of triples $(\mathcal{A}, (x_I)_{I \in \mathcal{A}}, (\rho_I^{\mathcal{A}})_{I \in \mathcal{A}})$ where $\mathcal{A} \in \mathcal{P}_n$ and, for every I in $\mathcal{A}$, $x_I \in \mathbb{R}^2$ and $\rho_I^{\mathcal{A}} > 0$. We put

$$\delta_{\mathcal{A}} = \min_{I \neq J \in \mathcal{A}} |x_I - x_J|, \quad \forall I \in \mathcal{A}: \quad r_I^{\mathcal{A}} = \frac{\delta_{\mathcal{A}}}{2} \vee \rho_I^{\mathcal{A}}, \quad R_{\mathcal{A}} = \vee \max_{I \in \mathcal{A}} r_I^{\mathcal{A}};$$

we simply write ρ_I (resp. r_I) instead of $\rho_I^{\mathcal{A}}$ (resp. $r_I^{\mathcal{A}}$) when no ambiguity may occur.

Empty–Intersection Property (Δ). $(\mathcal{A}, (x_I)_{I \in \mathcal{A}}, (\rho_I)_{I \in \mathcal{A}}) \in \Gamma$ verifies (Δ) if

$$\forall I, J \in \mathcal{A} \quad I \neq J \Rightarrow B(x_I, \rho_I) \cap B(x_J, \rho_J) = \emptyset \text{ and } x_J \notin B(x_I, 2\rho_I)$$

The interest in (Δ) relies on the fact that it allows to enlarge the balls $B(x_I, \rho_I)$ without affecting the empty–intersection property.

Proposition 2.1. *Assume that* $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(\rho_I)_{I\in\mathcal{A}})\in\Gamma$ *satisfies* (Δ). *Then the triple* $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(r_I)_{I\in\mathcal{A}})$ *satisfies* (Δ).

Proof. Let $I\neq J$ in $\mathcal{A}$. Since $2r_I=\delta_{\mathcal{A}}\vee 2\rho_I$ then $x_J\notin B(x_I,2\rho_I)$.
Assume that $B(x_I,r_I)\cap B(x_J,r_J)\neq\emptyset$; then necessarily we have $|x_I-x_J|<r_I+r_J$ so that $\delta_{\mathcal{A}}<r_I+r_J$ and thus either $r_I=\rho_I>\delta_{\mathcal{A}}/2$ or $r_J=\rho_J>\delta_{\mathcal{A}}/2$. Property (Δ) implies that $(\rho_I,\rho_J)\neq(r_I,r_J)$; if for instance $r_I=\rho_I>\delta_{\mathcal{A}}/2$ and $r_J=\delta_{\mathcal{A}}/2$ then we obtain $|x_I-x_J|<r_I+r_J=\rho_I+\delta_{\mathcal{A}}/2<2\rho_I$, contradicting (Δ). □

We shall denote by $|X|$ the cardinality of a set X. We introduce a relation in $\mathcal{P}_n$: we write $\mathcal{A}\succ\mathcal{B}$ if $|\mathcal{A}|>|\mathcal{B}|$ and if for each $I\in\mathcal{A}$ there exists $J\in\mathcal{B}$ such that $I\subset J$.

Proposition 2.2. *Let* $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(\rho_I^{\mathcal{A}})_{I\in\mathcal{A}})\in\Gamma$ *and* $|\mathcal{A}|\geq 2$. *There exist* $J_{\mathcal{B}}\subset I_n$, $x_{J_{\mathcal{B}}}\in\mathbb{R}^2$, $\rho_{J_{\mathcal{B}}}>0$ *such that for every* I *in* $\mathcal{A}$ *either* $I\subsetneqq J_{\mathcal{B}}$ *and* $B(x_I,R_{\mathcal{A}})\subset B(x_{J_{\mathcal{B}}},\rho_{J_{\mathcal{B}}})$ *or* $I\cap J_{\mathcal{B}}=\emptyset$ *and* $x_I\notin B(x_{J_{\mathcal{B}}},3\rho_{J_{\mathcal{B}}})$. *Moreover*

$$(\circ)\qquad 4R_{\mathcal{A}}\leq\rho_{J_{\mathcal{B}}}\leq 4^nR_{\mathcal{A}}.$$

Lemma 2.3. *Let* $x_1,\ldots,x_m\in\mathbb{R}^2, r_m>0, x_{m+1}\in B(\frac{x_1+\cdots+x_m}{m},3r_m)$. *Then*

$$B(\frac{x_1+\cdots+x_m}{m},r_m)\cup B(x_{m+1},r_m)\subset B(\frac{x_1+\cdots+x_{m+1}}{m+1},r_{m+1})\text{ where } r_{m+1}=4r_m.$$

Proof of Lemma 2.3. It is enough to apply the triangular inequality □

Proof of Proposition 2.2. Let $I_1\neq I_2$ in $\mathcal{A}$ be such that $|x_{I_2}-x_{I_1}|=\delta_{\mathcal{A}}$: then $x_{I_2}\in B(x_{I_1},3\frac{\delta_{\mathcal{A}}}{2})$ so that if we put $r_1=R_{\mathcal{A}}$ we have $x_{I_2}\in B(x_{I_1},3r_1)\setminus\{x_{I_1}\}$. For $k\geq 2$ we define inductively

$$\bar{x}_k=\frac{x_{I_1}+\cdots+x_{I_k}}{k},\quad r_k=4^{k-1}R_{\mathcal{A}},\quad E_k=B(\bar{x}_k,3r_k)\cap E\setminus\{x_{I_1},\ldots,x_{I_k}\}$$

and if E_k is not empty let $I_{k+1}\in\mathcal{A}\setminus\{I_1,\ldots,I_k\}$ be such that $x_{I_{k+1}}\in E_k$.
The set E being finite there exists $m\in\{2,\ldots,|\mathcal{A}|\}$ satisfying $E_m=\emptyset$: let

$$J_{\mathcal{B}}=I_1\cup\cdots\cup I_m,\quad x_{J_{\mathcal{B}}}=\bar{x}_m,\quad \rho_{J_{\mathcal{B}}}=r_m.$$

It follows directly from the definitions that $4R_{\mathcal{A}}\leq\rho_{J_{\mathcal{B}}}=4^{m-1}R_{\mathcal{A}}\leq 4^nR_{\mathcal{A}}$. Let $I\in\mathcal{A}$; then either $I\cap J_{\mathcal{B}}=\emptyset$ or $I=I_k\subsetneqq J_{\mathcal{B}}$ for some $k\in\{1,\ldots,m\}$: in the first case the above construction shows that $x_I\notin B(x_{J_{\mathcal{B}}},3\rho_{J_{\mathcal{B}}})$; otherwise by Lemma 2.3 we obtain

$$B(x_{I_1},r_1)\subset B(\bar{x}_2,r_2)\subset\cdots\subset B(\bar{x}_m,r_m)$$

and, for $k\in\{1,\ldots,m-1\}$,

$$B(x_{I_{k+1}},r_1)\subset B(x_{I_{k+1}},r_k)\subset B(\bar{x}_{k+1},r_{k+1})\subset\cdots\subset B(\bar{x}_m,r_m);$$

the conclusion follows. □

Corollary 2.3. *Let* $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(\rho_I^{\mathcal{A}})_{I\in\mathcal{A}})\in\Gamma$ *and* $|\mathcal{A}|\geq 2$. *There exists a triple* $(\mathcal{B},(x_I)_{I\in\mathcal{B}},(\rho_I^{\mathcal{B}})_{I\in\mathcal{B}})\in\Gamma$ *such that*

(2.1) $\mathcal{A}\succ\mathcal{B},\quad \delta_{\mathcal{A}}\leq\delta_{\mathcal{B}}$

(2.2) $\forall I\in\mathcal{A}\quad \exists J\in\mathcal{B}:\quad I\subset J,\quad B(x_I,r_I^{\mathcal{A}})\subset B(x_I,\rho_J^{\mathcal{B}})$;

(2.3) $\forall I\in\mathcal{B}:\quad \rho_I^{\mathcal{B}}\leq 4^n R_{\mathcal{A}}$.

Moreover if $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(\rho_I^{\mathcal{A}})_{I\in\mathcal{A}})$ *satisfies* (Δ) *then so does* $(\mathcal{B},(x_I)_{I\in\mathcal{B}},(\rho_I^{\mathcal{B}})_{I\in\mathcal{B}})$.

Proof of Corollary 2.3. Let $J_{\mathcal{B}},x_{J_{\mathcal{B}}},\rho_{J_{\mathcal{B}}}$ be as in Proposition 2.2 and set

$$\mathcal{B}=\{J_{\mathcal{B}}\}\cup\{\{I\}:I\in\mathcal{A},\,I\cap J_{\mathcal{B}}=\emptyset\};\quad \forall I\in\mathcal{B}:\rho_I^{\mathcal{B}}=\begin{cases} r_I^{\mathcal{A}} \text{ if } I\cap J_{\mathcal{B}}=\emptyset;\\ \rho_{J_{\mathcal{B}}} \text{ if } I=J_{\mathcal{B}}.\end{cases}$$

Clearly (2.3) follows directly from (∘).
Proof of (2.1). Let $I\in\mathcal{A}$; then either $I\in\mathcal{B}$ or $I\subsetneq J_{\mathcal{B}}\in\mathcal{B}$, hence $\mathcal{A}\succ\mathcal{B}$. Let $I_1\neq I_2\in\mathcal{B}$: if $I_1=J_{\mathcal{B}}$ and $I_2\in\mathcal{A}$ then $x_{I_2}\notin B(x_{J_{\mathcal{B}}},3\rho_{J_{\mathcal{B}}})$ so that $|x_{I_2}-x_{I_1}|\geq 3\rho_{J_{\mathcal{B}}}$ and thus, by (∘), $|x_{I_2}-x_{I_1}|>3R_{\mathcal{A}}>\delta_{\mathcal{A}}$; otherwise both I_1,I_2 belong to $\mathcal{A}$ and $|x_{I_2}-x_{I_1}|\geq\delta_{\mathcal{A}}$.
Proof of (2.2). Let $I\in\mathcal{A}$: either $I\cap J_{\mathcal{B}}=\emptyset$ so that $I\in\mathcal{B}$ and $\rho_I^{\mathcal{B}}=r_I^{\mathcal{A}}$ or $I\subset J_{\mathcal{B}}\in\mathcal{B}$ and $B(x_I,\rho_I^{\mathcal{A}})\subset B(x_{J_{\mathcal{B}}},\rho_{J_{\mathcal{B}}})$.
Finally, if $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(\rho_I^{\mathcal{A}})_{I\in\mathcal{A}})$ satisfies (Δ) then, by Proposition 2.1, so does the triple $(\mathcal{A},(x_I)_{I\in\mathcal{A}},(r_I)_{I\in\mathcal{A}})$: to prove the claim it is enough to show that

$$\forall I\in\mathcal{A}:\quad I\cap J_{\mathcal{B}}=\emptyset\Rightarrow B(x_I,2r_I^{\mathcal{A}})\cap B(x_{J_{\mathcal{B}}},2\rho_{J_{\mathcal{B}}})=\emptyset.$$

If the above property does not hold then $|x_I-x_{J_{\mathcal{B}}}|<2r_I^{\mathcal{A}}+2\rho_{J_{\mathcal{B}}}\leq 2R_{\mathcal{A}}+2\rho_{J_{\mathcal{B}}}<3\rho_{J_{\mathcal{B}}}$, contradicting Proposition 2.2: the conclusion follows. □

3. A decreasing chain of partitions

Set $\mathcal{A}_0=\{\{1\},\ldots,\{n\}\}$ and, for every i in I_n, let $\rho_{\{i\}}^0=\rho$ and $x_{\{i\}}=x_i$.

Theorem 3.1. *Let* $n\geq 2$; *there exist* m $(<n)$ *and for every* i *in* $\{1,\ldots,m\}$ *a triple* $(\mathcal{A}_i,(x_I)_{I\in\mathcal{A}_i},(\rho_I^i)_{I\in\mathcal{A}_i})\in\Gamma$ *such that if we put* $\delta_{\mathcal{A}_i}=\delta_i$ *and* $r_I^{\mathcal{A}_i}=r_I^i$ *we have*

(3.1) $\mathcal{A}_0\succ\cdots\succ\mathcal{A}_m=\{I_n\},\quad \delta_0\leq\cdots\leq\delta_{m-1}$;

(3.2) $\forall I\in\mathcal{A}_{i-1}\quad \exists J\in\mathcal{A}_i:\quad I\subset J,\quad B(x_I,r_I^{i-1})\subset B(x_J,\rho_J^i)\quad (i=1,\ldots,m)$;

(3.3) $\forall I,J\in\mathcal{A}_i:\quad I\neq J\Rightarrow B(x_I,r_I^i)\cap B(x_J,r_J^i)=\emptyset\quad (i=0,\ldots,m)$;

(3.4) $\forall I\in\mathcal{A}_i:\quad \rho_I^i\leq 4^{ni}\frac{\delta_{i-1}}{2}\quad (i=1,\ldots,m)$.

Remark 3.2. Let $i\in\{0,\ldots,m\}$ and $I\in\mathcal{A}_i$. Then $B(x_k,\rho)\subset B(x_I,\rho_I^i)$ if $k\in I$ and $B(x_k,\rho)\cap B(x_I,\rho_I^i)=\emptyset$ otherwise.

Remark 3.3. It is interesting to note that

$$\forall i\in\{1\ldots,m\}\quad 1\leq\rho_I^i/\delta_{i-1}\leq 4^{n^2},\qquad \forall i\in\{0,\ldots,m-1\}\quad 1/2\leq r_I^i/\delta_i\leq 4^{n^2}.$$

Proof of Theorem 3.1.
The initial assumption (1.2) implies that $(\mathcal{A}_0, (x_i)_{i\in I_n}, (\rho)_{i\in I_n})$ satisfies (Δ). The iterated application of Corollary 2.3 and Proposition 2.1 yields the existence of a chain of triples in Γ satisfying (3.1), (3.2) and (3.3) (since $n = |\mathcal{A}_0| \geq |\mathcal{A}_{i-1}| > |\mathcal{A}_i|$ there exists $m < n$ such that $|\mathcal{A}_m| = 1$). To prove (3.4) from (2.3) it is enough to show that $R_{\mathcal{A}_{i-1}} \leq 4^{n(i-1)}\frac{\delta_{i-1}}{2}$: we use induction on i. For $i = 1$ by (1.2) we have $R_{\mathcal{A}_0} = \frac{\delta_0}{2} \vee \rho = \frac{\delta_0}{2}$; assume that $R_{\mathcal{A}_{i-1}} \leq 4^{n(i-1)}\frac{\delta_{i-1}}{2}$ for some $i \geq 1$: then (2.3) implies that $\rho_I^i \leq 4^n R_{\mathcal{A}_{i-1}} \leq 4^{ni}\frac{\delta_{i-1}}{2}$ for every I in $\mathcal{A}_i$ and therefore $R_{\mathcal{A}_i} \leq \frac{\delta_i}{2} \vee 4^{ni}\frac{\delta_{i-1}}{2}$ so that, by (3.1), $R_{\mathcal{A}_i} \leq 4^{ni}\frac{\delta_i}{2}$. □

4. Proof of Han–Shafrir's Theorem

For x in $\mathbb{R}^2$ and $0 < r < R$ let $A_r^R(x) = B(x,R) \setminus B(x,r)$. The following elementary Lemma was stated in [1]; it will be widely used in the sequel.

Lemma 4.1. *Let $u \in C^1(A_r^R(x), S^1) \cap C(\partial A_r^R(x), S^1)$ and $d = \deg(u, \partial B(x,r))$. Then*

$$\int_{A_r^R(x)} |\nabla u|^2 \geq 2\pi d^2 \log(R/r).$$

Proof of Theorem 1.1. If $\mu \leq 2\rho$ then by (1.1) and (1.2) we have $B(x_i,\rho) \subset G$ and $B(x_i, 2\rho) \cap B(x_j, 2\rho) = \emptyset$ for $i \neq j$ in I_n; Lemma 4.1 then yields the conclusion:

$$\int_\Omega |\nabla u|^2 \geq \sum_{i=1}^n \int_{A_\rho^{2\rho}(x_i)} |\nabla u|^2 \geq 2\pi \sum_{i=1}^n d_i^2 \log 2 \geq 2\pi F(d_1,\ldots,d_n)\log(\mu/\rho).$$

In what follows we assume that $\mu \geq 2\rho$. If $n = 1$ the result follows trivially from Lemma 4.1 with $r = \rho$, $R = \mu$ and $x = x_1$. Assume that $n \geq 2$ and let $(\mathcal{A}_i, (x_I)_{I\in\mathcal{A}_i}, (\rho_I^i)_{I\in\mathcal{A}_i}) \in \Gamma$ be such as in Theorem 3.1. Put $\delta_{-1} = \rho$ and

$$S = \{i \in \{0,\ldots,m\} : 4^{ni}\delta_{i-1} \leq \mu\}.$$

The set S being not empty ($0 \in S$) let $k = \max S$.
For $i \leq k$ and $j \in I \in \mathcal{A}_i$ Theorem 3.1 then gives $B(x_j,\rho) \subset B(x_I, \rho_I^i) \subset B(x_I, \mu/2)$ so that the triangular inequality yields $B(x_I, \mu/2) \subset B(x_j, \mu)$: (1.1) then implies

$$\forall i \in \{0,\ldots,k\} \quad \forall I \in \mathcal{A}_i : \quad B(x_I, \rho_I^i) \subset B(x_I, \mu/2) \tag{4.1}$$

and thus by Theorem 3.1 the annuli $(A_{\rho_I}^{r_I}(x_I))_{I\in\mathcal{A}_i}$ are disjoint subsets of Ω: set

$$\forall i \in \{0,\ldots,k-1\} \qquad \Omega_i = \cup_{I\in A_i} A_{\rho_I}^{r_I}(x_I).$$

By (3.2) we have $\Omega_i \subset \cup_{I\in\mathcal{A}_{i+1}} B(x_I,\rho_I^{i+1}) \setminus \cup_{I\in\mathcal{A}_i} B(x_I,\rho_I^i)$ and thus $\Omega_j \cap \Omega_i = \emptyset$ if $j \neq i$. For I in $\mathcal{A}_k$ we put

$$s_I = \begin{cases} \mu/2 \text{ if } k = m; \\ \mu/2 \wedge r_I^k \text{ if } k < m. \end{cases}$$

By Theorem 3.1 and (4.1) the annuli $(A_{\rho_I}^{s_I}(x_I))_{I\in\mathcal{A}_k}$ are disjoint subsets of Ω, set

$$\Omega_k = \cup_{I\in\mathcal{A}_k} A_{\rho_I}^{s_I}(x_I).$$

The sets $(\Omega_i)_{0\leq i\leq k}$ being disjoint we have

$$\int_\Omega |\nabla u|^2 \geq \sum_{i=0}^{k} \int_{\Omega_i} |\nabla u|^2.$$

Let $i \in \{0,\dots,k\}$ and $I \in \mathcal{A}_i$; since $B(x_I,\rho_I^i) \setminus \cup_{j\in I} B(x_j,\rho) \subset \Omega$ the excision property of the degree implies that $\deg(u, \partial B(x_I,\rho_I^i)) = d_I$: Lemma 4.1 then yields

$$\frac{1}{2\pi}\int_\Omega |\nabla u|^2 \geq \sum_{i=0}^{k-1} \sum_{I\in\mathcal{A}_i} d_I^2 \log(r_I/\rho_I) + \sum_{I\in\mathcal{A}_k} d_I^2 \log(s_I/\rho_I). \tag{4.2}$$

To estimate the right-hand side of (4.2) remark that if $i \in \{0,\dots,k-1\}$ and $I \in \mathcal{A}_i$ the inequalities (3.4) and the definition of r_I imply

$$\forall I \in \mathcal{A}_i : \qquad r_I/\rho_I \geq 4^{-n^2}\delta_i/\delta_{i-1}.$$

Moreover for $I \in \mathcal{A}_k$ either $s_I = \mu/2$ or $k < m$ and $s_I = r_I \geq \delta_k$; however if $k = \max S < m$ we have $4^{n(k+1)}\delta_k \geq \mu$ and thus in both cases $s_I \geq 4^{-n^2}\mu/2$ so that by (3.4) we obtain

$$\forall I \in \mathcal{A}_k : \qquad s_I/\rho_I \geq 4^{-n^2}\mu/\delta_{k-1}.$$

We point out that $\sum_{I\in\mathcal{A}} d_I^2 \leq \|d\|^2$ for every $\mathcal{A}$ in $\mathcal{P}_n$; going back to (4.2) we deduce that

$$\frac{1}{2\pi}\int_\Omega |\nabla u|^2 \geq \sum_{i=0}^{k-1} \sigma(\mathcal{A}_i)\log(\delta_i/\delta_{i-1}) + \sigma(\mathcal{A}_k)\log(\mu/\delta_{k-1}) - C$$

where $C = n^3\|d\|^2\log 4$. Now $\rho \leq \delta_0 \leq \dots \leq \delta_{k-1} \leq \mu$ and $\sigma(\mathcal{A}_i) \geq F(d_1,\dots,d_n)$ for $i = 0,\dots,k$. The above inequality finally yields

$$\begin{aligned}\frac{1}{2\pi}\int_\Omega |\nabla u|^2 &\geq F(d_1,\dots,d_n)\,[\log(\delta_0/\rho) + \dots + \log(\mu/\delta_{k-1})] - C = \\ &= F(d_1,\dots,d_n)\log(\mu/\rho) - C\end{aligned}$$

which is the desired conclusion. □

References

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