Winter School on Fluid Dynamics, Dispersive equations and Quantum fluids

Variational Methods in Hyperbolic Balance Laws

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Pressureless Gas Dynamics

Metric Projection (1/2)

Hilbert space $\mathscr{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and $C \subset \mathscr{H}$ **closed and convex**.

Definition

For any $Y \in \mathscr{H}$ we will denote by $\mathrm{P}_C(Y)$ the metric projection of Y onto the cone C, so that $\mathrm{P}_C(Y)$ satisfies

$$\|Y - \mathrm{P}_C(Y)\| = \inf \Big\{ \|Y - Z\| : Z \in C \Big\}.$$

Metric Projection (2/2)

The metric projection $\mathrm{P}_C(Y)$ exists and is **uniquely determined** for all $Y \in \mathscr{H}$. Moreover, it is characterized by the following property:

$$Y^* = \mathrm{P}_C(Y) \quad \Longleftrightarrow \quad \begin{cases} Y^* \in C, \\ \langle Y - Y^*, Y^* - Z \rangle \geqslant 0 \quad \text{for all } Z \in C. \end{cases}$$

If C is a **closed convex** cone, this is equivalent to

$$\langle Y - Y^*, Y^* \rangle = 0, \quad \langle Y - Y^*, Z \rangle \leqslant 0 \quad \text{for all } Z \in C.$$

The metric projection is a **contraction**:

$$\|\mathrm{P}_C(Y_1) - \mathrm{P}_C(Y_2)\| \leqslant \|Y_1 - Y_2\| \quad \text{for all } Y_1, Y_2 \in \mathscr{H}.$$

Lipschitz continuity

For given $\bar{X}, \bar{V} \in \mathscr{H}$ we now consider the map

$$t \mapsto X_t := \mathrm{P}_C(\bar{X} + t\bar{V}) \quad \text{for } t \in \mathbb{R},$$

which is well-defined and **Lipschitz continuous**. We have

$$\|X_{t+h} - X_t\| \leqslant |h| \|\bar{V}\| \quad \text{for all } t, h \in \mathscr{H}.$$

The velocity

$$V_t := \lim_{h \to 0} (X_{t+h} - X_t)/h \quad \textbf{exists strongly} \text{ for a.e. } t \in \mathbb{R}$$

and satisfies the inequality $\|V_t\| \leqslant \|\bar{V}\|$.

Tangent Cone

For any $Z \in C$ we define the **tangent cone**

$$\mathrm{Tan}_Z\, C := \overline{T_Z C}, \qquad T_Z C := \bigcup_{h>0} h(C - Z),$$

which is a **convex set**:

if $h_1, h_2 > 0$ and $Y_1, Y_2 \in C$ are given, then

$$(1-\lambda)h_1(Y_1 - Z) + \lambda h_2(Y_2 - Z) = h\Big(\big((1-\mu)Y_1 + \mu Y_2\big) - Z\Big) \in T_Z C$$

for all $\lambda \in [0,1]$, where

$$h := (1-\lambda)h_1 + \lambda h_2 > 0 \quad \text{and} \quad \mu := \lambda h_2 / h \in [0,1].$$

For any $Y \in \mathscr{H}$ we denote by $[Y]^\perp$ the **orthogonal complement** of $\mathbb{R}Y$.

Velocities (1/2)

Theorem (Haraux)

For fixed $t \in \mathbb{R}$*, let* V_t *be any weak limit point of*

$$V_t(h_n) := (X_{t+h_n} - X_t)/h_n$$

as $h_n \to 0+$*. Then* $V_t \in \Sigma_{X_t} C$ *and* $\langle \bar{V} - V_t, V_t \rangle \geqslant 0$*, where*

$$\Sigma_{X_t} C := \mathrm{Tan}_{X_t} C \cap [(\bar{X} + t\bar{V}) - X_t]^{\perp}.$$

Moreover, we have

$$\langle \bar{V} - V_t, W \rangle \leqslant 0 \quad \text{for all } W \in T_{X_t} C \cap [(\bar{X} + t\bar{V}) - X_t]^{\perp}.$$

Velocities (2/2)

If the cone C is polyhedric, so that

$$\overline{T_{X_t}C \cap [(\bar{X} + t\bar{V}) - X_t]^\perp} = \Sigma_{X_t}C$$

for all $t \in \mathbb{R}$, then the map $t \mapsto X_t$ is **strongly right-differentiable** and, denoting the right derivative again by V_t, we have

$$V_t = \mathrm{P}_{\Sigma_{X_t}C}(\bar{V}) \quad \text{for all } t \in \mathbb{R}.$$

Consider now $\mathscr{H} = \mathscr{L}^2(\mathbb{R}, \mu)$ for some $\mu \in \mathscr{P}_2(\mathbb{R})$.

The cone is defined as

$$C := \left\{X \in \mathscr{L}^2(\mathbb{R}, \mu) : X \text{ is monotone}\right\}.$$

We call X monotone if the support of the induced transport plan

$$\gamma_X := (\mathrm{id}, X)\#\mu$$

is a **monotone set** in $\mathbb{R} \times \mathbb{R}$, where $\#$ denotes the push-forward.

A subset $\Gamma \subset \mathbb{R} \times \mathbb{R}$ is monotone if for any $(m_i, x_i) \in \Gamma$ we have

$$(m_1 - m_2) \cdot (x_1 - x_2) \geqslant 0.$$

A Technical Lemma

For a Borel measure ν on a topological space Ω, we say $z \in \operatorname{spt} \nu$ if

$$\nu(N) > 0 \quad \text{for every open neighborhood } N \text{ of } z.$$

Lemma

For given $\mu \in \mathscr{P}_2(\mathbb{R})$ and $X \in C$ we define $\gamma_X := (\mathrm{id}, X)\#\mu$. Then there exists a Borel set $N_X \subset \mathbb{R}$ with $\mu(N_X) = 0$ such that

$$(m, X(m)) \in \operatorname{spt} \gamma_X \quad \textit{for all } m \in \mathbb{R} \setminus N_X.$$

Closed Convex Cone

Lemma

Let $\mathscr{H} = \mathscr{L}^2(\mathbb{R}, \mu)$ for some $\mu \in \mathscr{P}_2(\mathbb{R})$. The set

$$C := \{X \in \mathscr{L}^2(\mathbb{R}, \mu) \colon X \text{ is monotone}\}.$$

is a closed convex cone in $\mathscr{L}^2(\mathbb{R}, \mu)$. For any $X \in C$ and any smooth, strictly increasing function $\zeta \colon \mathbb{R} \longrightarrow \mathbb{R}$ that coincides with the identity map outside a compact set, we have $\zeta \circ X \in C$.

Theorem (Global Existence)

Let initial data $\bar{\varrho} \in \mathscr{P}_2(\mathbb{R})$ and $\bar{v} \in \mathscr{L}^2(\mathbb{R}, \bar{\varrho})$ be given. For some reference measure $\mu \in \mathscr{P}_2(\mathbb{R})$, let $\bar{X} \in C$ be the unique monotone transport with $\bar{X}\#\mu = \bar{\varrho}$. For $\bar{V} := \bar{v} \circ \bar{X}$ we define

$$X_t := \mathrm{P}_C(\bar{X} + t\bar{V}) \quad \text{for all } t \in \mathbb{R}.$$

Then X_t is differentiable for a.e. $t \in \mathbb{R}$ and $V_t := \dot{X}_t$ can be written in the following form: there exists a velocity $v_t \in \mathscr{L}^2(\mathbb{R}, \varrho_t)$ with

$$\varrho_t := X_t\#\mu, \quad \text{such that } V_t = v_t \circ X_t.$$

Then (ϱ_t, v_t) is a weak solution of the pressureless gas equations.

Space of Velocities

Lemma

For any X_t as above we define

$$\mathscr{H}_{X_t} := \mathscr{L}^2(\mathbb{R}, \mu)\text{-closure of } \{\varphi \circ X_t \colon \varphi \in \mathscr{D}(\mathbb{R})\}$$

and $\varrho_t := X_t \# \mu$. Then the following statement is true:

the function $W \in \mathscr{L}^2(\mathbb{R}, \mu)$ is in $\mathscr{H}_{X_t}$

if and only if

there exists $w \in \mathscr{L}^2(\mathbb{R}, \varrho_t)$ such that $W = w \circ X_t$.

Velocities are in the Tangent Cone

Lemma

Let $\mathscr{H}_{X_t}$ be defined as above. Then $\mathscr{H}_{X_t} \subset \mathbb{S}_{X_t}C$, with

$$\mathbb{S}_{X_t}C := \mathscr{L}^2(\mathbb{R}, \mu)\text{-closure of } T_{X_t}C \cap [(\bar{X} + t\bar{V}) - X_t]^{\perp}.$$

Recall that $X_t := \mathrm{P}_C(\bar{X} + t\bar{V})$ for $t \in \mathbb{R}$.

Lemma

With the notation above, we have $V_t =: \dot{X}_t \in \mathscr{H}_{X_t}$ for a.e. $t \in \mathbb{R}$.

Uniqueness and Semi-Group Property

Special case: For $\mu = \bar{\varrho}$ we can choose $(\bar{X}, \bar{V}) = (\mathrm{id}, \bar{v})$.

If there are two sets of initial data $(\bar{\varrho}_i, \bar{v}_i)$, $i = 1..2$, and if $(\bar{X}_i, \bar{V}_i)$ are the monotone transport maps and initial velocities corresponding to the reference measure $\mu \in \mathscr{P}_2(\mathbb{R})$, then the transport maps $X_{i,t}$ satisfy

$$\|X_{1,t} - X_{2,t}\|_{\mathscr{L}^2(\mathbb{R},\mu)} \leqslant \|\bar{X}_1 - \bar{X}_2\|_{\mathscr{L}^2(\mathbb{R},\mu)} + |t| \|\bar{V}_1 - \bar{V}_2\|_{\mathscr{L}^2(\mathbb{R},\mu)}$$

for all $t \in \mathbb{R}$ since the metric projection is a **contraction**. This implies the uniqueness and semi-group property of the transport map X_t.

The Eulerian velocities v_t are determined by the **orthogonal projection** of $\bar{V}$ onto the space $\mathscr{H}_{X_t}$, which is also unique.

One-dimensional granular system with memory effects

Evolution of Solid/Liquid Mixture [Lefebvre-Lepot & Maury]

$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2) + \partial_x p &= \varrho f \\ \partial_t \gamma + u \partial_x \gamma &= -p \end{aligned}\right\} \quad \text{in } [0, \infty) \times \mathbb{R} \tag{1}$$

$$\boxed{0 \leqslant \varrho \leqslant 1}$$

$$(1 - \varrho)\gamma = 0, \quad \gamma \leqslant 0$$

Physical quantities

- ϱ density
- u velocity
- γ adhesion potential
- p pressure
- f external force

Different Flow Behavior

$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2) + \partial_x p &= \varrho f \\ \partial_t \gamma + u \partial_x \gamma &= -p \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbb{R}$$

$$\boxed{0 \leqslant \varrho \leqslant 1}$$

$$(1-\varrho)\gamma = 0, \quad \gamma \leqslant 0$$

1. **Free Zones** $\varrho < 1$

   Pressureless dynamics of a compressible flow
   Both p and γ vanish

2. **Congested Zones** $\varrho = 1$

   "Incompressible flow" $\partial_x u = 0$
   Lagrange multiplier p is recorded into γ

Conservative Form

Differentiating with respect to x we obtain

$$\partial_t(\partial_x\gamma) + \partial_x\big(u\partial_x\gamma\big) = -\partial_x p \quad \text{in } [0,\infty)\times\mathbb{R}$$

Subtracting from momentum equation:

$$\partial_t(\varrho u - \partial_x\gamma) + \partial_x\big((\varrho u - \partial_x\gamma)u\big) = 0 \quad \text{in } [0,\infty)\times\mathbb{R}$$

Additional momentum $\partial_x\gamma$

Exclusion relation $(1-\varrho)\gamma = 0$ implies that $\partial_x\gamma \ll \varrho$. Writing

$$\boxed{\partial_x\gamma =: \varrho v}, \quad w := u - v,$$

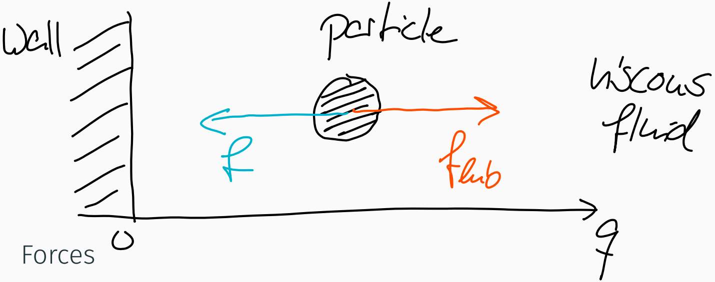
we obtain

$$\partial_t(\varrho w) + \partial_x(\varrho u w) = \varrho f \quad \text{in } [0,\infty)\times\mathbb{R}$$

(cf. two-velocity models by [Brenner])

Memory Effects

Spherical solid particle immersed in viscous fluid
in the presence of a wall [Maury]



Forces

- external force f
- lubrication force $f_{\text{lub}} \approx -\varepsilon\frac{\dot{q}}{q}$ **dominant as $q \to 0$**

where q is the distance to the wall
and ε is the viscosity

Vanishing Viscosity Limit $\varepsilon \to 0$

Sticky particle case. Hybrid system

$$\begin{cases} \dot{q} + \gamma = \bar{u} + \int_0^t f(s)\,ds \\ q \geqslant 0 \\ q\gamma = 0, \quad \gamma \leqslant 0 \end{cases}$$

Two possible system states: *free* $q > 0$ and *stuck* $q = 0$

Macroscopic model. Aligned solid particles

$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2) - \partial_x\left(\frac{\varepsilon}{1-\varrho}\partial_x u\right) &= \varrho f \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbb{R}$$

Lubrication force represented by *singular viscous force* [Lefebvre-Lepot & Maury]

Global Existence

Weak solutions of the following system:

$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0 \\ \partial_t(\varrho u - \partial_x \gamma) + \partial_x\big((\varrho u - \partial_x \gamma) u\big) &= \varrho f \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbb{R}$$

$$\boxed{0 \leqslant \varrho \leqslant 1}$$

$$(1-\varrho)\gamma = 0, \quad \gamma \leqslant 0$$

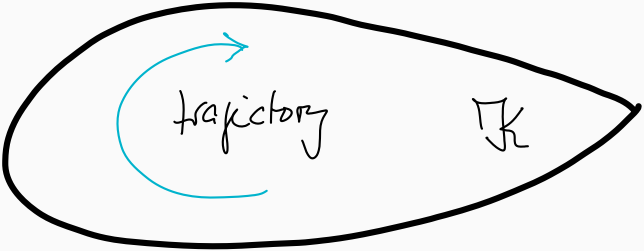
Related work on

- pressureless gas dynamics equation [Natile-Savaré]
- pressureless system with maximal density constraint [Bouchut et al.]

Maximal density constraint relevant e.g. for traffic flow, crowd motion

Finite-Dimensional Example

Consider a particle of unit mass moving under the influence of some force field $f(t,x)$ inside a convex set $\mathcal{K} \subset \mathbb{R}^N$:



Newton's Law

Most of the time we have $\ddot{x}(t) = f(t, x(t))$.

Collisions

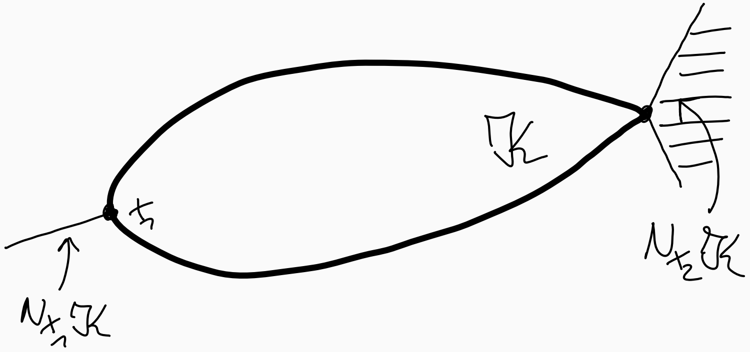
Upon collision with the boundary, an instantaneous force changes the velocity so that the particle remains inside $\mathcal{K}$.

Normal Cone

For any $x \in \mathcal{K}$ we define the closed convex set

$$N_x\mathcal{K} := \left\{ w \in \mathbb{R}^d \colon w \cdot (\tilde{x} - x) \leqslant 0 \text{ for all } \tilde{x} \in \mathcal{K} \right\}.$$

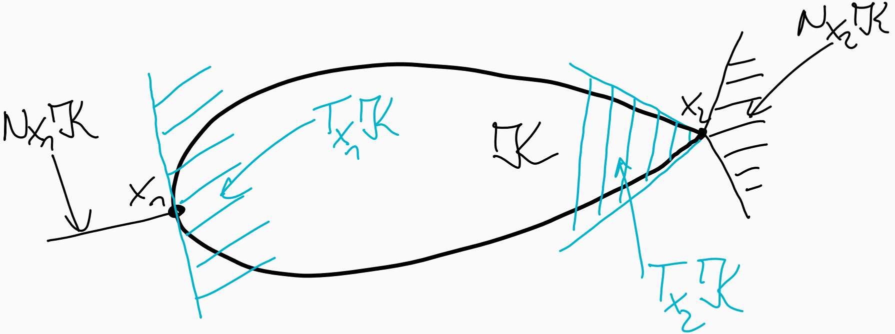
We have that $N_x\mathcal{K} = \partial I_{\mathcal{K}}(x)$, where $I_{\mathcal{K}}$ is the indicator function of $\mathcal{K}$.



Tangent Cone (Admissible Velocities)
For any $x \in \mathcal{K}$ we define the tangent cone by polarity:

$$T_x\mathcal{K} := (N_x\mathcal{K})^* = \left\{ u \in \mathbb{R}^d : u \cdot w \leqslant 0 \text{ for all } w \in N_x\mathcal{K} \right\}.$$

Since both set are closed convex cones, we also have $N_x\mathcal{K} = (T_x\mathcal{K})^*$.

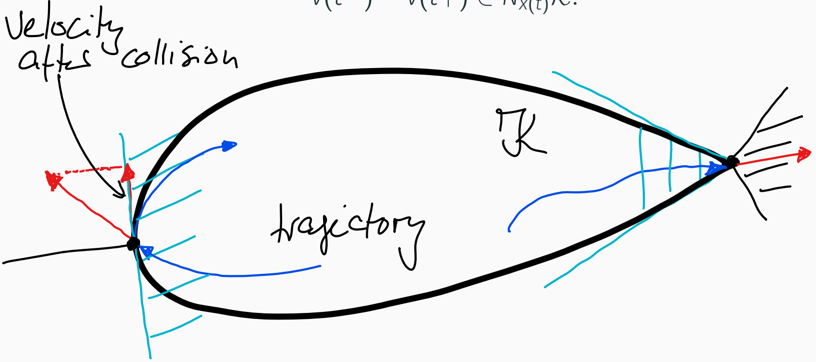


When the particle hits the boundary, its velocity is changed to an admissible velocity (particle “bounces off”).

Impact Law

If $x(t) \in \partial K$, then we assume that $v(t+) := \mathrm{P}_{T_{x(t)}\mathcal{K}} v(t-)$, where $\mathrm{P}_{T_{x(t)}\mathcal{K}}$ is the metric projection onto $T_{x(t)}\mathcal{K}$, which is well-defined. Then

$$v(t-) - v(t+) \in N_{x(t)}\mathcal{K}.$$



Second Order Differential Inclusion

We can model the above behavior using the differential inclusion

$$\dot{x} = v \quad \text{and} \quad \dot{v} + \partial I_{\mathcal{K}}(x) \ni f(x)$$

a.e. in $[0, \infty)$. This makes sense e.g. if

$$x \in \mathrm{Lip}\big([0,\infty), \mathbb{R}^d\big) \quad \text{and} \quad v \in \mathrm{BV}\big([0,\infty), \mathbb{R}^d\big).$$

The acceleration measure $\dot{v}$ is supported in the set of times for which $x(t) \in \partial\mathcal{K}$. At these times, we assume that the impact law holds.

Remark

This framework allows for particle motions that preserve the energy, such as a particle sliding along the boundary of a circle.

Pressureless Gas Dynamics

Let $\Omega := (0,1)$ and

$$\mathcal{K} := \left\{ X \in \mathscr{L}^2(\Omega) \colon X \text{ nondecreasing} \right\}$$

There exists an bijection between $\varrho \in \mathscr{P}_2(\mathbb{R})$ and $X \in \mathcal{K}$ given by

$$\varrho = X\#(\mathcal{L}^1|_\Omega).$$

Here $\mathcal{L}^1$ is the Lebesgue measure and $\#$ denotes the push-forward. The elements of $\mathcal{K}$ are optimal transport maps.

Lagrangian Formulation

To any solution (ϱ, u) of the pressureless gas dynamics equations we associate a uniquely determined curve of optimal transport maps

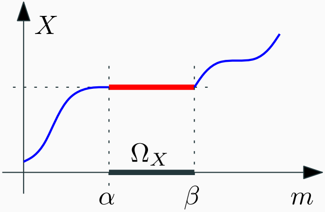
$$t \mapsto X(t,\cdot) \in \mathcal{K}.$$

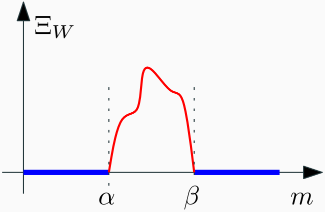
Boundary of $\mathcal{K}$

We find that $X \in \partial\mathcal{K}$ if and only if

$$\Omega_X := \Big\{ m \in \Omega \colon X \text{ is constant in a neighborhood of } m \Big\} \neq \varnothing.$$

Notice that if (α, β) is a maximal interval contained in Ω_X, then the push-forward $\varrho := X\#(\mathcal{L}^1|_\Omega)$ has a Dirac measure of mass $\beta - \alpha$ at position $X(m)$ for any generic $m \in (\alpha, \beta)$ (where X is constant).





Lemma (Normal Cone $N_X\mathcal{K} = \partial I_{\mathcal{K}}(X)$)

Let $X \in \mathcal{K}$ be given. For any $W \in \mathscr{L}^2(\Omega)$ we denote by

$$\Xi_W(m) := \int_0^m W(s)\,ds \quad \textit{for all } m \in \Omega,$$

its primitive. Then $W \in N_X\mathcal{K}$ if and only if $\Xi_W \in \mathcal{N}_X$, where

$$\mathcal{N}_X := \Big\{ \Xi \in \mathscr{C}(\Omega) \colon \Xi \geqslant 0 \textit{ in } \Omega \textit{ and } \Xi = 0 \textit{ in } \Omega \setminus \Omega_X \Big\}.$$

Lemma (Tangent Cone $T_X\mathcal{K} = (N_X\mathcal{K})^*$)

Let $X \in \mathcal{K}$ be given. Then

$$T_X\mathcal{K} = \Big\{ U \in \mathscr{L}^2(\Omega) \colon U \textit{ is nondecreasing in each } (\alpha, \beta) \subset \Omega_X \Big\}.$$

Space of Velocities

For any $X \in \mathcal{K}$ we also define

$$\mathscr{H}_X = \Big\{ U \in \mathscr{L}^2(\Omega) \colon U \text{ is constant in each } (\alpha, \beta) \subset \Omega_X \Big\}.$$

This is a closed subspace (not merely a cone). We denote by $\mathrm{P}_{\mathscr{H}_X}$ the orthogonal projection onto $\mathscr{H}_X$. For any $V \in \mathscr{L}^2(\Omega)$ we find that

$$\mathrm{P}_{\mathscr{H}_X}(V) = \begin{cases} V & \text{a.e. in } \Omega \setminus \Omega_X, \\ \displaystyle\fint_\alpha^\beta V(s)\, ds & \text{in each maximal interval } (\alpha, \beta) \subset \Omega_X. \end{cases}$$

Eulerian Velocity

If $V \in \mathscr{H}_X$, then V is constant wherever X is constant. Therefore there exists a $u \in \mathscr{L}^2(\mathbb{R}, \varrho)$ with $V = u \circ X$ a.e., where $\varrho = X_\#(\mathcal{L}^1|_\Omega)$.

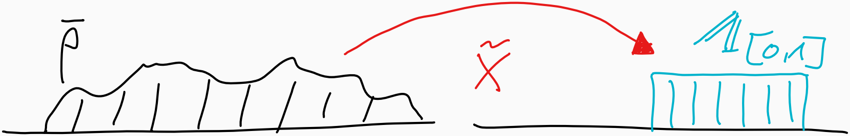
Admissible Transport Maps

Consider initial density $\bar{\varrho} \in \mathscr{P}_2(\mathbb{R})$ with $0 \leqslant \bar{\varrho} \leqslant 1$.

Maximal Compression

Let $\tilde{X} \in \mathscr{L}^2(\mathbb{R}, \bar{\varrho})$ be a monotone map such that

$$\tilde{X}\#\bar{\varrho} = \mathbb{1}_{[0,1]}.$$



Admissible transport maps compress $\bar{\varrho}$ not more than $\tilde{X}$:

$$\boxed{\tilde{\mathcal{K}} := \tilde{X} + \mathcal{K}} \quad \text{with} \quad \mathcal{K} := \left\{ X \in \mathscr{L}^2(\mathbb{R}, \bar{\varrho}) : X \text{ nondecreasing} \right\}$$

To any $X_t \in \tilde{\mathcal{K}}$ we associate an monotone transport map

$$S_t := X_t - \tilde{X} \in \mathcal{K}.$$

Lagrangian Dynamics

Free velocity.

For initial velocity $\bar{U} \in \mathscr{L}^2(\mathbb{R}, \bar{\varrho})$, we define

$$U_t^{\mathrm{free}}(y) := \bar{U} + \int_0^t f\big(s, X_s(y)\big)\, ds,$$

integrating along trajectories $t \mapsto X_t(y)$ starting at $y \in \mathbb{R}$.

Free trajectory.

For initial transport $\bar{X} := \mathrm{id} \in \tilde{\mathcal{K}}$, we define

$$X_t^{\mathrm{free}}(y) := \bar{X}(y) + \int_0^t U_s^{\mathrm{free}}(y)\, ds.$$

Lagrangian solution.

We are looking for a curve $t \mapsto X_t \in \tilde{\mathcal{K}}$ such that

$$\boxed{X_t = \mathrm{P}_{\tilde{\mathcal{K}}}(X_t^{\mathrm{free}})} \quad \text{for } t \in [0, \infty).$$

Coupled system: U_t^{free} depends on X_t.

The associated Lagrangian velocity $\boxed{U_t := \dot{X}_t}$ not only belongs to the tangent cone of $\mathcal{K}$ at $S_t := X_t - \tilde{X}$, which is defined as

$$T_{S_t}\mathcal{K} := \mathscr{L}^2(\mathbb{R}, \bar{\varrho})\text{-closure of } \bigcup_{h>0} h(\mathcal{K} - S_t),$$

it has to be constant on each congested block and thus belong to

$$\mathscr{H}_{S_t} := \left\{U \in \mathscr{L}^2(\mathbb{R}, \bar{\varrho})\colon U \text{ constant on maximal intervals in } \Omega_{S_t}\right\},$$

where Ω_{S_t} is the union of intervals on which S_t is constant.

Adhesion potential

For all $y \in \mathbb{R}$ and $t \in [0, \infty)$ we define

$$\Gamma_t(y) := \int_{-\infty}^{y} \Big(U_t(z) - U_t^{\mathrm{free}}(z)\Big)\, dz.$$

Solution Concept

Given suitable initial data $(\bar{\varrho}, \bar{u})$, a triple (ϱ, u, γ) is called a weak solution of the system (1) provided that

- the triple (ϱ, u, γ) satisfies

$$\varrho_t \in \mathscr{P}_2(\mathbb{R}), \quad u_t \in \mathscr{L}^2(\mathbb{R}, \varrho_t) \quad \text{for a.e. } t \in [0, \infty),$$
$$\gamma \in \mathscr{L}^\infty\big([0,\infty); W^{1,1}(\mathbb{R})\big);$$

- the density constraints $0 \leqslant \varrho_t \leqslant 1$ holds a.e.;
- the exclusion principle $(1-\varrho_t)\gamma_t = 0$ and $\gamma_t \leqslant 0$ holds a.e.;
- for all $\varphi \in \mathcal{C}_c^\infty\big([0,T) \times \mathbb{R}\big)$ we have

$$\int_0^T \int_{\mathbb{R}} \Big(\partial_t\varphi(t,x) + u_t(x)\partial_x\varphi(t,x)\Big)\big(\rho_t(x)u_t(x) - \partial_x\gamma_t(x)\big)\,dx\,dt$$
$$+ \int_0^T \int_{\mathbb{R}} \varphi(t,x)\,\rho_t(x)f_t(x)\,dx\,dt = -\int_{\mathbb{R}} \varphi(0,x)\,\bar{\rho}(x)\bar{u}(x)\,dx,$$

with a similar statement for the continuity equation.

Main Result

Theorem (Global Existence)

Let external force $f \in L^\infty([0,\infty); \mathrm{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ be given. Suppose that $\bar\rho \in \mathcal{P}_2(\mathbb{R})$ with $0 \leqslant \varrho \leqslant 1$ a.e., and that $\bar u \in L^2(\mathbb{R}, \bar\rho)$. Define

$$\rho_0 := \bar\rho, \quad U_0 := \bar u, \quad X_0 := \mathrm{id}.$$

There exists a curve $t \mapsto X_t \in \tilde K$ that is differentiable for a.e. t and solves the Lagrangian dynamics equations introduced above.

Lagrangian velocity U_t and adhesion potential Γ_t are well-defined.

There exist $(u_t, \gamma_t) \in \mathcal{L}^2(\mathbb{R}, \rho_t) \times W^{1,1}(\mathbb{R})$ for a.e. t, such that

$$U_t = u_t \circ X_t, \quad \Gamma_t = \gamma_t \circ X_t \quad \textit{where} \quad \rho_t := (X_t)_\# \rho_0.$$

The triple (ρ, u, γ) is a global weak solution of system (1).

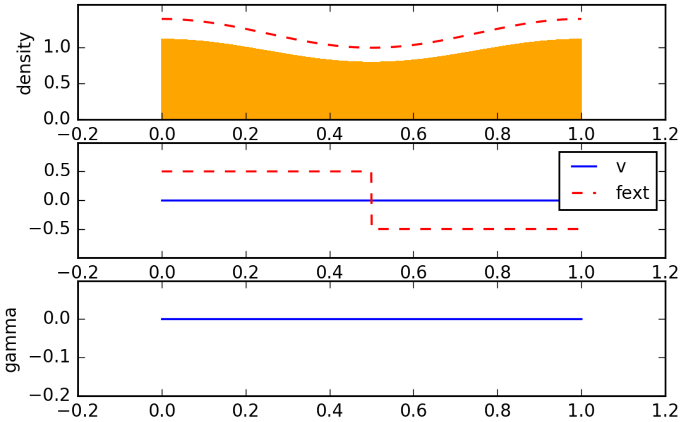
Heterogeneous Maximal Constraint

The same approach works if the maximal density constraint is given by a function ϱ^* that is transported with the flow:

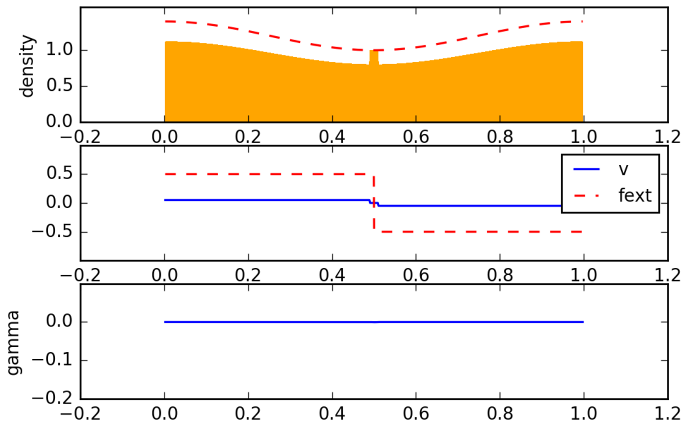
$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0 \\ \partial_t(\varrho u - \partial_x \gamma) + \partial_x\big((\varrho u - \partial_x\gamma) u\big) &= \varrho f \\ \partial_t \varrho^* + u \partial_x \varrho^* &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbb{R}$$

$$\boxed{0 \leqslant \varrho \leqslant \varrho^*}$$

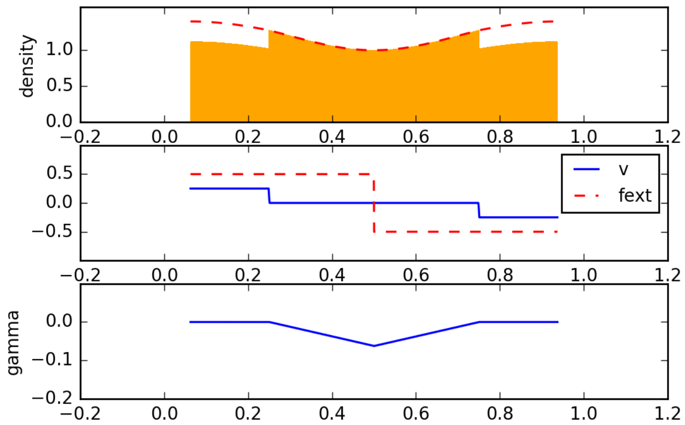
$$(\varrho^* - \varrho)\gamma = 0, \quad \gamma \leqslant 0$$



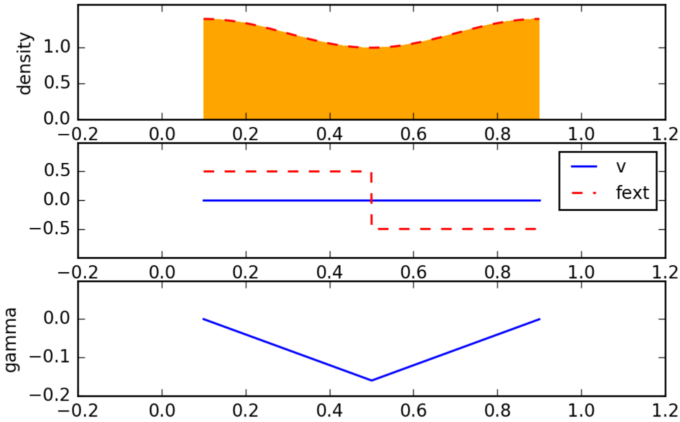
Solution at time $t = 0$.



Solution at time $t = 0.1$.



Solution at time $t = 0.5$.



Solution at time $t = 0.8$.

Monotone Functions

Set-Valued Maps

For a set-valued map $u\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we define

the **domain** of u:

$$\mathrm{dom}(u) := \{x \in \mathbb{R}^n \colon u(x) \neq \varnothing\},$$

the **image** of u:

$$\mathrm{im}(u) := \{y \in \mathbb{R}^n \colon \text{there exists } x \in \mathbb{R}^n \text{ with } y \in u(x),$$

the **graph** of u:

$$\Gamma u := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \colon y \in u(x)\},$$

and the **inverse** of u:

$$u^{-1}(x) := \{y \in \mathbb{R}^n \colon x \in u(y)\}.$$

We say that a set-valued map $u \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is **monotone** if

$$(y_1 - y_2) \cdot (x_1 - x_2) \geqslant 0 \quad \text{for all } x_i \in \mathbb{R}^n,\, y_i \in u(x_i),\, i = 1..2.$$

A monotone map u is called **maximal** if it is maximal with respect to inclusion in the class of monotone functions:

$$v \supset u,\, v \text{ monotone} \quad \Longrightarrow \quad v = u.$$

Let $\Phi\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ be the **Cayley transform**:

$$\Phi(x, y) := \frac{1}{\sqrt{2}}(x + y, x - y) \quad \text{for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Proposition

Let u be a maximal monotone function. Then $(u + \mathrm{id})^{-1}$ is defined on the whole $\mathbb{R}^n$ and $\Phi(\Gamma u)$ is the graph of a 1-Lipschitz function $Fu\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$Fu(z) := z - \sqrt{2}(u + \mathrm{id})^{-1}(\sqrt{2}z) \quad \text{for all } z \in \mathbb{R}^n.$$

Conversely, for any 1-Lipschitz function $\phi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the set $\Phi^{-1}(\Gamma\phi)$ is the graph of a maximal monotone function.

Properties of Monotone Functions (1/2)

Proposition

Let u be a maximal monotone function. Then

1. *if u is maximal, then its graph Γu is closed, and $u(x)$ is a* ***convex, closed*** *(possibly empty) set for every $x \in \mathbb{R}^n$;*
2. *u is maximal if and only if the domain of $u + \mathrm{id}$ is $\mathbb{R}^n$;*
3. *$u + \mathrm{id}$ and $(u + \mathrm{id})^{-1}$ are monotone functions and $(u + \mathrm{id})^{-1}$ is 1-Lipschitz continuous;*
4. *for any set $X \subset \mathrm{dom}(u)$, $\bar{x}$ in the interior of $\mathrm{conv}(X)$ and $\bar{y} \in u(\bar{x})$*

$$|\bar{y}| \leqslant \frac{C}{\mathrm{dist}(\bar{x}, \mathbb{R}^n \setminus \mathrm{conv}(X))}$$

where $C := \left(\sup_{x \in X} \inf_{y \in u(x)} |y|\right) \mathrm{diam}(X)$.

Properties of Monotone Functions (2/2)

Corollary

Let u be a maximal monotone function. Then

1. *u is upper semicontinuous:*

$$x_n \longrightarrow x, y_n \longrightarrow y, y \in u(x_n) \quad \Longrightarrow \quad y \in u(x);$$

2. *the domain of u contains the interior of its convex hull:*

$$\operatorname{int}\operatorname{conv}\operatorname{dom}(u) \subset \operatorname{dom}(u) \subset \operatorname{conv}\operatorname{dom}(u);$$

3. *$u(B)$ is **bounded** if B is relatively compact in $\operatorname{int}\operatorname{dom}(u)$;*
4. *if $u(x)$ consists of exactly one point y, then x belongs to $\operatorname{int}\operatorname{dom}(u)$ and u is continuous at x:*

$$x_n \longrightarrow x, y_n \in u(x_n) \quad \Longrightarrow \quad y_n \longrightarrow y.$$

Singular Sets

Definition

Let u be a maximal monotone function and $k = 1, \ldots, n$ an integer. We define

$$\Sigma^k(u) := \{x \in \mathbb{R}^n :\ \dim u(x) \geqslant k\}$$

where $\dim u(x)$ is the dimension of the set $u(x)$.

Recall that $u(x)$ is a closed, convex set (possibly empty). We note that

$$\Sigma^n(u) \subset \Sigma^{n-1}(u) \subset \cdots \subset \Sigma^1(u).$$

Size of Singular Sets

Theorem

The Hausdorff dimension of the set $\Sigma^k(u)$ is at most $(n-k)$.
More precisely $\Sigma^k(u)$ is countably $\mathcal{H}^{n-k}$-rectifiable: we can find countably many $\mathscr{C}^1$-submanifolds $\Gamma_i \subset \mathbb{R}^n$ of dimension $n-k$ that cover $\mathcal{H}^{n-k}$-almost all of $\Sigma^k(u)$, i.e.,

$$\mathcal{H}^{n-k}\left(\Sigma^k(u) \setminus \bigcup_i \Gamma_i\right) = 0.$$

In particular, $\Sigma^n(u)$ is at most countable.

Size Estimates for the Graph

Proposition

For any monotone function u and for any ball $B \subset \mathbb{R}^n \times \mathbb{R}^n$ of radius r we have

$$\mathcal{H}^n(\Gamma u \cap B) \leqslant 2^{n/2}\omega_n r^n$$

for some constant ω_n, and for every Borel set $A \subset \mathbb{R}^n$

$$\mathcal{H}^n\big(\Gamma u \cap (A \times \mathbb{R}^n)\big) \leqslant 2^{n/2}\omega_n\big(\mathrm{diam}(A) + \mathrm{osc}(u, A)\big)^n,$$

where $\mathrm{osc}(u, A) := \sup\{|y_1 - y_2| : y_1, y_2 \in u(A)\}$.

Differentiability Properties

Theorem

Let u be a maximal monotone function and let D be the set of points x such that $u(x)$ is a singleton (that is, $u(x)$ consists of exactly one point, which we still denote by $u(x)$).
Then u is differentiable at almost every $\bar{x} \in D$, i.e., there exists an $(n \times n)$-matrix $\nabla u(\bar{x})$ such that

$$\lim_{\substack{x \to \bar{x} \\ y \in u(x)}} \frac{y - u(\bar{x}) - \nabla u(\bar{x}) \cdot (x - \bar{x})}{|x - \bar{x}|} = 0.$$

Moreover, the determinants of all minors of ∇u are integrable on every bounded set B such that $u(B)$ is bounded.

Proposition

Let u be a monotone function and Ω an open set relatively compact in $\operatorname{int}\operatorname{dom}(u)$. The function u, viewed as an element in $\mathscr{L}^\infty(\Omega;\mathbb{R}^n)$, belongs to $\mathrm{BV}(\Omega;\mathbb{R}^n)$. Moreover, we have

$$\int_\Omega |Du| \leqslant C_n\big(\mathrm{diam}(\Omega) + \mathrm{osc}(u,\Omega)\big)^n$$

where C_n is a constant that depends on n only.

The inequality above can be improved to

$$\int_\Omega |Du| \leqslant C_n \mathrm{diam}(\Omega)^{n-1}\,\mathrm{osc}(u,\Omega).$$

Distributional Derivatives (2/2)

Theorem

Let Ω be an open convex set in $\mathbb{R}^n$.

1. *If u is a maximal monotone function such that $\mathrm{dom}(u) \supset \Omega$, then $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and Du is a positive, matrix-valued, and locally bounded measure.*
2. *Conversely, if $u \in \mathscr{L}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and Du is a positive, matrix-valued distribution on Ω, then there exists a maximal monotone function v such that $\mathrm{dom}(v) \supset \Omega$ and $v = u$ a.e. in Ω.*

A matrix-valued distribution Λ is positive (resp. symmetric) if $\langle \Lambda, \phi \rangle$ is a positive (resp. symmetric) matrix for positive test functions ϕ.

Approximation of Monotone Maps

For any $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ we define

$$C_\varrho = \left\{ \mathbf{t} \in \mathscr{L}^2(\mathbb{R}^d, \varrho) \colon \operatorname{spt}\left((\mathrm{id}, \mathbf{t})\#\varrho\right) \text{ is monotone} \right\}.$$

Lemma

For every $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ absolutely continuous with respect to the Lebesgue measure and all $\mathbf{t} \in \mathscr{C}_\varrho$ there is a sequence of Lipschitz continuous, monotone maps $\mathbf{t}_k$ defined on all of $\mathbb{R}^d$, such that

$$\lim_{k \to \infty} \|\mathbf{t}_k - \mathbf{t}\|_{\mathscr{L}^2(\mathbb{R}^d, \varrho)} = 0.$$

Polar Cone of the Set of Monotone Maps

Some Notation (1/3)

Let $\mathrm{Mat}_d(\mathbb{R})$ be the space of real $(d \times d)$-matrices and

$$\mathrm{Mat}_d(\mathbb{R}, \Box) := \left\{ A \in \mathrm{Mat}_d(\mathbb{R}) \colon v \cdot (Av) \,\Box\, 0 \text{ for all } v \in \mathbb{R}^d \right\}$$

where $\Box$ stands for either $\geqslant$ or $>$.

The analogous spaces of **symmetric** matrices will be denoted by

$$\mathrm{Sym}_d(\mathbb{R}) \quad \text{and} \quad \mathrm{Sym}_d(\mathbb{R}, \Box).$$

For all $A \in \mathrm{Mat}_d(\mathbb{R})$ we define the symmetric/antisymmetric parts

$$A^{\mathrm{sym}} := (A + A^{\mathrm{T}})/2 \quad \text{and} \quad A^{\mathrm{anti}} := (A - A^{\mathrm{T}})/2.$$

We denote by $\mathrm{Skew}_d(\mathbb{R})$ the space of **antisymmetric** matrices.

Some Notation (2/3)

Let $\mathscr{C}_*(\mathbb{R}^d;\mathbb{R}^D)$ be the space of all continuous functions

$$f\colon \mathbb{R}^d \longrightarrow \mathbb{R}^D \quad \text{for which} \quad \lim_{|x|\to\infty} f(x) \in \mathbb{R}^D \text{ exists.}$$

We identify $\mathscr{C}_*(\mathbb{R}^d;\mathbb{R}^D)$ with the space $\mathscr{C}(\dot{\mathbb{R}}^d;\mathbb{R}^D)$ of continuous functions on the one-point compactification $\dot{\mathbb{R}}^d$ of $\mathbb{R}^d$.

We adjoin to $\mathbb{R}^d$ a point ∞ and define a distance

$$d(x,y) := \begin{cases} \min\{|x-y|, h(x)+h(y)\} & \text{if } x,y \in \mathbb{R}^d, \\ h(x) & \text{if } x \in \mathbb{R}^d \text{ and } y = \infty, \\ 0 & \text{if } x,y = \infty, \end{cases}$$

where $h(x) := 1/(1+|x|)$ for all $x \in \mathbb{R}^d$. Then $|x| \to \infty$ is equivalent to $d(x,\infty) \to 0$. To $g \in \mathscr{C}_*(\mathbb{R}^d;\mathbb{R}^D)$ we associate $\dot{g} \in \mathscr{C}(\dot{\mathbb{R}}^d;\mathbb{R}^D)$ as

$$\dot{g}(x) := \begin{cases} g(x) & \text{if } x \in \mathbb{R}^d, \\ \lim_{|x|\to\infty} g(x) & \text{if } x = \infty. \end{cases}$$

For $u \in \mathscr{C}^1(\mathbb{R}^d;\mathbb{R}^d)$ we refer to the symmetric part $\nabla u(x)^{\mathrm{sym}}$ for all $x \in \mathbb{R}^d$ as its **deformation tensor**, which is in $\mathscr{C}(\mathbb{R}^d;\mathrm{Sym}_d(\mathbb{R}))$. Let

$$\mathscr{C}^1_*(\mathbb{R}^d;\mathbb{R}^d) := \big\{u \in \mathscr{C}^1(\mathbb{R}^d;\mathbb{R}^d) \colon \nabla u \in \mathscr{C}_*\big(\mathbb{R}^d;\mathrm{Mat}_d(\mathbb{R})\big)\big\},$$
$$\mathrm{Mon}(\mathbb{R}^d) := \big\{u \in \mathscr{C}^1_*(\mathbb{R}^d;\mathbb{R}^d) \colon u \text{ is monotone}\big\}.$$

The cone $\mathrm{Mon}(\mathbb{R}^d)$ contains, in particular, all **linear maps**

$$u(x) := Ax \quad \text{with} \quad A \in \mathrm{Mat}_d(\mathbb{R},\geqslant).$$

Stress Tensor

Theorem

Assume that there exist a measure $\mathsf{F} \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ *with finite first moment and a measure* $\mathsf{P} \in \mathscr{M}(\mathbb{R}^d;\mathrm{Sym}_d(\mathbb{R},\geqslant))$ *with*

$$G(u) := -\int_{\mathbb{R}^d} \langle u(x), \mathsf{F}(dx)\rangle - \int_{\mathbb{R}^d} \mathrm{tr}\big(\nabla u(x)\mathsf{P}(dx)\big) \geqslant 0$$

for all $u \in \mathrm{Mon}(\mathbb{R}^d)$. *There exists* $\mathsf{R} \in \mathscr{M}(\dot{\mathbb{R}}^d;\mathrm{Sym}_d(\mathbb{R},\geqslant))$ *with*

$$G(u) = \int_{\dot{\mathbb{R}}^d} \mathrm{tr}\big(\nabla u(x)\mathsf{R}(dx)\big) \quad \textit{for all } u \in \mathscr{C}^1_*(\mathbb{R}^d;\mathbb{R}^d),$$

$$\int_{\dot{\mathbb{R}}^d} \mathrm{tr}\big(\mathsf{R}(dx)\big) = -\int_{\mathbb{R}^d} \langle x, \mathsf{F}(dx)\rangle - \int_{\mathbb{R}^d} \mathrm{tr}\big(\mathsf{P}(dx)\big).$$

In fact, we can choose $\mathsf{R} \in \mathscr{M}(\mathbb{R}^d;\mathrm{Sym}_d(\mathbb{R},\geqslant))$.

Positive Cones

Let E a **normed vector space**. We call positive cone any subset $C \subset E$ with $C \neq E$ with the following properties:

$$C + C \subset C, \qquad \lambda C \subset C \quad \text{for all } \lambda > 0, \qquad C \cap (-C) = \{0\}.$$

The positive cone C induces a partial ordering $\geqslant$ on the space E by

$$y \geqslant x \quad \Longleftrightarrow \quad y - x \in C.$$

A linear map $F\colon L \longrightarrow \mathbb{R}$ on a **subspace** $L \subset E$ is called positive if

$$F(x) \geqslant 0 \quad \text{for all } x \in L \cap C.$$

A linear map $F\colon E \longrightarrow \mathbb{R}$ is called functional if it is continuous.

Proposition

Let E be a Banach space, partially ordered by a positive cone C. If some subspace $L \subset E$ contains an interior point of C, then every positive linear map $F_0 : L \longrightarrow \mathbb{R}$ can be ***extended*** *to a positive functional $F\colon E \longrightarrow \mathbb{R}$.*

Gradient Young Measures

Elementary Gradient Young Measures

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\boldsymbol{t} \in \mathrm{BV}(\Omega;\mathbb{R}^d)$.
Let B_d be the open unit ball in $\mathrm{Mat}_d(\mathbb{R})$ and ∂B_d its boundary.

We associate to the derivative $D\boldsymbol{t}$ a triple $\upsilon = (\nu, \sigma, \mu)$ with

$$\nu \in \mathscr{L}^\infty_{\mathrm{w}}\Big(\Omega; \mathscr{P}\big(\mathrm{Mat}_d(\mathbb{R})\big)\Big), \quad \sigma \in \mathscr{M}_+(\bar{\Omega}), \quad \mu \in \mathscr{L}^\infty_{\mathrm{w}}\big(\bar{\Omega}, \sigma; \mathscr{P}(\partial B_d)\big)$$

as follows: Consider the **Lebesgue-Radon-Nikodým decomposition**

$$D\boldsymbol{t} = \nabla \boldsymbol{t}\, \mathcal{L}^d + D^s\boldsymbol{t}, \quad D^s\boldsymbol{t} \perp \mathcal{L}^d,$$

and define $\nu_x := \delta_{\nabla \boldsymbol{t}(x)}$ for a.e. $x \in \Omega$ and $\sigma := |D^s\boldsymbol{t}|$. Let further

$$D^s\boldsymbol{t} = \frac{\mathrm{d}D^s\boldsymbol{t}}{\mathrm{d}|D^s\boldsymbol{t}|}\,|D^s\boldsymbol{t}|, \quad p := \frac{\mathrm{d}D^s\boldsymbol{t}}{\mathrm{d}|D^s\boldsymbol{t}|} \in \mathscr{L}^1(\Omega, |D^s\boldsymbol{t}|; \partial B_d).$$

be the polar decomposition of $D^s\boldsymbol{t}$ and define $\mu_x = \delta_{p(x)}$ for $|D^s\boldsymbol{t}|$-a.e. $x \in \Omega$. We call $\upsilon = (\nu, \sigma, \mu)$ an **elementary gradient Young measure**.

Weak*-Precompactness (1/2)

Consider a sequence of uniformly bounded maps $\boldsymbol{t}^k \in \mathrm{BV}(\Omega;\mathbb{R}^d)$. Extracting a subsequence, we may assume that

$$\begin{aligned} \boldsymbol{t}^k &\longrightarrow \boldsymbol{t} \quad \text{in } \mathscr{L}^1(\Omega;\mathbb{R}^d), \\ D\boldsymbol{t}^k &\rightharpoonup D\boldsymbol{t} \quad \text{weak* in } \mathscr{M}(\Omega;\mathrm{Mat}_d(\mathbb{R})), \end{aligned}$$

for some $\boldsymbol{t} \in \mathrm{BV}(\Omega;\mathbb{R}^d)$, i.e., $\boldsymbol{t}^k$ **converges weak*** to $\boldsymbol{t}$ in $\mathrm{BV}(\Omega;\mathbb{R}^d)$.

Let $\upsilon^k = (\nu^k, \sigma^k, \mu^k)$ be the elementary gradient Young measure of $D\boldsymbol{t}^k$. Since the spaces above are (contained in) dual spaces, there exists a subsequence and a **triple** $\upsilon = (\nu, \sigma, \mu)$ such that

$$\begin{aligned} [\![f, \upsilon^k]\!] &:= \int_\Omega [f(x,\cdot), \nu_x^k]\, dx + \int_{\bar{\Omega}} [f^\infty(x,\cdot), \mu_x^k]\, \sigma^k(dx) \\ &:= \int_\Omega \int_{\mathrm{Mat}_d(\mathbb{R})} f(x,M)\, \nu_x^k(dM)\, dx + \int_{\bar{\Omega}} \int_{\partial B_d} f^\infty(x,M)\, \mu_x^k(dM)\, \sigma^k(dx) \end{aligned}$$

converge to $[\![f, \upsilon]\!]$ (defined analogously) as $k \to \infty$ (...)

Weak*-Precompactness (2/2)

(...) for all test functions $f \in \mathscr{R}(\Omega; \mathrm{Mat}_d(\mathbb{R}))$ with

$$\mathscr{R}\big(\Omega; \mathrm{Mat}_d(\mathbb{R})\big) := \begin{cases} f \colon \bar{\Omega} \times \mathrm{Mat}_d(\mathbb{R}) \longrightarrow \mathbb{R} \ : \\ \quad \text{the map } f \text{ is a } \textbf{Carathéodory function} \text{ with} \\ \quad \text{linear growth at infinity, and there exists} \\ \quad f^\infty \in \mathscr{C}\big(\bar{\Omega} \times \mathrm{Mat}_d(\mathbb{R})\big) \end{cases}$$

We denote by f^∞ the recession function of f, defined as

$$f^\infty(x, M) := \lim_{\substack{x' \to x \\ M' \to M \\ t \to \infty}} \frac{f(x', tM')}{t} \quad \text{for a.e. } x \in \bar{\Omega} \text{ and all } M \in \mathrm{Mat}_d(\mathbb{R}).$$

The recession function is positively 1-homogeneous in M.

We call a triple $\upsilon = (\nu, \sigma, \mu)$ a **gradient Young measure** and denote the space of gradient Young measures by $\mathscr{G}(\Omega; \mathrm{Mat}_d(\mathbb{R}))$. Then

$$D\boldsymbol{t} = [\mathrm{id}, \nu]\,\mathcal{L}^d + [\mathrm{id}, \mu]\,\sigma,$$

by construction. Moreover, we have

$$\|\nabla \boldsymbol{t}^k\|\,\mathcal{L}^d + \left\|\frac{\mathrm{d}D^s\boldsymbol{t}^k}{\mathrm{d}|D^s\boldsymbol{t}^k|}\right\|\,|D^s\boldsymbol{t}^k| \longrightarrow [\|\cdot\|, \nu]\,\mathcal{L}^d + [\|\cdot\|, \mu]\,\sigma$$

weak* in $\mathscr{M}(\bar{\Omega})$ as $k \to \infty$, which implies that $[\|\cdot\|, \nu] \in \mathscr{L}^1(\Omega)$.

Monotone Gradient Young Measures

Proposition (Gradient Young Measures)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and suppose that $\mathbf{t}^k \rightharpoonup \mathbf{t}$ weak in $\mathrm{BV}(\Omega; \mathbb{R}^d)$ with $\mathbf{t}^k, \mathbf{t} \in \mathrm{BV}(\Omega; \mathbb{R}^d)$ monotone. We denote by υ^k the elementary gradient Young measure of $D\mathbf{t}^k$. There exists a subsequence and a* ***gradient Young measure***

$$\upsilon \in \mathscr{G}(\Omega; \mathrm{Mat}_d(\mathbb{R}))$$

such that $[\![f, \upsilon^k]\!] \longrightarrow [\![f, \upsilon]\!]$ for all $f \in \mathscr{R}_+(\Omega; \mathrm{Mat}_d(\mathbb{R}))$, where

$$\mathscr{R}_+(\Omega; \mathrm{Mat}_d(\mathbb{R})) := \begin{cases} f\colon \bar{\Omega} \times \mathrm{Mat}_d(\mathbb{R}) \longrightarrow \mathbb{R} \ : \\ \quad \text{the map } f \text{ is a Carathéodory function with} \\ \quad \text{linear growth at infinity, and there exists} \\ \quad f^\infty \in \mathscr{C}(\bar{\Omega} \times \mathrm{Mat}_d(\mathbb{R}, \geqslant)) \end{cases}$$

Internal Energy

An Unbounded, Convex Function on Matrices

Lemma

For any $\gamma > 1$, the map $h\colon \mathrm{Mat}_d(\mathbb{R}) \longrightarrow [0,\infty]$ defined by

$$h(M) := \begin{cases} \det(M^{\mathrm{sym}})^{1-\gamma} & \text{if } M \in \mathrm{Mat}_d(\mathbb{R}, >), \\ +\infty & \text{otherwise,} \end{cases}$$

*is **lower semicontinuous**, **proper**, and **convex**.*
For all $M \in \mathrm{Mat}_d(\mathbb{R})$, we have

$$h^\infty(M) := \lim_{t\to\infty} \frac{h(\mathbb{1} + tM) - h(\mathbb{1})}{t} = \begin{cases} 0 & \text{if } M \in \mathrm{Mat}_d(\mathbb{R}, \geqslant), \\ +\infty & \text{otherwise.} \end{cases}$$

inf-Convolution

Lemma

For $n \in \mathbb{N}$ and $M \in \mathrm{Mat}_d(\mathbb{R})$, we define the inf-convolution

$$h_n(M) := \inf_{B \in \mathrm{Mat}_d(\mathbb{R})} \Big\{ n\|M - B\| + h(B) \Big\}.$$

1. *The map h_n is **lower semicontinuous**, **proper**, and **convex**, and $h_n(M) \longrightarrow h(M)$ **monotonically** from below.*
2. *The map h_n is Lipschitz continuous with Lipschitz constant n and has linear growth at infinity:*

$$h_n(M) \leqslant 1 + n\sqrt{d} + n\|M\| \quad \text{for all } M \in \mathrm{Mat}_d(\mathbb{R}).$$

3. *For all $M \in \mathrm{Mat}_d(\mathbb{R})$, we have that*

$$h_n^\infty(M) := \lim_{t \to \infty} \frac{h_n(\mathbb{1} + tM) - h_n(\mathbb{1})}{t} = n\, \mathrm{dist}(M, \mathrm{Mat}_d(\mathbb{R}, \geqslant)).$$

Internal Energy Functional

Proposition

Let $\Omega \subset \mathbb{R}^d$ be open and convex, and h given above.
For $U \in \mathscr{L}^1(\Omega)$ non-negative and $\mathbf{t} \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ we define

$$\mathcal{U}[\mathbf{t}] := \begin{cases} \displaystyle\int_\Omega U(x) h\big(\nabla \mathbf{t}(x)\big)\, dx & \text{if } \mathbf{t} \text{ monotone,} \\ +\infty & \text{otherwise,} \end{cases}$$

using again the Lebesgue-Radon-Nikodým decomposition.

1. *The functional $\mathcal{U}$ is* ***convex****.*
2. *For $\mathbf{t}^k \rightharpoonup \mathbf{t}$ weak* in $\mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ with $\mathbf{t}^k, \mathbf{t} \in \mathrm{BV}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ monotone, there exists a subsequence such that*

$$\mathcal{U}[\mathbf{t}] \leqslant \liminf_{k \to \infty} \mathcal{U}[\mathbf{t}^k].$$

Lemma about Determinants

Lemma

*Let S be a real, **positive semidefinite, symmetric** $(d \times d)$-matrix. For any real **skew-symmetric** $(d \times d)$-matrix A we have*

$$\det(S + A) \geqslant \det S \geqslant 0.$$

For $\det(S) > 0$ there is a unique $R \in \mathrm{Sym}_d(\mathbb{R}, >)$ such that $R^2 = S$.
With $C := R^{-1}AR^{-1}$ **skew-symmetric**, we obtain the following identity:

$$\left(\frac{\det(S + A)}{\det(S)}\right)^{\gamma} = 1 + \gamma \int_0^1 \det(\mathbb{1} + tC)^{\gamma} \operatorname{tr}\big((\mathbb{1} + tC)^{-1} C\big)\, dt,$$

where the integral on the right-hand side is non-negative.

Compressible Euler Equations

Compressible Fluid Equations

Hyperbolic System of Conservation Laws

$$\left.\begin{aligned} \partial_t \varrho + \nabla \cdot (\varrho \boldsymbol{u}) &= 0 \\ \partial_t (\varrho \boldsymbol{u}) + \nabla \cdot (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla \pi &= 0 \\ \partial_t \varepsilon + \nabla \cdot \big((\varepsilon + \pi) \boldsymbol{u} \big) &= 0 \end{aligned} \right\} \quad \text{in } [0, \infty) \times \mathbb{R}^d$$

Physical Quantities

- Density $\varrho(t, \cdot) \in \mathscr{P}_2(\mathbb{R}^d)$
- Velocity $\boldsymbol{u}(t, \cdot) \in \mathscr{L}^2\big(\mathbb{R}^d, \varrho(t, \cdot)\big)$
- Total Energy $\varepsilon(t, \cdot) \in \mathscr{M}_+(\mathbb{R}^d)$

An **Equation of State** determines the pressure π in terms of $(\varrho, \boldsymbol{u}, \varepsilon)$.

Isentropic Euler Equations

The pressure is a function of the density only:

$$\left.\begin{aligned} \partial_t \varrho + \nabla \cdot (\varrho \boldsymbol{u}) = 0 \\ \partial_t(\varrho \boldsymbol{u}) + \nabla \cdot (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla P(\varrho) = 0 \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbb{R}^d \tag{2}$$

Total energy $\varepsilon = \frac{1}{2}\varrho|\boldsymbol{u}|^2 + U(\varrho)$ (with $P(\varrho) = U'(\varrho)\varrho - \mathcal{U}(\varrho)$).
Energy conservation follows formally from (2).

The flow can become discontinuous in finite time!

Entropy Condition (Mathematical Entropy)

$$\partial_t\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + U(\varrho)\right) + \nabla \cdot \left(\left(\frac{1}{2}\varrho|\boldsymbol{u}|^2 + U'(\varrho)\varrho\right)\boldsymbol{u}\right) \leqslant 0$$

$\Longrightarrow$ Dissipation of **Total Energy**

Initial value problem: for given initial data

$$(\varrho, \boldsymbol{u})(0, \cdot) = (\bar{\varrho}, \bar{\boldsymbol{u}}).$$

Global existence of **weak solutions** in one space dimension. Uniqueness is open.

Several space dimensions: numerical evidence that there is no convergence to one particular weak solution: New small-scale features appear upon every mesh refinement.

$\Longrightarrow$ Consider **measure-valued solutions** (statistics of solutions)?

Isentropic Euler Equations

On the analysis side, there is a massive nonuniqueness problem:
Infinitely many weak (“wild”) solutions for isentropic Euler.
Much worse than instability/chaotic behavior!
(De Lellis-Székelyhidi)

Strategy: linear pde plus pointwise constraint

1. construct subsolution (captures the macroscopic features)
2. successively superimpose waves that are highly oscillatory in the divergence-free component of the momentum
3. limit is an extreme point of the convex set of subsolutions.

The density is not changed in this construction.

The entropy inequality is insufficient to ensure uniqueness!
$\Longrightarrow$ What is the right entropy condition?

Dafermos' Entropy Rate Admissibility Condition

Among all weak solutions seek the one that dissipates the total energy (i.e., the mathematical entropy) at maximal rate.

$\boxed{\text{BUT}}$ This seems to pick the "wild" solutions!

Vortex-sheet initial data (quasi-1d)

- there exists a quasi-1d solution: Riemann problem
- there exist *infinitely many* **genuinely 2d** solutions...
- **that dissipate total energy faster than the quasi-1d solution**

This is not a problem of non-smooth initial data!
(Chiodaroli-De Lellis-Kreml)

$\Longrightarrow$ Avoid steepest descent for *total* energy?
(Anomalous dissipation: weak convergence effect)

Variational Time Discretization

$\boxed{\text{GOAL}}$ Find a variational time discretization in the spirit of minimizing movements for **curves of maximal slope**.

Timestep $\tau > 0$, discrete times $t^k = k\tau$ with $k \in \mathbb{N}_0$.

Approximate the solution at time t^k by $(\varrho^k, \boldsymbol{u}^k)$.

Update $(\varrho^{k+1}, \boldsymbol{u}^{k+1})$ is the minimizer of an optimization problem:

- **decrease some energy** as much as possible,
- while minimizing the **cost associated to the step size**.

Lagrangian formulation in terms of **transport maps**.

Maximize Entropy Production

For given timestep $\tau > 0$ and data $(\varrho, \boldsymbol{u})$ we minimize

$$\boxed{\text{Work + Internal Energy}}$$

- **Optimal** work and internal energy: $\mathscr{W}_\tau$ and $\mathcal{U}_\tau$
- **No work** (free transport): $\mathscr{W}_0 = 0$ and $\mathcal{U}_0$

From the inequality $\mathscr{W}_\tau + \mathcal{U}_\tau \leqslant \mathscr{W}_0 + \mathcal{U}_0$ we obtain

$$\mathscr{W}_\tau \leqslant \Delta\mathcal{U} := \mathcal{U}_0 - \mathcal{U}_\tau \quad \text{(“2nd law”).}$$

In fact, maximize the difference $\Delta\mathcal{U} - \mathscr{W}_\tau$.

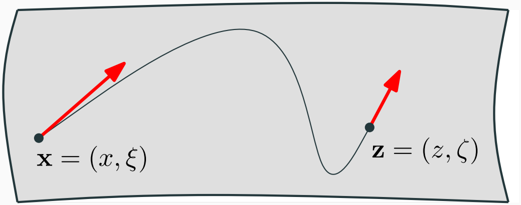
By *analogy* with classical thermodynamics: $\Delta\mathcal{U} - \mathscr{W}_\tau \sim \theta\Delta S$
$\implies$ maximize entropy production $\Delta S \geqslant 0$.

Curves of Minimal Acceleration

Among all smooth curves $X\colon [0,\tau] \longrightarrow \mathbb{R}^d$ such that

$$(X,\dot{X})(0) = (x,\xi) \quad \text{and} \quad (X,\dot{X})(\tau) = (z,\zeta),$$

find the one that minimizes the acceleration $\tau \int_0^\tau |\ddot{X}(t)|^2\,dt$.
$\Longrightarrow$ Uniquely determined cubic polynomials.



Minimal Acceleration (2/2)

Acceleration: For $\boldsymbol{x} = (x, \xi)$ and $\boldsymbol{z} = (z, \zeta)$ let

$$a(\boldsymbol{x}, \boldsymbol{z})^2 = \frac{3}{4\tau^2} |(x + \tau\xi) - z|^2 + \left| \zeta - \left(\xi - \frac{3}{2\tau} \Big((x + \tau\xi) - z \Big) \right) \right|^2.$$

Note that

- the first term measures how much the final position z differs from $x + \tau\xi$, which would be the position after a free transport;
- the second term measures the difference between ζ and

  $$w(\boldsymbol{x}, z) = \xi - \frac{3}{2\tau} \Big((x + \tau\xi) - z \Big),$$

  which is different from the **transport velocity**

  $$v(\boldsymbol{x}, z) = \frac{z - x}{\tau}$$

  (convex combination $v(\boldsymbol{x}, z) = \frac{2}{3} w(\boldsymbol{x}, z) + \frac{1}{3}\xi$).

Minimal Work

We express fluid states as measures on $\mathbb{R}^{2d}$:

$$\mathscr{P}_\varrho(\mathbb{R}^{2d}) = \left\{\boldsymbol{\gamma} \in \mathscr{P}_2(\mathbb{R}^{2d}) \colon \mathtt{x}\#\boldsymbol{\gamma} = \varrho\right\} \quad \text{with } \varrho \in \mathscr{P}_2(\mathbb{R}^d).$$

If $\boldsymbol{\mu} \in \mathscr{P}_\varrho(\mathbb{R}^{2d})$ with disintegration $\boldsymbol{\mu}(dx, d\xi) = \mu_x(d\xi)\, \varrho(dx)$, then μ_x describes the *distribution of velocities*.

Monokinetic states

$$\boldsymbol{\mu}(dx, d\xi) = \delta_{\boldsymbol{u}(x)}(d\xi)\, \varrho(dx) \quad \text{with } \boldsymbol{u} \in \mathscr{L}^2(\mathbb{R}^d, \varrho).$$

Definition

For $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathscr{P}_2(\mathbb{R}^{2d})$ define the minimal work as

$$\mathscr{W}(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)^2 := \inf\left\{\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} a(\mathbf{x}_1, \mathbf{x}_2)^2\, \gamma(d\mathbf{x}_1, d\mathbf{x}_2) \colon \mathrm{p}^i\#\gamma = \boldsymbol{\mu}^i\right\}.$$

Note that $\mathscr{W}(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ has the dimensions of an *energy*.

Internal Energy I

Definition

Let $U(r) := \kappa r^\gamma$ for all $r \geqslant 0$ (with constants $\kappa > 0$ and $\gamma > 1$).
For $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ we define

$$\mathcal{U}[\varrho] := \begin{cases} \int_{\mathbb{R}^d} U\big(r(z)\big)\, dz & \text{if } \varrho = r\mathcal{L}^d, \\ +\infty & \text{otherwise.} \end{cases}$$

For any $\boldsymbol{t} \in \mathscr{L}^2(\mathbb{R}^d, \varrho)$ smooth, injective we have
(by change of variables formula)

$$\mathcal{U}[\boldsymbol{t}\#\varrho] = \int_{\mathbb{R}^d} U\big(r(x)\big)\, \operatorname{det}\big(\nabla \boldsymbol{t}(x)\big)^{1-\gamma}\, dx.$$

Minimization Problem (1st try)

Internal energy does not depend on velocity.
$\Longrightarrow$ **Optimal velocity update**

$$\zeta = w(\boldsymbol{z}, z)$$

Simplified minimization problem (monokinetic state):

$$\min \left\{ \frac{3}{4\tau^2} \int_{\mathbb{R}^d} \varrho |\boldsymbol{t} - (\mathrm{id} + \tau \boldsymbol{u})|^2 + \int_{\mathbb{R}^d} U(\varrho) \det(\nabla \boldsymbol{t})^{1-\gamma} \right\}$$

over all $\boldsymbol{t} \in \mathscr{C}^1(\mathbb{R}^d) \cap \mathscr{L}^2(\mathbb{R}^d, \varrho)$ with $\det(\nabla \boldsymbol{t}) > 0$ ϱ-a.e.

Integrand is **unbounded, x-dependent, not coercive, not convex**.
$\Longrightarrow$ Need to modify the problem

Definition

Let $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ with $\varrho = r\,\mathcal{L}^d$ and $\mathcal{U}[\varrho] < +\infty$.
Then we define

$$\mathcal{U}[t|\varrho] := \begin{cases} \int_{\mathbb{R}^d} U(r(x)) \det\left(\nabla t(x)^{\mathrm{sym}}\right)^{1-\gamma} dx & \text{if } \nabla t \in \mathrm{Mat}_d(\mathbb{R}, >) \text{ a.e.,} \\ +\infty & \text{otherwise.} \end{cases}$$

Here $\mathrm{Mat}_d(\mathbb{R}, >)$ is the set of strictly positive definite matrices (not necessarily symmetric). In particular, we use the **deformation**

$$\nabla t(x)^{\mathrm{sym}} := \frac{\nabla t(x) + \nabla t(x)^{\mathrm{T}}}{2}.$$

Note that for $t =: \mathrm{id} + \tau v$, we have $\det(\nabla t) = 1 + \tau \mathrm{tr}(\nabla v) + \ldots$

Configuration Manifold

Matter is not allowed to interpenetrate (cf. pressureless case)
$\Longrightarrow$ The transport map $\boldsymbol{t}$ should be (essentially) **injective**.

For existence we need to find the right functional setting:

1. choose transport maps $\boldsymbol{t} \in \mathscr{L}^2(\mathbb{R}^d, \varrho)$ that are monotone,
2. and **stresses** $\sigma \in \mathscr{M}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ such that

$$\nabla \cdot \sigma = \varrho \boldsymbol{a} \quad \text{in } \mathscr{D}'(\mathbb{R}^d), \text{ for some } \boldsymbol{a} \in \mathscr{L}^2(\mathbb{R}^d, \varrho).$$

Definition

For any $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ we define

$$C_\varrho = \Big\{ \boldsymbol{t} \in \mathscr{L}^2(\mathbb{R}^d, \varrho) \colon \operatorname{spt}\big((\mathrm{id}, \boldsymbol{t})\#\varrho\big) \text{ is monotone} \Big\}.$$

Monotone Transport Maps

A subset $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ is called **monotone** if

$$(x_1 - x_2) \cdot (y_1 - y_2) \geqslant 0 \quad \text{for any pair of } (x_i, y_i) \in \Gamma.$$

Such a set is called **maximal monotone** if for any monotone set $\Gamma' \subset \mathbb{R}^d \times \mathbb{R}^d$ with $\Gamma \subset \Gamma'$ we have that $\Gamma = \Gamma'$.

Every monotone set has a maximal monotone extension.

Note that

- compositions of monotone maps are not monotone (only in 1d);
- in our case, transports will be perturbations of identity: the map $\mathrm{id} + \tau \boldsymbol{u}$ is monotone if $\|(D\boldsymbol{u})_-\| < 1/\tau$ (cf. CFL condition).

Extension of Monotone Maps

No assumptions on ϱ, so $\boldsymbol{t} \in C_\varrho$ may not be defined in all of $\mathbb{R}^d$.
We can associate to $\boldsymbol{t}$ a maximal monotone map with domain

$$\Omega = \operatorname{int} \operatorname{conv} \operatorname{spt} \varrho$$

If $\varrho \ll \mathcal{L}^d$ then we have $\varrho(\mathbb{R}^d \setminus \Omega) = 0$.

Properties of Maximally Monotone Maps defined on Ω
(see Alberti-Ambrosio):

- single-valued except for a codimension-one rectifiable set (which is negligible w.r.t. Lebesgue measure)
- bounded in $\mathscr{L}^\infty_{\mathrm{loc}}(\Omega)$;
- in $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ (total variation of the derivative can be controlled by the oscillation, which is finite).

Closed Convex Cone

Lemma

For every $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ such that $\varrho \ll \mathcal{L}^d$ and for all $\mathbf{t} \in C_\varrho$ there exists a sequence of Lipschitz continuous (or $\mathscr{C}^1(\mathbb{R}^d)$), monotone maps $\mathbf{t}_k$ defined on all of $\mathbb{R}^d$, such that

$$\lim_{k\to\infty} \|\mathbf{t}_k - \mathbf{t}\|_{\mathscr{L}^2(\mathbb{R}^d,\varrho)} = 0.$$

Lemma

For any $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$, the set C_ϱ of monotone maps in $\mathscr{L}^2(\mathbb{R}^d,\varrho)$ is a closed convex cone, which includes constants and rigid motions.

For all $f \in \mathscr{L}^2(\mathbb{R}^d,\varrho)$ there exist uniquely determined maps $\mathbf{t} \in C_\varrho$ and $\boldsymbol{a} \in C_\varrho^\perp$ (polar cone; see below) such that

$$f = \mathbf{t} \oplus \boldsymbol{a} \quad \text{(orthogonal decomposition)}.$$

Polar Cone

Lemma

For any $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ *we denote by*

$$C_\varrho^\perp := \left\{ \boldsymbol{a} \in \mathscr{L}^2(\mathbb{R}^d, \varrho) \colon \int_{\mathbb{R}^d} \varrho \boldsymbol{a} \cdot \boldsymbol{t} \leqslant 0 \text{ for all } \boldsymbol{t} \in C_\varrho \right\}$$

the polar cone of the cone C_ϱ *of monotone maps in* $\mathscr{L}^2(\mathbb{R}^d, \varrho)$*. For all* $\boldsymbol{a} \in C_\varrho^\perp$ *there exists* $\omega \in \mathscr{M}(\mathbb{R}^d; \mathrm{Sym}_d(\mathbb{R}, \geqslant))$ *such that*

$$\boxed{\nabla \cdot \omega = \varrho \boldsymbol{a} \text{ in } \mathscr{D}'(\mathbb{R}^d),} \qquad \int_{\mathbb{R}^d} \mathrm{tr}(\omega) = -\int_{\mathbb{R}^d} \varrho \boldsymbol{a} \cdot \mathrm{id}.$$

For any $\varrho \boldsymbol{a} = \nabla \cdot \omega$ and $\boldsymbol{t} \in C_\varrho$ we can **“integrate by parts”**: there exists a nonnegative measure $\mathrm{tr}\big(\omega(D\boldsymbol{t})\big)$ such that

$$0 \leqslant \int_{\beta\mathbb{R}^d} \mathrm{tr}\big(\omega(D\boldsymbol{t})\big) = -\int_{\mathbb{R}^d} \varrho \boldsymbol{a} \cdot \boldsymbol{t}.$$

Properties of Internal Energy

Lemma

Let ϱ and $\mathbf{t} \mapsto \mathcal{U}[\mathbf{t}|\varrho]$ be given as above. Then the internal energy is convex and lower semicontinuous with respect to the weak convergence of monotone maps in $\mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, with*

$$\Omega := \operatorname{int}\operatorname{conv}\operatorname{spt}\varrho.$$

Use Jensen inequality with gradient Young measures generated by sequences in $\mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ (see Kristensen-Rindler), and a standard regularization of g using the $\inf$-convolution; see Rockafellar.

Note: *Monotonicity* substitutes for coercivity

$$\text{bounded in } \mathscr{L}^2(\mathbb{R}^d,\varrho) \quad\Longrightarrow\quad \text{bounded in } \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$$

Control of Internal Energy after Transport

Lemma

Suppose that $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ is given with $\varrho =: r\mathcal{L}^d$ and $\mathcal{U}[\varrho] < \infty$. For any $\mathbf{t} \in C_\varrho$ with $\mathcal{U}[\mathbf{t}|\varrho] < \infty$ there exists a Borel set $\Sigma \subset \mathbb{R}^d$ with $\varrho(\Sigma) = 0$ and $\mathbf{t}|_{\mathbb{R}^d \setminus \Sigma}$ injective. Then

$$\mathcal{U}[\mathbf{t}\#\varrho] \leqslant \mathcal{U}[\mathbf{t}|\varrho].$$

Let $R(x) \in \mathrm{Sym}_d(\mathbb{R}, >)$ such that $R(x)^2 = \nabla \mathbf{t}(x)^{\mathrm{sym}}$ for ϱ-a.e. $x \in \mathbb{R}^d$. Defining $C(x) := R(x)^{-1}\nabla \mathbf{t}(x)^{\mathrm{anti}} R(x)^{-1}$, we have

$$\begin{aligned} &\mathcal{U}[\mathbf{t}\#\varrho] - \mathcal{U}[\mathbf{t}|\varrho] \\ &\quad = -\int_{\mathbb{R}^d} P(r, S) \left(\det(\nabla \mathbf{t}^{\mathrm{sym}})^{1-\gamma} \int_0^1 \det(\mathbb{1} + tC)^{1-\gamma} \mathsf{T}(t, C)\, dt \right) dx, \end{aligned}$$

$$\mathsf{T}(t, C) := \mathrm{tr}\big((\mathbb{1} + tC)^{-1} C\big) \quad \text{for all } t \geqslant 0.$$

The difference vanishes **if and only if** $\nabla \mathbf{t}(x)^{\mathrm{anti}} = 0$ for ϱ-a.e. $x \in \mathbb{R}$.

Existence of Minimizers

Proposition

Consider density $\varrho \in \mathscr{P}_2(\mathbb{R}^d)$ *and velocity* $\boldsymbol{u} \in \mathscr{L}^2(\mathbb{R}^d, \varrho)$. *Assume that* $\varrho =: r\mathcal{L}^d$ *and* $\mathcal{U}[\varrho] < \infty$. *Given any timestep* $\tau > 0$, *there **exists** a **unique*** $\boldsymbol{t}_\tau \in C_\varrho$ *that minimizes the functional*

$$\Psi_\tau[\boldsymbol{t}|\varrho, \boldsymbol{u}] := \frac{3}{4\tau^2} \int_{\mathbb{R}^d} \left|\big(x + \tau \boldsymbol{u}(x)\big) - \boldsymbol{t}(x)\right|^2 \varrho(dx) + \mathcal{U}[\boldsymbol{t}|\varrho].$$

This minimum is finite. For all Borel maps $\boldsymbol{v}\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *with the property that* $\boldsymbol{t}_\tau + \varepsilon \boldsymbol{v} \in C_\varrho$ *for some* $\varepsilon > 0$, *we have the following inequality: Let* $P(r) := U'(r)r - U(r)$ *for* $r \geqslant 0$. *Then*

$$\begin{aligned} &- \frac{3}{2\tau^2} \int_{\mathbb{R}^d} \big\langle \big(x + \tau \boldsymbol{u}(x)\big) - \boldsymbol{t}_\tau(x), \boldsymbol{v}(x) \big\rangle \, \varrho(dx) \\ &\quad - \int_{\mathbb{R}^d} P\big(r(x)\big) \, \det\big(\nabla \boldsymbol{t}_\tau(x)^{\mathrm{sym}}\big)^{1-\gamma} \mathrm{tr}\Big(\big(\nabla \boldsymbol{t}_\tau(x)^{\mathrm{sym}}\big)^{-1} \nabla \boldsymbol{v}(x)\Big) \, dx \geqslant 0. \end{aligned}$$

Stress Tensor

Proposition

There exists $\mathsf{R}_\tau \in \mathscr{M}(\mathbb{R}^d; \mathrm{Sym}_d(\mathbb{R}, \geqslant))$ *with*

$$\int_{\mathbb{R}^d} \mathrm{tr}\big(\nabla u(x) \mathsf{R}_\tau(dx)\big) = -\frac{3}{2\tau^2} \int_{\mathbb{R}^{2d}} \big\langle \big(x + \tau \boldsymbol{u}(x)\big) - \boldsymbol{t}_\tau(x), u(x) \big\rangle \, \varrho(dx) - \int_{\mathbb{R}^d} P\big(r(x)\big) \ \det \left(\nabla \boldsymbol{t}_\tau(x)^{\mathrm{sym}}\right)^{1-\gamma} \mathrm{tr}\Big(\left(\nabla \boldsymbol{t}_\tau(x)^{\mathrm{sym}}\right)^{-1} \nabla u(x)\Big) \, dx$$

for all $u \in \mathscr{C}^1_*(\mathbb{R}^d; \mathbb{R}^d)$*. In particular, we have the* ***size control***

$$\int_{\mathbb{R}^d} \mathrm{tr}\big(\mathsf{R}_\tau(dx)\big) = -\frac{3}{2\tau^2} \int_{\mathbb{R}^{2d}} \big\langle \big(x + \tau \boldsymbol{u}(x)\big) - \boldsymbol{t}_\tau(x), x \big\rangle \, \varrho(dx) - \int_{\mathbb{R}^d} P\big(r(x)\big) \ \det \left(\nabla \boldsymbol{t}_\tau(x)^{\mathrm{sym}}\right)^{1-\gamma} \mathrm{tr}\Big(\left(\nabla \boldsymbol{t}_\tau(x)^{\mathrm{sym}}\right)^{-1}\Big) \, dx.$$

Energy Balance

Proposition

With total energy $\mathcal{E}[\varrho, \mathbf{u}] := \int_{\mathbb{R}^d} \frac{1}{2}\varrho|\mathbf{u}|^2 + \mathcal{U}[\varrho]$*, we have*

$$\mathcal{E}[\varrho_\tau, \mathbf{u}_\tau] + \int_{\mathbb{R}^d} \frac{1}{6}\varrho|\mathbf{w}_\tau - \mathbf{u}|^2 + \int_{\mathbb{R}^d} \Big(P(r,S)\, \mathsf{D}^2\big(\nabla \mathbf{t}_\tau - \mathbb{1}\big)\Big)\, dx + \mathrm{tr}\big(\mathsf{R}_\tau(dx)\big)\Big) = \mathcal{E}[\varrho, \mathbf{u}].$$

For all matrices $\mathbb{1} + S \in \mathrm{Sym}_d(\mathbb{R}, >)$ *and* $A \in \mathrm{Skew}_d(\mathbb{R})$ *we have*

$$\mathsf{D}^2(S + A) := \int_0^1 \det(\mathbb{1} + tS)^{1-\gamma}\Big((\gamma - 1)\mathsf{T}(t,S)^2 + \mathsf{T}_2(t,S)\Big)t\, dt + \det(\mathbb{1} + S)^{1-\gamma}\int_0^1 \det(\mathbb{1} + tC)^{1-\gamma}\mathsf{T}(t,C)\, dt \geqslant 0,$$

$$\mathsf{T}_2(t,S) := \mathrm{tr}\Big(\big((\mathbb{1} + tS)^{-1}S\big)^2\Big) \quad \textit{for all } t \geqslant 0.$$

Here $C := R^{-1}AR^{-1}$ *and* $R \in \mathrm{Sym}_d(\mathbb{R}, >)$ *such that* $\mathbb{1} + S =: R^2$.

Approximate Solutions

The transport maps are **interpolated linearly** in time, the transport velocities are **piecewise constant**.

We patch transports together to obtain a global transport map

$$X_{\tau,t} = \boldsymbol{t}_{\tau,t} \circ \boldsymbol{t}_\tau^k \circ \cdots \circ \boldsymbol{t}_\tau^1, \quad \Xi_{\tau,t} = \boldsymbol{w}_\tau^{k+1} \circ \boldsymbol{t}_\tau^k \circ \cdots \circ \boldsymbol{t}_\tau^1$$

where $k = \lfloor t/\tau \rfloor$ (the largest integer not bigger than t/τ).

Approximate solutions have **total energy uniformly bounded**.

Lemma

Consider approximate solutions $(\varrho_\tau, \boldsymbol{u}_\tau)$ as above. Then

$$\sup_{\tau>0} \|\varrho_\tau\|_{\mathrm{Lip}([0,\infty),\mathscr{P}_2(\mathbb{R}^d))} \leqslant (2\bar{\mathcal{E}})^{1/2},$$

$$\left(\int_{\mathbb{R}^d} |x|^2 \varrho_{\tau,t}(dx) \right)^{1/2} \leqslant \left(\int_{\mathbb{R}^d} |x|^2 \bar{\varrho}(dx) \right)^{1/2} + t(2\bar{\mathcal{E}})^{1/2}$$

for all $\tau > 0$ and $t \in [0,\infty)$. Here $\bar{\mathcal{E}}$ is the initial total energy.

The Space $\mathrm{BL}_1(\mathbb{R}^d; \mathbb{R}^N)$

Definition

We denote by $\mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^N)$ the vector space of Lipschitz continuous maps $\zeta \colon \mathbb{R}^d \longrightarrow \mathbb{R}^N$. The Lipschitz constant of $\zeta \in \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^N)$ is

$$\|\zeta\|_{\mathrm{Lip}(\mathbb{R}^d)} := \sup_{x_1 \neq x_2} \frac{|\zeta(x_1) - \zeta(x_2)|}{|x_1 - x_2|}.$$

We denote by $\mathrm{BL}(\mathbb{R}^d; \mathbb{R}^N)$ the subspace of **bounded** functions in $\mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^N)$. It is a Banach space when equipped with the norm

$$\|\zeta\|_{\mathrm{BL}(\mathbb{R}^d)} := \max\left\{\|\zeta\|_{\mathscr{L}^\infty(\mathbb{R}^d)}, \|\zeta\|_{\mathrm{Lip}(\mathbb{R}^d)}\right\}.$$

Let $\mathrm{BL}_1(\mathbb{R}^d; \mathbb{R}^N)$ be the space of $\zeta \in \mathrm{BL}(\mathbb{R}^d; \mathbb{R}^N)$ with $\|\zeta\|_{\mathrm{BL}(\mathbb{R}^d)} \leqslant 1$.

We denote by $\mathscr{M}_{\mathrm{K}}(\mathbb{R}^d;\mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued Borel measures $\boldsymbol{m}$ with **zero mean** and **finite first moment**, equipped with norm

$$\|\boldsymbol{m}\|_{\mathscr{M}_{\mathrm{K}}(\mathbb{R}^d)} := \sup\bigg\{ \int_{\mathbb{R}^d} \zeta(x)\cdot \boldsymbol{m}(dx) \colon \zeta \in \mathrm{BL}_1(\mathbb{R}^d;\mathbb{R}^N) \bigg\}.$$

The Monge-Kantorovich norm is bounded by the total variation.

Lemma

Consider approximate solutions $(\varrho_\tau, \boldsymbol{u}_\tau)$ as above. Then

$$\sup_{\tau>0} \|\boldsymbol{m}_\tau\|_{\mathrm{Lip}([0,T];\mathscr{M}_{\mathrm{K}}(\mathbb{R}^d;\mathbb{R}^d))} \leqslant C\bar{\mathcal{E}}$$

*for all $t \in [0,\infty)$. Here $\boldsymbol{m}_\tau$ is the **approximate momentum**.*

Global Existence (1/2)

Theorem

*Suppose that initial data $\bar{\varrho} \in \mathscr{P}_2(\mathbb{R}^d)$ and $\bar{\mathbf{v}} \in \mathscr{L}^2(\mathbb{R}^d, \bar{\varrho})$ is given with **vanishing total momentum** and **finite internal energy**.*
For any $T > 0$ there exist curves

$$\varrho \in \mathrm{Lip}\big([0,T]; \mathscr{P}_2(\mathbb{R}^d)\big), \quad \mathbf{m} \in \mathrm{Lip}\big([0,T]; \mathscr{M}_{\mathrm{K}}(\mathbb{R}^d; \mathbb{R}^d)\big)$$

with the following properties:

1. *The initial data is attained:*

$$\varrho(0,\cdot) = \bar{\varrho}, \quad \mathbf{m}(0,\cdot) = \bar{\varrho}\bar{\mathbf{v}}.$$

2. *We have $\mathbf{m} =: \varrho\mathbf{v}$ with*

$$\mathbf{v}(t,\cdot) \in \mathscr{L}^2\big(\mathbb{R}^d, \varrho(t,\cdot)\big), \quad \text{for all } t \in [0,T].$$

Global Existence (2/2)

Theorem (cont.)

3. *There exist two* ***Young measures***

$$\nu^1, \nu^2 \in \mathscr{L}^\infty_w([0,T]; \mathscr{M}_+(\dot{\mathbb{R}}^d \times \mathfrak{X})),$$

where $\mathfrak{X}$ is a suitable compactification of the set

$$X := [0,\infty) \times \mathbb{R}^d$$

of admissible $(\varrho, \mathbf{v})$, such that

$$\left.\begin{aligned} \partial_t \varrho + \nabla \cdot [\varrho \mathbf{v}] &= 0 \\ \partial_t(\varrho \mathbf{v}) + \nabla \cdot [\varrho \mathbf{v} \otimes \mathbf{v}] + \nabla [\![P(\varrho)]\!] &= 0 \end{aligned}\right\} \quad \textit{distributionally.}$$

Here the brackets $[\cdot]$ and $[\![\cdot]\!]$ denote the integration of ν^1 and ν^2, respectively, against suitable functions of $(\varrho, \mathbf{v})$.

Final Remarks

The stress tensor $\mathbb{R}_\tau$ (Lagrange multiplier of monotonicity contraint) **does not appear** in the momentum equation. The term $[\![P(\varrho)]\!]$ is only determined by the Young measure for the pressure.

The measure-valued solution satisfies a local **energy inequality**.

Weak-Strong-Uniqueness: Using the **relative entropy method** one can show that the measure-valued solution coincides with the unique strong solution as long as the latter exists.

A similar time discretization works for the **full Euler case** and the **pressureless gas dynamics case**.

Thank You