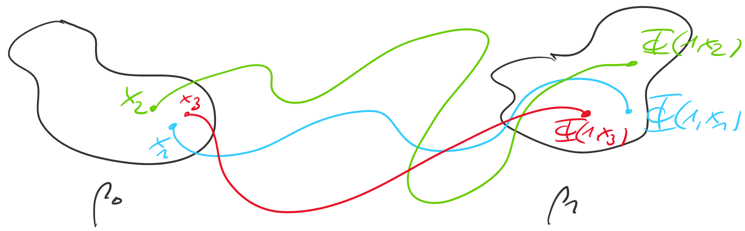
Wasserstein distance: $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$

$$W(\rho_0,\rho_1)^2 = \inf \Big\{ \int_{\mathbb{R}^d} \underbrace{\int_0^1 |\dot{\Phi}(t,x)|^2 dt}_{\text{energy of curves}} \, \rho_0(dx) : \Phi(1,\cdot)\#\rho_0 = \rho_1 \Big\}$$

transport ρ_0 into ρ_1



pick up mass at x_i, transport along the trajectory

$t \longmapsto \Phi(t,x_i)$

$$\text{Cost} = \int_0^1 |\dot{\Phi}(t,x_i)|^2 dt$$

minimize!

The minimizing curve connecting x and y is a straight line

and

$$\inf\left\{ \int_0^1 |\dot{\gamma}(t)|^2 \, dt : t \mapsto \gamma(t),\ t\in[0,1],\ \gamma(0)=x,\ \gamma(1)=y \right\}$$

$$= \underbrace{|y-x|^2}_{\text{transport cost for Wasserstein distance}}$$

Need a generalization for Euler eqns.

State variables
- density ρ (as before)
- velocity v
- specific entropy S

formally transported along the flow

Consider measures

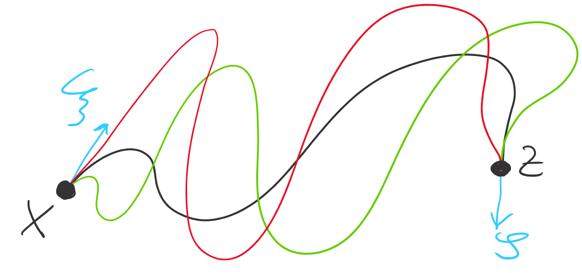
$$\mu \in \mathcal{P}_\rho(\mathbb{R}^{2d}) := \left\{ \mu \in \mathcal{P}_2(\mathbb{R}^{2d}) : p^1 \# \mu = \rho \right\},\ \rho \in \mathcal{P}_2(\mathbb{R}^d)$$

where $p^k(x_1, \dots, x_N) = x^k$, $x_i \in \mathbb{R}^d$, $i,k = 1 \dots N$ (projection)

$$\underbrace{\int_{\mathbb{R}^d} |x|^2 \mu(dx, d\xi)}_{= \int_{\mathbb{R}^d} |x|^2 \rho(dx)} + \underbrace{\int_{\mathbb{R}^{2d}} |\xi|^2 \mu(dx, d\xi)}_{\text{kinetic energy finite}} < \infty$$

Spatial moment

Goal: find suitable cost function(al) for measures in $\mathcal{P}_\rho(\mathbb{R}^{2d})$



locations x, z

velocities ξ, ς

$\hookrightarrow$ minimize <u>acceleration</u> $\quad \tau > 0$ timestep

second deri

$$\inf \left\{ \int_0^\tau |\ddot{\gamma}(t)|^2 dt : \ t \mapsto \gamma(t),\ t \in [0,\tau], \ \gamma(0) = x,\ \gamma(\tau) = z, \ \boxed{\dot{\gamma}(0) = \xi,\ \dot{\gamma}(\tau) = \zeta} \right\}$$

Quadratic functional (check convex), consider <u>first variation</u>

$$\int_0^\tau \ddot{\gamma}(t) \cdot \ddot{a}(t) dt = -\int_0^\tau \gamma^{(4)}(t) \cdot a(t) dt \overset{!}{=} 0$$

for all $a : [0,\tau] \longrightarrow \mathbb{R}^d$ s.t.

$$\begin{cases} a(0) = a(\tau) = 0 \\ \dot{a}(0) = \dot{a}(\tau) = 0 \end{cases}$$

$\Rightarrow$ minimizing curve satisfies

$$\gamma^{(4)}(t) = 0, \quad t \in (0,\tau)$$

Hence γ is a cubic polynomial

$$\gamma(t) = x + t\xi + \left[3(z - x) + \tau(\zeta + 2\xi) \right] \frac{t^2}{\tau^2}$$
$$+ \left[-2(z - x) + \tau(\zeta + \xi) \right] \frac{t^3}{\tau^3}$$

The minimal acceleration can be computed explicitly as

different sign!

$$\inf\left\{ \int_0^\tau |\ddot{y}(t)|^2 \, dt : \dots \right\} = 3\left| \frac{z-x}{\tau} - \frac{\zeta+\xi}{2} \right|^2 + \frac{1}{4}\left|\zeta-\xi\right|^2$$

$$= \frac{3}{4\tau^2}\left| z-(x+\tau\xi)\right|^2 + \left|\zeta - \left(\xi + \frac{3}{2\tau}\left(z-(x+\tau\xi)\right)\right)\right|^2$$

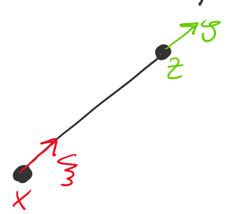
$$=: C\big((x,\xi),(z,\zeta)\big)$$

Cost functional

optimal new velocity $V_\tau\big((x,\xi),z\big)$

Remarks

(i) $C\big((x,\xi),(z,\zeta)\big) \geq 0$, $\quad = 0$ if and only if



$\zeta = \xi$, $\quad z = x + \tau\xi$

no change in velocity

free transport

(ii) $C\big((x,\xi),(z,\zeta)\big) \neq C\big((z,\zeta),(x,\xi)\big)$ Keine Symmetrie!

Def. (Minimal Work) $\quad \mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^{2d})$

dimension of energy

$$W(\mu_0,\mu_1)^2 := \inf\left\{ \iint_{\mathbb{R}^{2d}\,\mathbb{R}^{2d}} C\big((x,\xi),(z,\zeta)\big)\, \alpha\big(d(x,\xi),d(z,\zeta)\big) : \; P^{1,2}\#\alpha = \mu_0, \; P^{3,4}\#\alpha = \mu_1 \right\}$$

Similar properties like Wasserstein distance, but <u>not symmetric</u> and $W(\mu_0,\mu_1) = 0$ iff

$$\mu_1 = F_\tau \# \mu_0$$

with free transport map $F_\tau(x,\xi) = (x+\tau\xi, \xi)$
for $(x,\xi) \in \mathbb{R}^{2d}$.

$\hookrightarrow$ replace Wasserstein distance in JKO scheme / minimizing movement schemes by minimal work.

<u>Remark</u> Minimization of acceleration for curves on Riemannian manifolds leads to <u>Riemannian cubic polynomials</u>

metric

covariant derivative

$$\inf_u \frac{1}{2}\int_0^T \langle u,u\rangle_c \, dt, \qquad \frac{dc}{dt} = V, \quad \frac{DV}{dt} = u$$

curve $c : [0,T] \to \mathcal{M}$

$\Longrightarrow$ Minimizers satisfy

curvature tensor

$$\frac{D^3 V}{dt} + R\left(\frac{DV}{dt}, V\right)V = 0, \quad t \in [0,T];$$

see [Camarinha Silva Leite].