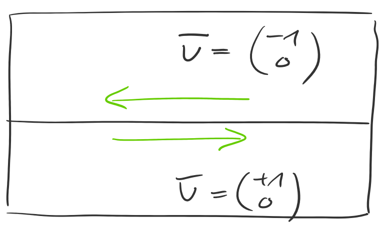
Reasonable assumption: no spontaneous generation of energy
(but energy can be lost in the form of heat)

not part of the model (no temperature)

$\boxed{\text{Bad News}}$ Still no uniqueness!

[Chiodaroli–DeLellis–Kreml]

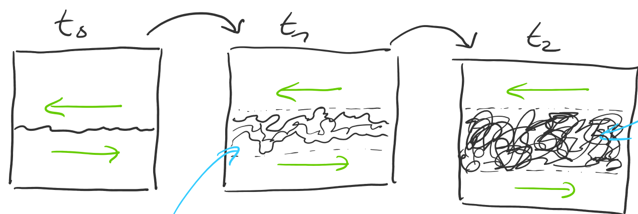
periodic b.c.



initial data: shear flow

admissible weak solution (stationary) ✓

But there are other solutions:



widely oscillating solutions

band grows in time

Connection to turbulence?

[DeLellis–Székelyhidi]

For a large class of initial data there exist infinitely many weak solutions.

And: entropy condition (differential inequality) does not prevent this!

$\boxed{\text{What now?}}$

Different entropy conditions have been proposed e.g. Dafermos maximal dissipation rate condition (similar approaches in physics)

Consider $\partial_t u + \partial_x f(u) = 0$, $u(t,x) \in K$ (CL)

Assume there exists a convex entropy $\eta : K \to \mathbb{R}$

$\hookrightarrow$ Select the solution u that, among all weak solutions w of (CL), decreases the total entropy as fast as possible:

steepest descent

for all weak solutions w of (CL) and $\tau \geq 0$

$$u(t,\cdot) = w(t,\cdot) \quad \text{for } t \in [0,\tau]$$

$$\Longrightarrow \frac{d}{dt}\Big|_{t=\tau} \int_{\mathbb{R}} \eta(u(\tau,\cdot)) < \frac{d}{dt}\Big|_{t=\tau} \int_{\mathbb{R}} \eta(w(\tau,\cdot))$$

Does not seem to rule out wild solutions for isentropic Euler eqns.

Q: What is a good entropy condition?

3. Full Euler equations

for polytropic gases

Pressure $\quad p = (\gamma - 1)\left(\varepsilon - \frac{1}{2}\rho|v|^2\right)$

Thermodynamical entropy $\quad S := \log\left(\frac{p}{c\rho^\gamma}\right), \quad c = \kappa(\gamma-1), \; \gamma > 1.$

(entropy density: $\sigma = \rho S$, $S \in L^1(\mathbb{R}^d, \rho)$)

<u>Def</u> Internal energy (polytropic gases) $\gamma > 1$, $\kappa \geq 0$

$$U(r,S) = \frac{\kappa}{\gamma - 1} e^{S} r^{\gamma}, \quad r \in [0,\infty), \; S \in \mathbb{R}$$

For all pairs of measures $(\rho, \sigma) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)$

$$\mathcal{U}[\rho,\sigma] := \begin{cases} \int_{\mathbb{R}^d} U(r(x), S(x))\, dx & \text{if } \rho = r\mathcal{L}^d, \; \sigma = \rho S, \\ +\infty & \text{otherwise.} \end{cases}$$

$\rho \ll \mathcal{L}^d$
$\sigma \ll \rho$

Pressure $\quad p(t,\cdot) = P(r(t,\cdot), S(t,\cdot))\,\mathcal{L}^d$

with $\quad P(r,S) = U'(r,S)\,r - U(r,S)$

derivative w.r.t. r

Formally / if (ρ, v, ε) are smooth solutions of (FE), then

$$\partial_t \varepsilon + \nabla\cdot\big((\varepsilon + p)v\big) = 0$$

can be replaced by a transport equation

$$\partial_t \sigma + \nabla\cdot(\sigma v) = 0$$

or $$\partial_t S + v\cdot\nabla S = 0$$

use continuity eqn. $\partial_t \rho + \nabla\cdot(\rho v) = 0$

Proof $\quad \dfrac{p}{\gamma-1} = \mathcal{E} - \frac{1}{2}\rho|v|^2$

We have $\quad \partial_t \mathcal{E} + \nabla\cdot(\mathcal{E} v) = \boxed{-\nabla\cdot(pv)}$ $\quad$ total energy

$$\partial_t\left(\tfrac{1}{2}\rho|v|^2\right) + \nabla\cdot\left(\left(\tfrac{1}{2}\rho|v|^2\right)v\right) = \tfrac{1}{2}|v|^2\Big(\underbrace{\cancel{\partial_t\rho + \nabla\cdot(\rho v)}}_{=0}\Big) \quad \text{continuity}$$

$$+ \rho v\cdot\underbrace{\Big(\partial_t v + (v\cdot\nabla)v\Big)}_{= -\frac{1}{\rho}\nabla p} \quad \text{Kinetic energy} \quad \text{momentum}$$

$$= \boxed{-v\cdot\nabla p}$$

$$\hookrightarrow \quad \partial_t p + \nabla\cdot(pv) = (\gamma-1)\Big[-\nabla\cdot(pv) + v\cdot\nabla p\Big]$$

$$= -(\gamma-1)\,p\,(\nabla\cdot v) \qquad (PT)$$

Consider now $\quad S = \log\left(\dfrac{p}{c\rho^\gamma}\right)$. Then

$$\partial_t S + v\cdot\nabla S = \left(\frac{p}{c\rho^\gamma}\right)^{-1}\Bigg[\frac{1}{c\rho^\gamma}\underbrace{\Big(\partial_t p + v\cdot\nabla p\Big)}_{=\underbrace{\partial_t p + \nabla\cdot(pv)}_{= -(\gamma-1)p(\nabla\cdot v)} - p(\nabla\cdot v)} - \frac{\gamma p}{c\rho^{\gamma+1}}\underbrace{\Big(\partial_t\rho + v\cdot\nabla\rho\Big)}_{= -\rho(\nabla\cdot v)}\Bigg]$$

$$= \frac{1}{p}\Big[-(\gamma-1)p(\nabla\cdot v) - p(\nabla\cdot v) - \frac{\gamma p}{\rho}\big(-\rho(\nabla\cdot v)\big)\Big]$$

$$= 0$$

Thermodynamics. Entropy nondecreasing!

$\hookrightarrow$ Full Euler system

$$\left.\begin{aligned} \partial_t \rho + \nabla\cdot(\rho v) &= 0 \\ \partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) + \nabla p &= 0 \\ \partial_t \mathcal{E} + \nabla\cdot((\mathcal{E}+p)v) &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbb{R}^d$$

Equation of state: $p = (\gamma-1)\left(\mathcal{E} - \frac{1}{2}\rho|v|^2\right)$ polytropic gases, $\gamma > 1$

Entropy condition: specific entropy $\boxed{S = \log\left(\frac{p}{c\rho^\gamma}\right),\ c = k(\gamma-1)}$

$$\partial_t S + v\cdot\nabla S \geq 0 \qquad \text{distributionally}$$

(hence rhs $= \mu$ nonnegative measure)

$$\left(\text{or: } \partial_t \sigma + \nabla\cdot(\sigma v) = \sigma\mu \geq 0\right)$$

Entropy lower bound:

$$\inf_{x\in\mathbb{R}^d} S(t,x) \geq \inf_{x\in\mathbb{R}^d} \bar{S}(x)$$

initial specific entropy

W.l.o.g. we will assume $S \geq 0$ (hence $\sigma(t,\cdot) = (\rho S)(t,\cdot) \in \mathcal{M}_+(\mathbb{R}^d)$)

Note [Brezina–Feireisl] proposed to select weak solutions that maximize entropy production μ

(can be implemented in the framework of measure valued solutions)