System of hyperbolic conservation laws

$$\left.\begin{aligned} \partial_t \rho + \nabla\cdot(\rho v) &= 0 \\ \partial_t(\rho v) + \nabla\cdot(\rho v \otimes v) + \nabla p &= 0 \\ \partial_t \mathcal{E} + \nabla\cdot\big((\mathcal{E}+p)v\big) &= 0 \end{aligned}\right\} \text{ in } \underbrace{[0,\infty)}_{\text{time } t} \times \underbrace{\mathbb{R}^d}_{\text{space } x}$$

Cauchy problem: initial data

$$(\rho, v, \mathcal{E})(t=0,\cdot) =: (\bar\rho, \bar v, \bar{\mathcal{E}})$$

Density: map $t \longmapsto \rho(t,\cdot) \in \mathcal{P}_2(\mathbb{R}^d)$

Borel probability measures (wlog) with finite second moments

$\hookrightarrow$ Continuity equation: local conservation of mass

Eulerian velocity: $v(t,\cdot) \in L^2(\mathbb{R}^d; \rho(t,\cdot))$ vector field

taking values in $\mathbb{R}^d$

Momentum: $m(t,\cdot) = \rho(t,\cdot)\, v(t,\cdot)$

$\hookrightarrow$ Momentum equation: local conservation of momentum

p pressure (discussed below)

Energy: $\mathcal{E}(t,\cdot) \in \mathcal{M}_+(\mathbb{R}^d)$, nonnegative finite Borel measure

should be a.c. wrt. $\rho(t,\cdot)$

$\hookrightarrow$ Energy equation: local conservation of total energy

Conservation of total mass / total energy:

$$\begin{cases} \dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^d} \rho(t,dx) = 0 \\ \dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^d} \varepsilon(t,dx) = 0 \end{cases}$$

possibly relax to inequality ≤ 0 (i.e., no spontaneous generation of energy)

provides first a priori-bounds!

(1.) Pressureless gases $p = 0$ $\rightsquigarrow$ total energy = kinetic energy

$$\varepsilon = \tfrac{1}{2}\rho|v|^2$$

$$\left.\begin{aligned} \partial_t \rho + \nabla\cdot(\rho v) &= 0 \\ \partial_t(\rho v) + \nabla\cdot(\rho v \otimes v) &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbb{R}^d$$

Note that the energy equation follows (formally) by product rule.

$$\partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) = 0$$

$$\Longleftrightarrow \quad v\,\Big(\underbrace{\cancel{\partial_t\rho + \nabla\cdot(\rho v)}}_{=0 \text{ continuity equation}}\Big)$$

$$+\rho\,\Big(\underbrace{\partial_t v + v\cdot\nabla v}_{\overset{!}{=}0}\Big) = 0$$

velocity is anyway only defined where $\rho > 0$ (outside vacuum) since $v \in L^2(\mathbb{R}^d,\rho)$

Then

$$\partial_t\left(\tfrac{1}{2}\rho|v|^2\right) = \tfrac{1}{2}|v|^2\left(\partial_t\rho\right) + \rho v\cdot\partial_t v$$

$$= \tfrac{1}{2}|v|^2\left(-\nabla\cdot(\rho v)\right) + \underbrace{\rho v\cdot\left(-(v\cdot\nabla)v\right)}_{= -\rho v\cdot\nabla\left(\frac{1}{2}|v|^2\right)}$$

$$= -\nabla\cdot\left(\rho v\left(\tfrac{1}{2}|v|^2\right)\right)$$

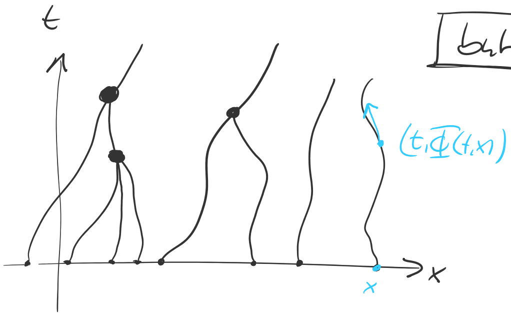
Momentum equation: only transport, no interaction (no pressure!)

Consider flow:

$$\begin{cases} \dot{\Phi}(t,x) = v(t,\Phi(t,x)), & t \in \mathbb{R} \\ \Phi(t=0,x) = x \end{cases}$$

Then $\rho(t,\cdot) = \Phi(t,\cdot)\#\bar{\rho}$, $t \in \mathbb{R}$ — initial density

Momentum is transported along with the flow



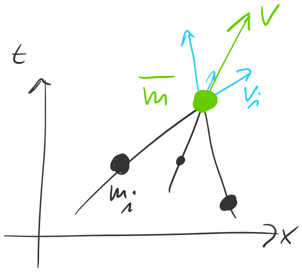
$\boxed{\text{but}}$ Sticky particle dynamics: — additional assumption

if particles collide (= meet at the same location) they stick together to form larger particles

$\hookrightarrow$ local conservation of mass and momentum still satisfied ✓

But energy may be lost in the (inelastic) collision.

assume particles with masses m_i, $i=1\dots N$,
and velocities v_i collide:



- total momentum before collision
$$\sum_{i=1}^{N} m_i v_i =: \overline{M}$$
- after collision there is only one particle with
$$\text{mass} \quad \overline{m} = \sum_{i=1}^{N} m_i$$
$$\text{velocity} \quad \overline{v} = \frac{\overline{M}}{\overline{m}} = \frac{\sum_{i=1}^{N} m_i v_i}{\sum_{i=1}^{N} m_i}$$

By Jensen's inequality:

$$\frac{1}{2}\overline{m}|\overline{v}|^2 = \frac{1}{2}\left(\sum_{k=1}^{N} m_k\right)\Bigg|\sum_{i=1}^{N}\underbrace{\frac{m_i}{\sum_{j=1}^{N} m_j}}_{=:\,\lambda_i \,\geq\, 0,\ \sum_{i=1}^{N}\lambda_i = 1} v_i\Bigg|^2$$

$$\leq \frac{1}{2}\left(\sum_{k=1}^{N} m_k\right)\sum_{i=1}^{N}\frac{m_i}{\left(\sum_{j=1}^{N} m_j\right)}|v_i|^2$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i |v_i|^2$$

usually strict

cancel

convex combination

$\Longrightarrow$ energy equation is related to inequality

"entropy condition"

$$\partial_t\left(\tfrac{1}{2}\rho|v|^2\right) + \nabla\cdot\left(\left(\tfrac{1}{2}\rho|v|^2\right)v\right) \leq 0 \quad \text{in distributional sense}$$

<u>Note</u> Stickyness assumption is a wanted feature of the model (without it, dynamics would be trivial)

- very simple model for formation of galaxies in the early universe
- building block in semiconductor modeling

Mathematical theory

pressureless gas dynamics with sticky particle assumption

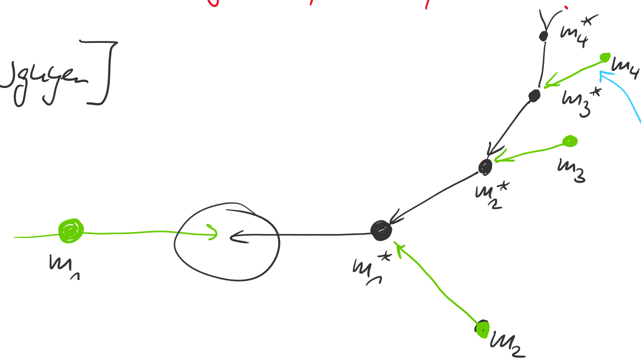
(i.) one space dimension:

existence, uniqueness, semigroup ... ✓

(ii.) multiple space dimensions:

very ill-posed problem!

[Bressan-Nguyen]



construction backward in time

each trajectory does not intersect with that of any other mass m_i

$\Longrightarrow$ countably many particles m_i, $i \in \mathbb{N}$, s.t.

- no collision between the trajectories of particles m_k and m_ℓ, $k \neq \ell$ can occur

  $\hookrightarrow$ free transport $\boxed{\text{is}}$ a solution

  sticky solution (empty condition)

- but if at time $t = -\infty$ collisions and stickyness "started to happen", then particles m_i^* are in the game

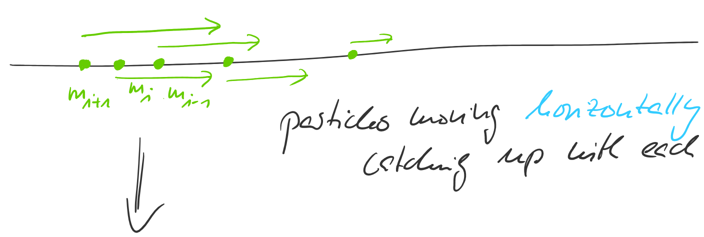
  $\hookrightarrow$ different, sticky solution

sticky!

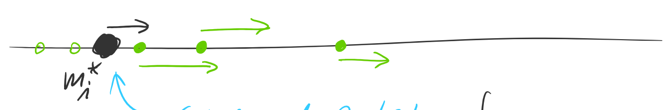
two different solutions to the same data

Nonuniqueness!

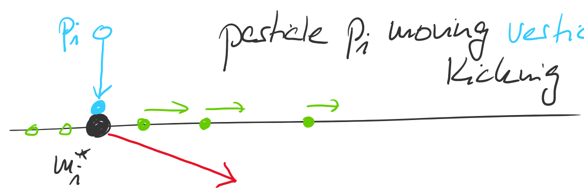
Second example



particles moving horizontally on the real line (1D) catching up with each other, sticking



compound particle from previous collisions



particle P_i moving vertically down kicking m_i^* off the real line

Can we have a sticky solution?

If there exists a compound particle m_i^*, then previous collisions of m_j, $j > i$, must have been sticky.

In particular, compound particle m_{i+1}^* must have existed.

$\boxed{\text{But}}$ m_{i+1}^* would have been knocked off the real line

$\Longrightarrow$ no m_i^* possible!

Nonexistence of sticky particle solution ← in multi-D

2. Isentropic Euler equations

Physical assumption: thermodynamical entropy constant in time/space

$\Longrightarrow$ pressure depends on density only

Internal energy functional ($L^1(\mathbb{R}^d)$-function)

$$\mathcal{U}[\rho] = \begin{cases} \int_{\mathbb{R}^d} U(r(x))\,dx & \text{if } \rho = r\mathcal{L}^d, \\ +\infty & \text{otherwise} \end{cases}$$

$\hookrightarrow$ pressure $P(r) = U'(r)r - U(r)$, $r \geq 0$

Examples

- Polytropic gases: $U(r) = \dfrac{\kappa}{\gamma - 1}\rho^\gamma$, $\kappa > 0$
  $P(r) = \kappa \rho^\gamma$

  $\gamma > 1$ adiabatic coefficient (Air: $\gamma = 1{,}4$)

- Isothermal gases: $U(r) = \kappa\, r \log(r)$, $\kappa > 0$
  $P(r) = r$

Continuity / momentum equation:

$$\left.\begin{aligned} \partial_t \rho + \nabla\cdot(\rho v) &= 0 \\ \partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) + \nabla P(\rho) &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbb{R}^d$$

Again energy equation follows from this for smooth (ρ, v):

$$\mathcal{E} = \tfrac{1}{2}\rho|v|^2 + U(\rho) \qquad \text{total energy (density)}$$

Note that

$$\partial_t v + v\cdot\nabla v + \underbrace{\frac{1}{\rho}\nabla P(\rho)}_{=\nabla U'(\rho)} = 0$$

Then

$$\begin{aligned} \partial_t \mathcal{E} &= \left(\tfrac{1}{2}|v|^2\,\partial_t\rho + \rho v\cdot\partial_t v\right) + U'(\rho)\,\partial_t\rho \\ &= -\tfrac{1}{2}|v|^2\,\nabla\cdot(\rho v) - \underbrace{\rho v\cdot\left((v\cdot\nabla)v + \nabla U'(\rho)\right)}_{=\rho v\cdot\nabla\left(\frac{1}{2}|v|^2\right)} - U'(\rho)\,\nabla\cdot(\rho v) \\ &= -\nabla\cdot\left(\left(\tfrac{1}{2}\rho|v|^2\right)v\right) - \nabla\cdot\left(\underbrace{U'(\rho)\rho}_{=P(\rho)+U(\rho)}\, v\right) \end{aligned}$$

Hence

$$\partial_t \mathcal{E} + \nabla\cdot\left((\mathcal{E}+P)v\right) = 0.$$

$\boxed{\text{But}}$ Solutions typically are not smooth!

Generic feature of hyperbolic conservation laws

$$\partial_t u + \partial_x f(u) = 0, \quad u(t=0,\cdot) = \bar{u}$$

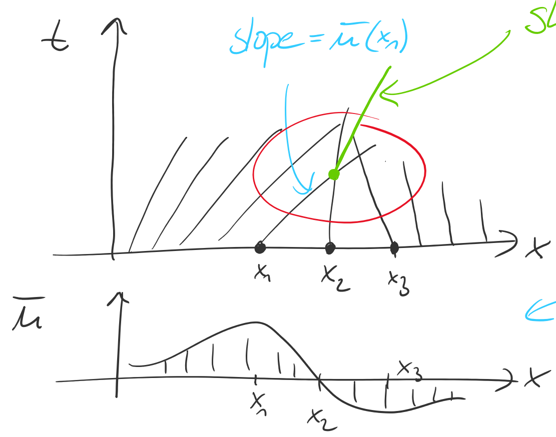
flux function, nonlinear

Simplest example:

Burgers equation $\quad \partial_t u + \partial_x\left(\tfrac{1}{2}u^2\right) = 0, \quad u = u(t,x)\in\mathbb{R}$

For smooth u equivalent to $\partial_t u + u\,\partial_x u = 0$

$\hookrightarrow$ solution constant along <u>characteristics</u>



along each characteristic line
solution = slope

$\rightsquigarrow$ intersections ?

initial data

$\Longrightarrow$ Formation of discontinuities (shocks)
solution jumps!

Weak solutions: for <u>all</u> smooth test functions

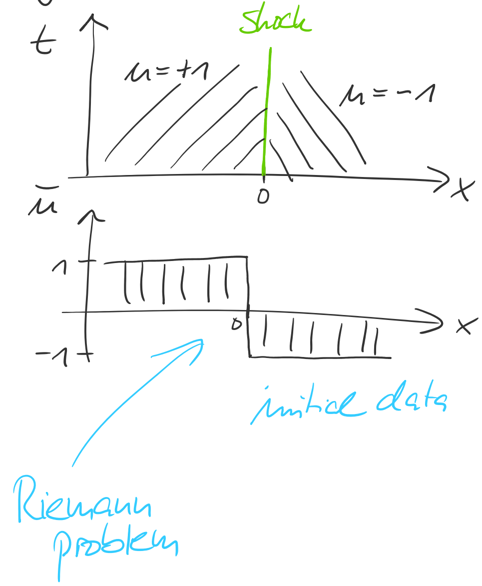
suitable class

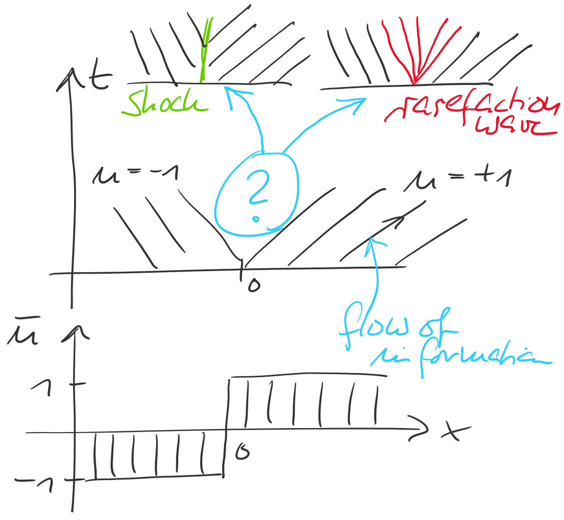
$$-\int_0^T\int_{\mathbb{R}} \left(\partial_t\varphi(t,x)\, u(t,x) + \partial_x\varphi(t,x)\, \tfrac{1}{2} u(t,x)^2 \right) dx\, dt = \int_{\mathbb{R}^d} \varphi(0,x)\, \bar{u}(x)\, dx$$

derivatives

(need sufficient <u>integrability</u> to make integrals defined)

Again: Burgers equation





Nonuniqueness !

Entropy condition: $\partial_t u + \partial_x f(u) = 0$ (CL)

with $u(t,x) \in K$ convex set (state space)

<u>Def.</u> A function $\eta : K \longrightarrow \mathbb{R}$ s.t.

additional conservation law

$$\partial_t \eta(u) + \partial_x q(u) = 0$$

for all smooth solutions of (CL) for some $q : K \longrightarrow \mathbb{R}$ is called entropy (with entropy flux q).

<u>Entropy condition</u>

For convex entropies $\eta : K \longrightarrow \mathbb{R}$ (with corresponding entropy flux q) we require that

relaxation

$$\partial_t \eta(u) + \partial_x q(u) \leq 0 \quad \text{distributionally}$$

i.e., for smooth test functions $\varphi \geq 0$

$$-\int_0^T \int_{\mathbb{R}} \Big(\partial_t \varphi(t,x)\, \eta(u(t,x)) + \partial_x \varphi(t,x)\, q(u(t,x)) \Big)\, dx\, dt$$

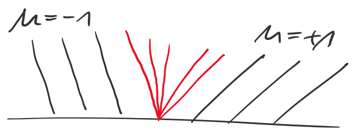
$$\leq \int_{\mathbb{R}} q(0,x)\, \eta(\bar{u}(x))\, dx .$$

<u>Note</u> For scalar conservation laws $u(t,x) \in \mathbb{R}$ (such as Burgers equation) there exists infinitely many convex entropies: $\eta : K \to \mathbb{R}$ convex

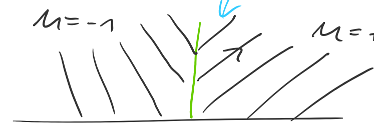
$$\hookrightarrow \quad q(u) := \int^{u} \eta'(s) f'(s)\, ds .$$

For Burgers:

characteristics must flow into the shock



admissible
(satisfies entropy condition)



not admissible

For isentropic Euler equations:

total energy is a (mathematical) entropy

$$\begin{cases} \eta(\rho, v) = \frac{1}{2}\rho|v|^2 + U(\rho) \\ q(\rho, v) = \left(\frac{1}{2}\rho|v|^2 + U'(\rho)\rho\right) v \end{cases}$$

convex in conserved quantities $(\rho, m = \rho v)$

$\hookrightarrow$ entropy condition (local version)

$$\partial_t \underbrace{\left(\tfrac{1}{2}\rho|v|^2 + U(\rho)\right)}_{= \eta(\rho,v)} + \nabla \cdot \Big(\underbrace{\left(\tfrac{1}{2}\rho|v|^2 + U'(\rho)\rho\right) v}_{= q(\rho,v)} \Big) \leq 0 \quad \text{distributionally}$$