2. Energy / entropy functional

Lebesgue measure

$$E : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow [0,\infty]$$

$$E(\rho) = \begin{cases} \int_{\mathbb{R}^d} U(r)\,dx & \text{if } \rho \ll \mathcal{L}^d \\ +\infty & \text{otherwise} \end{cases}$$

Radon Nikodym: If ρ absolutely continuous (a.c.) w.r.t. $\mathcal{L}^d$, then there exists

$$r \in \mathcal{L}^1(\mathbb{R}^d), \quad r \geq 0$$

s.t. $\boxed{\rho = r\mathcal{L}^d}$

Relation to P in porous medium equation $\partial_t \rho - \Delta P(\rho) = 0$:

$$P(r) = U'(r)r - U(r), \quad r \geq 0$$

Compute <u>gradient</u> $\nabla E(\rho)$:

Consider a curve $t \mapsto \rho_t \in \mathcal{P}_2(\mathbb{R}^d)$ s.t.

$$\partial_t \rho_t \big|_{t=0} = -\nabla\cdot(\rho v), \quad \rho_{t=0} = \rho$$

$v \in \mathrm{Tan}_\rho \mathcal{P}_2(\mathbb{R}^d)$

Then:

$$\frac{d}{dt}\Big|_{t=0} E(\rho_t) = dE_\rho(v) = \langle \nabla E(\rho), v\rangle_\rho$$

arguing geometrically

can be computed (formally)

we identify ρ_t and ρ_t

$$\frac{d}{dt}\Big|_{t=0} E(\rho_t) = \int_{\mathbb{R}^d} \partial_t\, U(\rho_t) \Big|_{t=0}$$

$$= \int_{\mathbb{R}^d} U'(\rho_t)\, \partial_t \rho_t \Big|_{t=0}$$

$\partial_t \rho_t = -\nabla\cdot(\rho v)$

integration by parts

$$= -\int_{\mathbb{R}^d} \rho \nabla U'(\rho)\cdot v$$

Now note that

$$\rho \nabla U'(\rho) = \rho U''(\rho)\nabla\rho = \underbrace{\nabla\big(U'(\rho)\rho - U(\rho)\big)}_{=\left(U''(\rho)\rho\nabla\rho + U'(\rho)\nabla\rho - U'(\rho)\nabla\rho\right)} = \nabla P(\rho)$$

cancel

$$\Longrightarrow \quad dE_\rho(v) = \langle \nabla E(\rho), v\rangle_\rho = -\int_{\mathbb{R}^d} \nabla P(\rho)\cdot v, \quad v \in \mathrm{Tan}_\rho \mathcal{P}_2(\mathbb{R}^d)$$

in particular, for all $\nabla\varphi$ with $\varphi \in \mathcal{D}(\mathbb{R}^d)$

- Gradient flows:

Assume that $t \mapsto \rho_t \in \mathcal{P}_2(\mathbb{R}^d)$ s.t.

(i) $\partial_t \rho_t = -\nabla\cdot(\rho_t u_t)$ and

(iii) $\quad u_t = -\nabla U'(\rho_t) \quad$ for all t

gradient! hence $u_t \in \mathrm{Tan}_{\rho_t} \mathcal{P}_2(\mathbb{R}^d)$

Then

$$\partial_t \rho_t = \nabla \cdot \left(\rho_t U'(\rho_t) \right) = \nabla \cdot \left(\nabla P(\rho_t) \right) = \Delta P(\rho_t)$$

see above

This is the porous medium equation ✓

Question: What about convexity of the functional

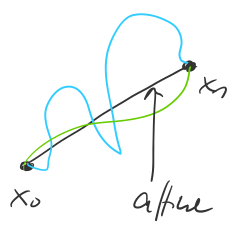
$$\rho \longmapsto E(\rho) = \begin{cases} \int_{\mathbb{R}^d} U(\rho) & \text{if } \rho = \rho \mathcal{L}^d \\ +\infty & \text{otherwise} \end{cases}$$

Related: what are the geodesics wrt. Wasserstein distance W?

(1.) Lagrangian point of view:

Heuristics: minimizers of $\int_0^1 |\dot\gamma(t)|^2 dt$,

$t \mapsto \gamma(t), \quad \gamma(0) = x_0, \ \gamma(1) = x_1$



are affine functions:

$$\gamma(t) = (1-t) x_0 + t x_1$$

$$\Rightarrow \min \left\{ \int_0^1 |\dot\gamma(t)|^2 dt : \gamma \text{ as above} \right\}$$
$$= |x_1 - x_0|^2$$

Therefore: Benamou–Brenier is equivalent to an optimal transport problem:

$$(MK) \qquad W(\rho_0,\rho_1)^2 = \inf \Big\{ \iint_{\mathbb{R}^d\times\mathbb{R}^d} |x_1-x_0|^2 \gamma(dx_0,dx_1) : \gamma \in \mathcal{P}_2(\mathbb{R}^d\times\mathbb{R}^d), \ \pi^1\#\gamma = \rho_0, \ \pi^2\#\gamma=\rho_1 \Big\}$$

Remarks

- γ is a transport plan:

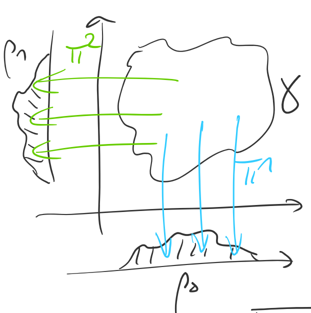
  $\gamma(dx_0,dx_1)$ = how much mass is moved from $[x_0, x_0+dx_0]$ to $[x_1,x_1+dx_1]$

- $|x_1-x_0|^2$ is the cost:

  how much does it cost to move mass from x_0 to x_1?

  $\Longrightarrow$ try to minimize the total cost!

- constraints: transport ρ_0 to ρ_1



$\pi^1: \mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ projection

$\pi^1(x_0,x_1) = x_1$

(π^2 analogously).

push forward

$\Longrightarrow \quad \pi^1\#\gamma = \rho_0$ first marginal

$\boxed{\text{Special case}}$ $\quad \gamma = (id, T)\#\rho_0$ for some transport map $T:\mathbb{R}^d \longrightarrow \mathbb{R}^d$

$\rightsquigarrow$ <u>Monge problem</u>:

$$\inf \left\{ \int_{\mathbb{R}^d} |T(x)-x|^2 \rho_0(dx) \; : \; T: \mathbb{R}^d \to \mathbb{R}^d \text{ s.t. } T\#\rho_0 = \rho_1 \right\}$$

$(T\#\rho_0)(A) := \rho_0(T^{-1}(A))$
for $A \subset \mathbb{R}^d$ Borel

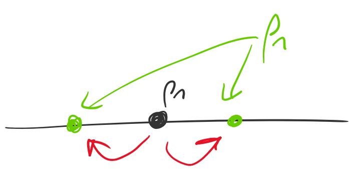
<u>Remarks</u>

- The constraint $T\#\rho_0 = \rho_1$ leads to a highly nonlinear pde for T:

$$\boxed{\det(\nabla T)\, \rho_0 = \rho_1 \circ T}$$

Monge-Ampère equation

- The Monge problem does not always have a solution! Consider $d=1$ and



$$\rho_0 = \delta_0 \quad , \quad \rho_1 = \frac{1}{2}\left(\delta_{-1} + \delta_1\right)$$

Dirac measure at $x=1$

need to split the mass:
- send half from $0 \to +1$
- send half from $0 \to -1$

Cannot be done with a <u>map</u>!

In contrast, the Monge-Kantorovich problem (MK) always has a solution

(transport plans allow <u>splitting of mass</u>)

Theorem The support of an optimal (= minimizing) transport plan is contained in a cyclically monotone subset of $\mathbb{R}^d \times \mathbb{R}^d$. Such sets are subdifferentials of proper, convex, and lower semicontinuous (l.s.c.) functions

(details later)

$\boxed{\text{Special case}}$ If T is a solution of Monge problem, then there exists $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{+\infty\}$ proper, convex, l.s.c. such that

$$T = \nabla \varphi$$

Moreover, the geodesic connecting μ_0, μ_1 are then given by

$$s \longmapsto T_s := (1-s)\underbrace{id}_{\text{identity map}} + s\underbrace{T}_{\text{optimal transport map}}$$

Γ check convexity of $\quad \rho \mapsto \int_{\mathbb{R}^d} u(\rho)$ for $\rho \ll \mathcal{L}^d$, suitable u

$$E(T_s \# \rho_0) = \int_{\mathbb{R}^d} u\big((T_s \# \rho_0)(y)\big)\, dy$$

push forward of ρ_0 under T_s

$$= \int_{\mathbb{R}^d} u\Big(\Big(\frac{\rho_0}{\det \nabla T_s}\Big)(\underbrace{T_s^{-1}(y)}_{=:x})\Big)\, dy$$

$dy = (\det \nabla T_s(x))\, dx$

If $\rho_0 \ll \mathcal{L}^d$ and T_s invertible (up to ρ_0-null set) and smooth,
then $T_s \# \rho_0 \ll \mathcal{L}^d$
with
$$T_s \# \rho_0 = \Big(\frac{\rho_0}{\det \nabla T_s}\Big) \circ T_s^{-1} \mathcal{L}^d$$

$$= \int_{\mathbb{R}^d} u\Big(\frac{\rho_0(x)}{\det \nabla T_s(x)}\Big) \det \nabla T_s(x)\, dx$$

$T_s = (1-s)\, id + s\, T, \quad T = \nabla \phi$

$$\Rightarrow \nabla T_s = (1-s)\, \mathbb{1} + s\, D^2 \phi$$

unit matrix

symmetric pos. semidefinite

$(\mathrm{Sym}_d(\mathbb{R}, \geq))$

Therefore
$$\nabla T_s \in \mathrm{Sym}_d(\mathbb{R}, >)$$
$$\hookrightarrow \quad \det \nabla T_s = \prod_{i=1}^d \lambda_{i,s}(x) > 0$$

eigenvalues positive

Hence

$$\frac{d}{ds} E(T_s \# \rho_0) = \frac{d}{ds} \int_{\mathbb{R}^d} U\left(\frac{\rho_0(x)}{\det \nabla T_s(x)}\right) \det \nabla T_s(x)\, dx$$

$$= \int_{\mathbb{R}^d} \left[U'\left(\frac{\rho_0}{\det \nabla T_s}\right) \frac{-\rho_0}{\det \nabla T_s} + U\left(\frac{\rho_0}{\det \nabla T_s}\right)\right] \partial_s \det \nabla T_s$$

$$= - \int_{\mathbb{R}^d} P\left(\frac{\rho_0}{\det \nabla T_s}\right) \underbrace{\partial_s \det \nabla T_s}$$

trace (= sum of eigenvalues)

$$= \det \nabla T_s \cdot \mathrm{tr}\left((\nabla T_s)^{-1} \partial_s \nabla T_s \right)$$

generic formula:

$\delta \det(X)$

$= \det(X)\, \mathrm{tr}(X^{-1} \delta X)$

derivative of det

$$= - \int_{\mathbb{R}^d} \underbrace{P\left(\frac{\rho_0}{\det \nabla T_s}\right) \det \nabla T_s} \cdot \mathrm{tr}\left((\nabla T_s)^{-1} \underbrace{(\nabla T - \mathbb{1})}_{= D^2\phi - \mathbb{1}} \right)$$

$p(r) = P'(r) r - P(r)$

Then

$$\frac{d^2}{ds^2} E(T_s \# \rho_0) = + \int_{\mathbb{R}^d} p\left(\frac{\rho_0}{\det \nabla T_s}\right) \det \nabla T_s \left(\mathrm{tr}\left((\nabla T_s)^{-1} (\nabla T - \mathbb{1}) \right) \right)^2$$

$$- \int_{\mathbb{R}^d} P\left(\frac{\rho_0}{\det \nabla T_s}\right) \det \nabla T_s \cdot \left(- \mathrm{tr}\left(\left((\nabla T_s)^{-1} (\nabla T - \mathbb{1}) \right)^2 \right) \right)$$

generic formula:

$\delta(X^{-1})$

$= -X^{-1} (\delta X) X^{-1}$

derivative of inverse

Question: Which condition on U (hence P, p) imply that $r.h.s \geq 0$ (and then convexity of $E(\rho)$)?

Since $T = \nabla\phi$ with ϕ convex we have

$$\nabla T = D^2\phi = \underbrace{Q}_{\text{orthogonal}}{}^T \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} Q$$

orthogonal

eigenvalues ≥ 0

Then also

$$(\nabla T_s)^{-1}(\nabla T - \mathbb{1}) = Q^T \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_d \end{pmatrix} Q$$

with $\mu_i = \dfrac{\lambda_i - 1}{(1-s) + s\lambda_i} \in \mathbb{R}$

By the geometric-arithmetic inequality

$$\left(\operatorname{tr}\left(\underbrace{(\nabla T_s)^{-1}(\nabla T - \mathbb{1})}_{\text{eigenvalues real}}\right)\right)^2 = \left(\sum_{i=1}^{d} \mu_i\right)^2$$

$$\leq d \sum_{i=1}^{d} \mu_i^2 = d \operatorname{tr}\left(\underbrace{\left((\nabla T_s)^{-1}(\nabla T - \mathbb{1})\right)^2}_{\text{eigenvalues} \geq 0 \text{ (squares)}}\right)$$

$\Rightarrow$ Sufficient condition: $P(r) \geq 0$ and

$$p(r) \geq -\frac{1}{d} P(r), \quad r \in [0, \infty)$$

$$\Gamma \underbrace{\det(\nabla T_s)}_{\geq 0} \left[\underbrace{p\left(\frac{\rho_0}{\det \nabla T_s}\right)}_{\geq -\frac{1}{d} P(\dots)} \underbrace{\left(\operatorname{tr}(\dots)\right)^2}_{\geq 0} + P\left(\frac{\rho_0}{\det(\nabla T_s)}\right) \operatorname{tr}\left((\dots)^2\right) \right]$$

$$\geq \det(\nabla T_s)\, P\left(\frac{\rho_0}{\det \nabla T_s}\right) \left[-\frac{1}{d} \left(\operatorname{tr}(\dots)\right)^2 + \operatorname{tr}\left((\dots)^2\right) \right] \geq 0 \quad \checkmark$$

Examples

- $U(r) = r \log(r) \rightsquigarrow P(r) = r$ ← heat equation
- $U(r) = \frac{r^m}{m-1}$, $m \geq 1 - \frac{1}{d}$ $\rightsquigarrow P(r) = r^m$ ← porous medium equation

$$\left[p(r) = (m-1) r^m \geq -\frac{1}{d} r^m = -\frac{1}{d} P(r) \right]$$

Eulerian way to check convexity

Recall the Riemannian metric on $\mathcal{P}_2(\mathbb{R}^d)$

$$\langle \delta\rho_0, \delta\rho_1 \rangle_\rho \mathrel{\hat{=}} \int_{\mathbb{R}^d} \rho \nabla\phi_0 \cdot \nabla\phi_1$$

perturbations $\int_{\mathbb{R}^d} \delta\rho_i = 0$

with $\delta\rho_i - \nabla \cdot (\rho \nabla\phi_i) = 0$ — tangent vector: gradient

$$= - \int_{\mathbb{R}^d} \delta\rho_0 \, \phi_1$$

$\delta\rho_0 = \nabla \cdot (\rho \nabla \phi_0)$ then integrate by parts

Variational characterization

$$\langle \delta\rho, \delta\rho \rangle_\rho \hat{=} \int_{\mathbb{R}^d} \rho \, |\delta\rho|^2$$

$$= \sup_\phi \left\{ -\int_{\mathbb{R}^d} \tfrac{1}{2} |\nabla\phi|^2_\rho - \int_{\mathbb{R}^d} \delta\rho \, \phi \right\}$$

sup is (formally) obtained for ϕ s.t.

$$\delta\rho - \nabla\cdot(\rho \nabla\phi) = 0$$

Consider $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$.

Induced distance

tangent vector

$$W(\rho_0,\rho_1)^2 = \inf \left\{ \int_0^1 \rho_s \, |\partial\rho_s|^2 ds \, ; \; s \mapsto \rho_s \in \mathcal{P}_2(\mathbb{R}^d), \; \rho_{s=0} = \rho_0, \rho_{s=1} = \rho_1 \right\}$$

(I) $$= \inf_{\rho_s} \sup_{\phi_s} \left\{ -\tfrac{1}{2} \int_0^1 \int_{\mathbb{R}^d} \rho_s |\nabla\phi_s|^2 ds - \int_0^1 \int_{\mathbb{R}^d} \phi_s \, \partial_s\rho_s \right\}$$

integrate by parts

(II) $$= \inf_{\rho_s} \sup_{\phi_s} \left\{ \int_0^1 \int_{\mathbb{R}^d} \rho_s \left[\partial_s\phi_s - \tfrac{1}{2} |\nabla\phi_s|^2 \right] + \int_{\mathbb{R}^d} \phi_0 \rho_0 - \int_{\mathbb{R}^d} \phi_1 \rho_1 \right\}$$

Saddle point problem

geodesics equations

Variation in ϕ_s (use (I))

Variation in ρ_s (use (II))

$$\boxed{\begin{aligned} &\partial_s\rho_s - \nabla\cdot(\rho_s \nabla\phi_s) = 0 \\ &\partial_s\phi_s - \tfrac{1}{2} |\nabla\phi|^2 = 0 \end{aligned}} \quad , \; \rho_{s=0} = \rho_0 \; \ldots$$

Hamilton Jacobi equation

Consider then the functional E along a geodesic:

$$\frac{d}{ds}E(\rho_s) = \int_{\mathbb{R}^d} U'(\rho)\, \partial_s\rho \qquad = \nabla\cdot(\rho\nabla\phi)$$

$$= -\int_{\mathbb{R}^d} \rho\nabla U'(\rho)\cdot\nabla\phi \qquad \nabla P(\rho)$$

$$= \int_{\mathbb{R}^d} \Delta P(\rho)\,\phi = \int_{\mathbb{R}^d} P(\rho)\,\Delta\phi\,,$$

$$\frac{d^2}{ds^2}E(\rho_s) = \int_{\mathbb{R}^d}\Big(P'(\rho)\,\partial_s\rho\,\Delta\phi + P(\rho)\,\Delta\,\partial_s\phi\Big)$$

$$= \int_{\mathbb{R}^d}\Big(P'(\rho)\,\nabla\cdot(\rho\nabla\phi)\,\Delta\phi + P(\rho)\,\Delta\big(\tfrac{1}{2}|\nabla\phi|^2\big)\Big)$$

integration by parts

$$= \int_{\mathbb{R}^d}\Big(P'(\rho)\rho\,(\Delta\phi)^2 + \nabla P(\rho)\cdot\nabla\phi\,\Delta\phi\Big) + P(\rho)\,\Delta\big(\tfrac{1}{2}|\nabla\phi|^2\big)$$

$$= \int_{\mathbb{R}^d}\rho P'(\rho)(\Delta\phi)^2 + P(\rho)\Big(\underbrace{-\nabla\cdot(\nabla\phi\,\Delta\phi)}_{=(\Delta\phi)^2+\nabla\phi\cdot\nabla\Delta\phi} + \Delta\big(\tfrac{1}{2}|\nabla\phi|^2\big)\Big)$$

$$= \int_{\mathbb{R}^d}\underbrace{\big(\rho P'(\rho) - P(\rho)\big)}_{=p(\rho)\text{ see above}}(\Delta\phi)^2 + P(\rho)\Big(-\nabla\phi\cdot\nabla\Delta\phi + \Delta\big(\tfrac{1}{2}|\nabla\phi|^2\big)\Big)$$

Bochner's formula:

$$-\nabla\phi\cdot\nabla\Delta\phi + \Delta\big(\tfrac{1}{2}|\nabla\phi|^2\big) = |D^2\phi|^2 \qquad \big(+\nabla\phi\cdot \mathrm{Ric}\,\nabla\phi\big)$$

Ricci curvature on manifolds

Therefore

$$\frac{d^2}{ds^2}E(\rho_s) = \int_{\mathbb{R}^d} p(\rho)\,(\Delta\phi)^2 + P(\rho)\,|D^2\phi|^2$$

If $\quad p(r) \geq -\frac{1}{d} P(r), \; r \in [0,\infty)$, then the rhs ≥ 0 because

$$(\Delta\phi)^2 \leq d \, |D^2\phi|^2$$

$\hookrightarrow$ The functional $\rho \longmapsto E(\rho)$ is convex
(which implies a contraction estimates for solutions of $\partial_t \rho - \Delta P(\rho) = 0$ wrt the Wasserstein distance)

hence uniqueness, global minimum, convergence to equilibrium ...

Remark

Other functionals can be considered

- potential energy $\quad \rho \longmapsto \int_{\mathbb{R}^d} V(x)\rho(dx)$, V convex
- interaction energy $\quad \rho \longmapsto \iint_{\mathbb{R}^d \mathbb{R}^d} W(x-y)\rho(dx)\rho(dy)$, W Lipschitz

...

Gradient flows $\iff$ steepest descent

along the solution curves energy / entropy decreases at maximal rate

- Variational characterization: consider curve $t \mapsto x_t \in \mathcal{M}$

$$\frac{d}{dt} E(x_t) = dE_{x_t}(\dot{x}_t)$$
$$= \langle \nabla E(x_t), \dot{x}_t \rangle_{x_t}$$
$$= -\frac{1}{2}|\nabla E(x_t)|^2 - \frac{1}{2}|\dot{x}_t|^2$$
$$+ \frac{1}{2}\underbrace{|\nabla E(x_t) + \dot{x}_t|^2}_{\geq 0}$$

vanishes if and only if $\dot{x} = -\nabla E(x_t)$

i.e., if $t \mapsto x_t$ is <u>gradient flow</u>.

$\hookrightarrow$ Since

$$\frac{d}{dt} E(x_t) \geq -\frac{1}{2}|\nabla E(x_t)|^2 + \frac{1}{2}|\dot{x}|^2$$

holds for any curve, gradient flows are characterized by the variational inequality

$$\frac{d}{dt} E(x_t) \leq -\frac{1}{2}|\nabla E(x_t)|^2 + \frac{1}{2}|\dot{x}|^2$$

<u>Integralform</u> $\quad t \longmapsto x_t \in \mathcal{M}$ s.t.

$$E(x_t) + \frac{1}{2}\int_0^t |\nabla E(x_s)|^2 ds + \frac{1}{2}\int_0^t |\dot{x}_s|^2 ds$$
$$\leq E(x_0) \quad , t \geq 0$$

good direction:

if there exists a sequence of approximate solutions x_t^n, one only need lower semicontinuity:

$$E(x_t) \leq \liminf_{n\to\infty} E(x_t^n)$$

$$\int_0^t |\nabla E(x_s)|^2 ds \leq \liminf \int_0^t |\nabla E(x_s^n)|^2 ds$$

$$\int_0^t |\dot{x}_s|^2 ds \leq \liminf \int_0^t |\dot{x}_s^n|^2 ds$$

<u>Note</u> No need for <u>linear space</u> !

- <u>Metric derivative</u>

$$|\dot{x}_t| := \limsup_{h\to 0} \frac{d(x_{t+h}, x_t)}{|h|}$$

speed, not velocity (no direction)

- <u>Slope</u>

$$|\partial E|(x) := \limsup_{y\to x} \frac{(E(y)-E(x))_+}{d(y,x)}$$

generalizes the norm of the gradient

Yet another alternative characterization for λ-convex functionals ($M = \mathbb{R}^d$)

$$\frac{1}{2}\frac{d}{dt}|x_t - y|^2 + \frac{1}{2}\lambda|x_t - y|^2 + E(x_t) \leq E(y) \quad \text{for all } y \in \mathbb{R}^d$$

Consider only $\lambda = 0$:

$$\frac{1}{2}\frac{d}{dt}|x_t - y|^2 = -\langle y - x_t, \dot{x}_t\rangle \leq E(y) - E(x_t) \quad \text{for all } y \in \mathbb{R}^d$$

$$= \langle \nabla E(x_t), y - x_t\rangle + o(1)$$

Consider $\pm(y - x_t) \to 0$

for $|x_t - y| \to 0$

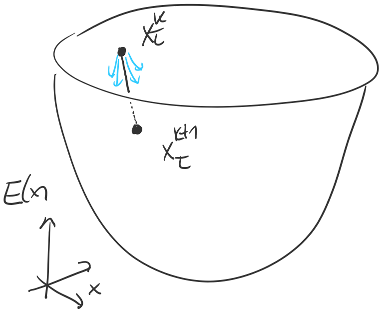
Can again be generalized to metric spaces.

replace $|x_t - y|^2$ by $d(x_t, y)^2$ ...

Natural time discretization

$[0,T]$ for some $\tau > 0$ timestep

initial data x_0



discrete times $t_\tau^k := k\tau$, $k = 0, 1, 2, \dots$

Define recursively: x_τ^n approximation of solution at time $t = t_\tau^k$

$\hookrightarrow$ choose

$$x_\tau^{k+1} \in \underset{x \in M}{\operatorname{argmin}} \left\{ \frac{d(x_\tau^k, x)^2}{2\tau} + E(x) \right\}$$

set of minimizers

JKO scheme, minimizing movements

Hence

- decrease energy/entropy $\Leftrightarrow$ steepest descent
- limiting constraint : step size

find the right balance ....

<u>Note</u> if $\tau \to 0$ then step size is strongly penalized s.t. minimizers will <u>converge to</u> x_τ^k

Given approximate solutions x_τ^k at discrete times, <u>interpolate in time</u>

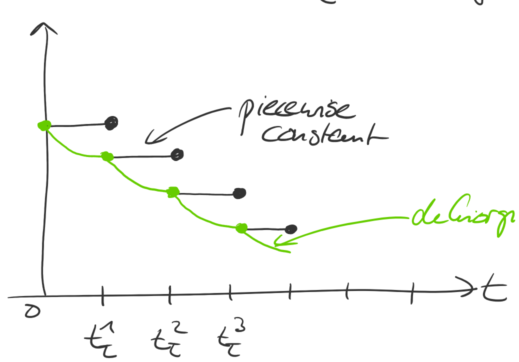
- piecewise constant

$$x_\tau(t) := x_\tau^{k+1} \quad \text{for } t \in (t_\tau^k, t_\tau^{k+1}], \quad k = 0, 1, \dots$$

- de Giorgi interpolation:

$$x_\tau(t) = \operatorname{argmin} \left\{ \frac{d(x_\tau^k, x)^2}{2(t - t_\tau^k)} + E(x) \right\}$$

for $t \in (t_\tau^k, t_\tau^{k+1}]$, $k = 0, 1, 2 \dots$



If M has a linear structure (or geodesics, ...) then one can also interpolate linearly ...

In general metric space, this is not possible.

One can show convergence of (x_τ) to gradient flow as $\tau \to 0$.