Uniform boundary stabilization of semi-linear wave eqts with non-linear boundary damping

(after the paper by the same title of I. Lasiecka & D. Tataru, Diff & Int. Eqts Vol 6, #3, May 93, 507–533)

1. Setting, assumptions, statements

Let Ω be a bounded region in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 & Γ_1 closed and disjoint; $\nu =$ unit outer normal to the boundary Γ, $\Gamma_0 \neq \emptyset$. We consider the semilinear problem

$$
\begin{cases}
y_{tt} = \Delta y - f_0(y) & \text{in } Q = (0,\infty)\times\Omega \qquad (1.1a)\\
y(0,\cdot) = y_0 \in H^1_{\Gamma_0}(\Omega) \equiv \{ h \in H^1(\Omega) : h|_{\Gamma_0} = 0 \} \text{ in } \Omega & (1.1b)\\
y_t(0,\cdot) = y_1 \in L_2(\Omega) & \\
y|_{\Sigma_0} \equiv 0, \quad \left.\dfrac{\partial y}{\partial \nu}\right|_{\Sigma_1} = -g\left(y_t|_{\Gamma_1}\right) & \text{on } \Sigma_i = (0,\infty)\times\Gamma_i \qquad (1.1c)
\end{cases}
$$

Remark 1.1 Actually I. L + D. T.'s paper consider the more general situation with

$$\left.\frac{\partial y}{\partial \nu}\right|_{\Sigma_1} = -g\left(y_t|_{\Gamma_1}\right) - f_1\left(y|_{\Gamma_1}\right) \quad \text{in } \Sigma_1 \qquad (1.1c')$$

under suitable assumptions on f_1 (f_1 is NOT assumed monotone, however). One could also have $\Gamma_0 = \emptyset$, but then $f_0(s)\, s \geq \varepsilon s^2$ is needed, compare with (H-2)(ii) below □

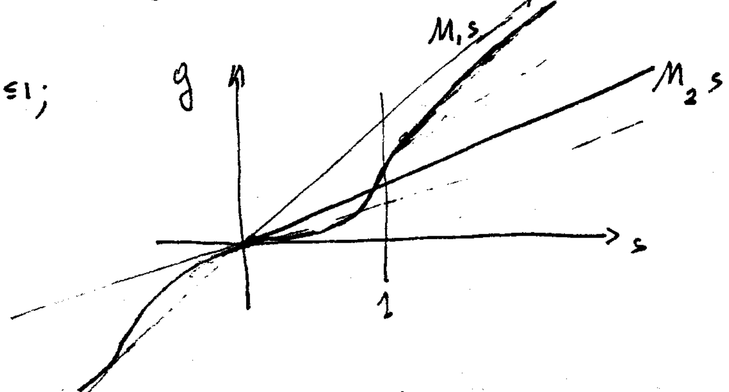
Assumptions on the non-linear terms f_0 and g:

(H-1) (i) $g(s)$ is a continuous, monotone increasing function on $\mathbb{R}$;

(ii) $g(s)s > 0$ for $s \neq 0$

(iii) $M_2 s^2 \leq g(s)s \leq M_1 s^2$, for $|s| \geq 1$, $0 < M_2 < M_1$;

e.g. $g(s) = s^3, s^5, |s| \leq 1$;



Remark 1.2 No growth assumptions on g near the origin □

(H-2) (i) $f_0(s)$ is a $W^{1,\infty}_{loc}(\mathbb{R})$, piecewise $C^1(\mathbb{R})$-function, differentiable at $s=0$;

(ii) $f_0(s)s \geq 0$, $s \in \mathbb{R}$ (so that $f_0(0)=0$ by (i), (ii));

(iii) $|f_0'(s)| \leq N(1+|s|^{k_0-1}$, $1 < k_0 < \frac{n}{n-2}$, $|s| > N =$ large, $n \geq 2$

[Can allow $k_0 = \frac{n}{n-2}$ as in Lasiecka + Eller + Chueshov

Main results. Theorem 1.1 Assume (H.1) – (H.3). Then, for each $\{y_0, y_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, problem (1.1) has a unique solution $\{y, y_t\}$:

$$y \in C_{loc}(0,\infty; H^1_{\Gamma_0}(\Omega)) \cap C^1_{loc}(0,\infty; L_2(\Omega)) \qquad (1.2)$$

such that, moreover

$$y_t\big|_{\Sigma_1} \in L_{2,loc}(0,\infty; L_2(\Gamma_1)); \quad \frac{\partial y}{\partial \nu}\Big|_{\Sigma_1} \in L_{2,loc}(0,\infty; L_2(\Gamma_1)) \quad \square \tag{1.3}$$

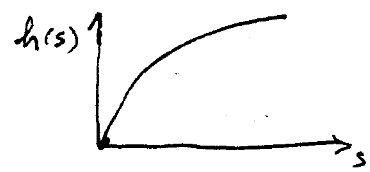
<u>Remark</u> 1.3 Under B.C. (1.1c') with $f_1 \neq 0$, uniqueness is not guaranteed, but existence of at least one solution is asserted in I.L. + D.T.'s paper, under suitable general assumptions on f_1 □

<u>Definition of</u> $h(s)$:

Before stating the corresponding stabilization result, we let $h(s)$ be a real-valued function for $s \geq 0$, concave, strictly increasing, with $h(0) = 0$ and satisfying

$$h(\underbrace{s\,g(s)}_{+}) \geq s^2 + g^2(s) \quad \text{for } |s| \leq N, \text{ for some } N > 0 \tag{1.4}$$

Such function $h(s)$ can always be constructed by virtue of hypoth. H.1



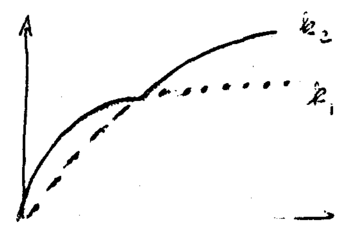
<u>Construction</u> Define the increasing functions k_1 and k_2 on $\mathbb{R}$ s.t.

$$k_1(s\,g(s)) \geq s^2 + g^2(s), \quad s \geq 0; \qquad k_2(s\,g(s)) \geq s^2 + g^2(s), \quad s \leq 0$$

$\Rightarrow$ Then take

$$h = \text{conc}\{\max(k_1, k_2)\}$$

= concave envelope



Notation needed for the stabilization result. Having the required function $h(s)$, we let

$$\tilde{h}(x) = h\left[\frac{x}{\text{meas}\,\Sigma_1}\right], \quad x \geq 0, \text{ is } h \text{ rescaled}; \tag{1.5}$$

$=$ monotone increasing

Let $c \geq 0$. Then $[c+\tilde{h}]$ is invertible. Define

$$p(x) = \left[cI + \tilde{h}\right]^{-1}(Kx), \quad K = \text{posit const.} \tag{1.6}$$

$=$ positive, continuous, strictly increasing with $p(0) = 0$.

Finally, let

$$q(x) = x - (I+p)^{-1}(x) = p(I+p)^{-1}(x), \quad \checkmark \; x>0 \tag{1.7}$$

$=$ positive ~~increasing~~, $q(0)=0$

[$q(x) + p(q(x)) = \cancel{x} + p(x) - \cancel{x} \xrightarrow{\text{diff}}$ $q'(x) = \dfrac{p'(x)}{1+p'(q(x))} > 0$ since $p(\cdot)$ increasing

Also $\underbrace{[I+p]}_{\text{invert.}}q(0) = p(0) = 0 \Rightarrow q(0) = 0$]

Define now the Energy of the solution $\{y, y_t\}$:

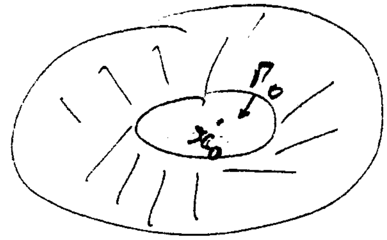
$$E(t) \equiv \frac{1}{2}\int_\Omega \left[|\nabla y(t)|^2 d\Omega + y_t^2(t)\right] d\Omega + \int_\Omega F_0(y)\, d\Omega \tag{1.8}$$

$$F_0(s) \equiv \int_0^s f_0(t)\, dt$$

We finally make a geometric assumption on the uncontrolled portion of the boundary Γ_0:

(H.3) $\exists\ x^0 \in \mathbb{R}^n$ s.t

$$(x - x_0) \cdot \nu(x) \leq 0 \text{ on } \Gamma_0 \qquad (1.9)$$



Theorem 1.2 (Stabilization) Assume (H.1) – (H.3). With reference to the solution $\{y, y_t\}$ of problem (1.1) which is guaranteed by Theorem 1.1, we have that for some $T_0 > 0$,

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right)(E(0)) \quad \text{for } t > T_0 \qquad (1.10)$$

$\to 0$ as $t \to \infty$

where the scalar function $S(t)$ is the solution of the following ODE: (non-linear contraction)

$$\frac{d}{dt} S(t) + q(S(t)) = 0 \qquad S(0) = E(0) \qquad (1.11)$$

where q is defined in (1.7) with K depending in general on $E(0)$ and $c = \frac{1}{\text{meas}\, \Sigma_1}(M_1 + M_2^{-1})$. Thus

$$E(t) \to 0 \qquad \text{as } t \to \infty \qquad (1.12)$$

with rates specified in the next corollary.

Corollary 1.3 In addition to hypotheses (H.1) – (H.3), we assume that for some positive constants a, b, we have

$$\begin{cases} g(s)s \le b s^2 & \text{for each real } s \\ g(s)s \ge a|s|^{r+1} & \text{for some } r \ge 1, \text{ for } |s| \le 1; \end{cases} \qquad (1.13)$$

$\Rightarrow$ Then

$$E(t) \le C e^{-\alpha t} \qquad \text{if } r = 1 \qquad (1.14)$$

$$E(t) \le C t^{\frac{2}{1-r}} \qquad \text{if } r > 1 \qquad (1.15)$$

where both constants $C > 0$ and $\alpha > 0$ depend generally on $E(0)$. $\square$

Proof It suffices to construct the required concave function $h(s)$ possessing property (1.4) with, say, $N = 1$. We'll indicate the procedure by taking $a = 1$. Take actually

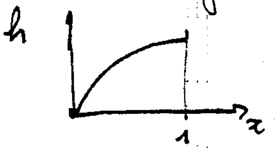
$$g(s) = s^r, \quad 0 \le s \le 1, \quad r \ge 1. \qquad (1.16)$$

We shall construct the functions $h \to p \to q$, as required by the algorithm leading to the statement of Theorem 1.2

Construction of $h(x)$. For $0 \le s \le 1$ the dominant term in $s^2 + g^2(s) = s^2 + s^{2r}$ is "s^2", since $r \ge 1$. By (1.4) we want to take $h(\cdot)$ s.t. $h(s g(s))$ is equal to $[s^2 + s^{2r}] \sim s^2$, so we take

$$h(s g(s)) = s^2 \qquad 0 \le s \le 1$$

Setting $x = s\,g(s) = s^{r+1}$, we have $h(x) = s^2 = x^{\frac{2}{r+1}}$, or



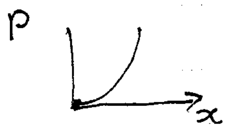
$$h(x) = x^m, \quad m = \frac{2}{r+1} \lesseqgtr 1 \text{ according to } r \gtreqless 1 \tag{1.17}$$

$= \underline{\text{concave}}$!

$\underline{\text{Construction of}}$ $p(x)$. We choose $c=1$, mean $\Sigma_1 = 1$, $K=1$, so $\tilde{h} = h$.
By (1.6), p is defined by

$$p(x) = [I + h]^{-1}(x) \quad \text{or} \quad p(x) + h(p(x)) = x, \text{ or by (1.17)}$$

$p(x) + p^m(x) = x$. For $x > 0$: small, since $p(0) = 0$, the dominant term is $p^m(x) \sim x$ or



$$p(x) \sim x^{1/m} = x^{\frac{r+1}{2}}, \quad 0 < x = \text{small} \tag{1.18}$$

increasing.

$\underline{\text{Construction of}}$ $q(x)$. By (1.7), we define $q(x) = p(I+p)^{-1}(x)$, $x > 0$. For x small, $p(\,)$ small and

$$q(x) \sim p(x) \sim x^{\frac{1}{m}} = x^{\frac{r+1}{2}}, \quad 0 < x = \text{small} \tag{1.19}$$

$\underline{\text{Solution}}$ $s(t)$ of ODE (1.11). We solve the ODE for $m < 1$:

$$s_t + q(s) = 0 \quad \text{or} \quad s_t + s^{\frac{1}{m}} = 0 \quad \text{or} \quad \frac{ds}{s^{1/m}} = -dt$$

$$\to \frac{s^{1-\frac{1}{m}}}{1-\frac{1}{m}} = -t + c; \to \boxed{\frac{s_0^{1-\frac{1}{m}}}{1-\frac{1}{m}} = c} \quad \text{hence} \to$$

$$s(t)^{\frac{m-1}{m}} = \frac{m-1}{m}\left[-t + \frac{s_0^{\frac{m-1}{m}}}{\frac{m-1}{m}}\right] = \frac{1-m}{m}\,t + s_0^{\frac{m-1}{m}}, \quad m < 1$$

$$S(t) = \left[\frac{1-m}{m} t + s_0^{\frac{m-1}{m}}\right]^{\frac{m}{m-1}} \qquad 0 < m < 1 \quad ~~\text{dc}~~$$

$$S(t) = \left[\frac{1}{t\left(\frac{1-m}{m}\right) + s_0^{\frac{m-1}{m}}}\right]^{\frac{m}{1-m}}, \qquad 0 < m < 1, \quad \text{or } r > 1$$

$$\frac{m}{1-m} = \frac{\frac{2}{r+1}}{1 - \frac{2}{r+1}} = \frac{2}{r-1}, \qquad r > 1 \quad \text{or}$$

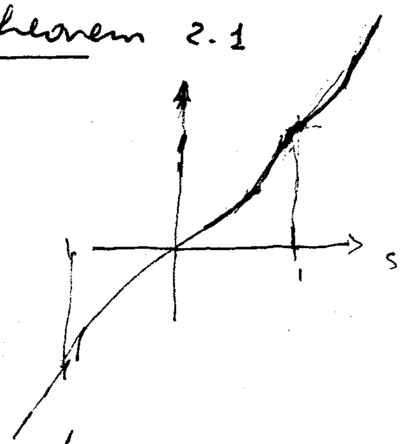
$$\boxed{S(t) = \left(\frac{1}{t\left(\frac{r-1}{2}\right) + s_0^{\frac{1-r}{2}}}\right)^{\frac{2}{r-1}}, \quad t \geq 0 \qquad r > 1 \qquad s_0 = E(0)} \tag{1.20}$$

Linear case $r=1$: $g(s) = s$. Then $m = 1$. The corresponding O.DE is $s_t + s = 0$ or

$$S(t) = E(0)\, e^{-t} \qquad (\text{expon. decay}) \tag{1.21}$$

recovering prior stabilization result in the linear case referenced in I.L + B.T paper (though many of them required the additional geometric condition $(x - x_0)\cdot \nu(x) \geq 0$ on Γ_1, which instead is redundant, see below at p. why$\square$

Example of $g(\cdot)$, $f_0(\cdot)$ covered by Theorem 2.1



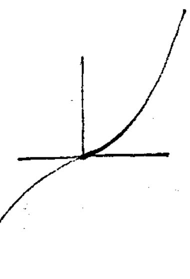
$$g(s) = \begin{cases} s^3 & 0 \le s \le 1 \\ \dfrac{1+s^2}{a s}, & s \ge 1 \quad (\text{increasing}) \end{cases}, \quad a = 2$$

symmetric $g(-s) = g(s)$, g continuous at $s = 1$ for $a = 2$.

For $s \ge 1 \rightarrow$ $g'(s) = \dfrac{(s^2-1)}{a s^2} > 0$ for $s > 1$.

Typical $f_0(s)$:

$f_0(s) = s^{(\text{odd power})}$ for $n = \dim \Omega = 2$



$f_0(s) = s^3$ for $n = 3$ $\quad k_0 = 2 = \dfrac{n}{n-1} = 2$ OK

We noted below (H.2)(iii) that the paper of I. Chueshov + M. Eller + I. Lasiecka allows $k_0 = \dfrac{n}{n-2}$.

2. Sketch of proof of Stabilization Theorem 1.2.: general strategy.

The following general strategy may be given to prove Thm 1.2
It has been applied to numerous other cases

It is based on two preliminary steps, where $E(t)$ is the "energy" of the system's solution

Step 1 Establish the "dissipative" identity

$$E(t) + \int_s^t \underbrace{\mathcal{D}(z)}\, dz = E(s) \qquad 0 \le s \le t \tag{2.1}$$

positive
dissipative term; in our case

$$\mathcal{D}(z) = \int_{\Gamma_1} \underbrace{g(y_t)\, y_t}_{+} \, d\Gamma_1 > 0 \tag{2.1'}$$

In I.L. + D.T.'s paper, this is Eq (3.6) p 523. Thus says:
$E(t) \le E(s)$, $t \ge s$, ie dissipativity.

Remark 2.1 Identity (2.1') can be ready obtained – formally by multiplying problem (1.1) by y_t and integrating by parts (Green theorem in space). This procedure is only formal, as the solution does not have the regularity properties required by the (formal) computations, on the basis of Thm 1.1.

In our present case (1.1) [with $f_1 \equiv 0$; the presence of f_1 greatly complicates the issue, see I.L. + D.T.'s paper], one starts with smooth initial data, say, $\{y_{0n}, y_{1n}\} \in \mathcal{D}(A)$, A = generator of the non-linear semigroup describing the dynamics, such that: $\{y_{0n}, y_{n1}\} \to \{y_0, y_1\} \in H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$. The corresponding solution $\{y_n, y_{nt}\}$ satisfies

$$\{y_n, y_{nt}\} \in L_\infty(0,T; H^2(\Omega) \times H^1(\Omega)) \to \frac{\partial u}{\partial \nu} \in L_\infty(0,T; H^{1/2}(\Gamma))$$

and this regularity is sufficient to carry out the application of Green's thm + integration by parts, thus yielding the dissipative energy identity (2.1'$_n$)

$$E_n(t) + \int_s^t \int_{\Gamma_1} g(y_{nt})\, y_{nt}\, d\Gamma_1\, dz = E_n(s) \le \text{const}, \quad 0 \le s \le t \quad (2.1'_n)$$

corresponding to this solution. We then need to take the limit in n. If we merely use that (2.1'$_n$) implies $\{y_n, y_{nt}\} \to$ some $\{y, y_t\}$, weakly in $H^1(\Omega) \times L_2(\Omega)$, a.e. and that the quadratic $E_n(\cdot)$ is convex, hence weakly lower semi-continuous, we would only get the corresponding <u>inequality</u> at the limit

$$E_n(t) \le E_n(s) \quad \to \quad E(t) \le E(s)$$

which is not good enough. We need equality as in (1.2'). To obtain (1.2'), i.e. equality, we need to establish that

$$y_n \to \text{some } y \quad \text{strongly in } H^1(\Omega)$$

To this end, we begin with a Cauchy sequence $\{y_n\}$, multiply the approximating problem

$$\begin{cases} (y_n - y_m)_{tt} = \Delta(y_n - y_m) = -[f_0(y_n) - f_0(y_m)] & \text{in } Q \\ \frac{\partial}{\partial \nu}(y_n - y_m) = -[g(y_{nt}) - g(y_{mt})] \text{ in } \Sigma_1; \quad y_n - y_m\big|_{\Sigma_0} = 0 \end{cases}$$

by $\tilde{y}_t$, where $\tilde{y} \equiv y_n - y_m$. Integrating by parts yields

$$\|\tilde{y}_t(t)\|^2_{L_2(\Omega)} + \|\nabla \tilde{y}(t)\|^2_{L_2(\Omega)} + \underbrace{\int_0^t \left(g(y_{nt}) - g(y_{mt}), \tilde{y}_t\right)_\Gamma dt}_{g \text{ monotone } \Rightarrow \text{ positive } \ge 0}$$

$$= \int_0^t \int_\Omega [f_0(y_m) - f_0(y_n)]\, \tilde{y}_t \, d\Omega\, dz$$

$$+ \underbrace{\|\tilde{y}_t(0)\|^2_{L_2(\Omega)} + \|\nabla \tilde{y}(0)\|^2_{L_2(\Omega)}}_{\to\; 0 \;+\; 0}$$

Using that g is monotone: $(g(x_1)-g(x_2),\ x_1-x_2) \geq 0$, we drop this term & get

$$\|\tilde{y}_t(t)\|^2_{L_2(\Omega)} + \|\nabla\tilde{y}(t)\|^2_{L_2(\Omega)} \leq \int_0^t\int_\Omega |f_0(y_n)-f_0(y_m)|^2 d\Omega\, dz$$

$$+\int_0^t\int_\Omega |\tilde{y}_t|^2 d\Omega\, dz$$

$$+\|\{\nabla\tilde{y}(0),\ \tilde{y}_t(0)\}\|^2_{L_2(\Omega)\times L_2(\Omega)}$$

and since f_0 satisfies the Lipschitz cond. (by (H.2)(ii)) and Poincare ineq holds true by $y_n - y_m|_{\Sigma_0} \equiv 0$, $\Gamma_0 \neq 0$, we obtain for n, m sufficiently large

$$\|\tilde{y}_t(t)\|^2_{L_2(\Omega)} + \|\nabla\tilde{y}(t)\|^2_{L_2(\Omega)} \leq \underbrace{(C_{y_0})}_{\uparrow} \int_0^t\int_\Omega \overbrace{|\nabla(y_n-y_m)|^2}^{|\nabla\tilde{y}|^2} + |\tilde{y}_t|^2 d\Omega\, dz + \varepsilon$$

$$C(|y_n|^k_{H^1} + |y_m|^k_{H^1}) \leq C|y_0|^k_{H^1}$$

$\underline{k = k_0 - 1}$, (H 2)(iii)

$$E(\tilde{y}(t)) \leq \varepsilon + c\int_0^t E(\tilde{y}(z))\, dz \qquad n, m \text{ large}$$

$\Rightarrow$ Gronwall

$$E(\tilde{y}(t)) \leq \underbrace{E(\tilde{y}(0))}\, e^{ct} \to 0, \text{ as } n, m \to \infty$$

<u>Step 2</u> Establish the following a-priori energy estimate

$$\int_0^T E(t)\,dt \le \underbrace{c_T(E(0))}_{} \, H\left(\int_0^T \mathcal{D}(z)\,dz \right) \quad (2.2)$$

constant c_T increasing with $E(0)$

concave increasing function H, with $H(0)=0$

$T\,E(T) \le$

by <u>Step 1</u>, (2.1)

In our present problem, this result reads as follows

$$T\,E(T) \le \int_0^T E(t)\,dt \le C_T(E(0))\, H\left(\int_0^T \int_{\Gamma_1} g(y_t)\, y_t \, d\Gamma_1 \, dt \right) \quad (2.2')$$

[A related result in I.L. + D.T. is Eq (3.38) p 531, as we shall see below]

Actually, at first one establishes the following energy estimate

$$\int_0^T E(t)\,dt \le c_T(E(0)) \int_{\Sigma_1} y_t^2 + g^2(y_t)\, d\Sigma_1 \qquad (2.3)$$

<u>Claim 1</u>

We now show that, under present assumptions, we have:

$$\left\{ \int_{\Sigma_1} \left[y_t^2 + g^2(y_t) \right] d\Sigma_1 \le H\left(\int_{\Sigma_1} y_t\, g(y_t)\, d\Sigma_1 \right) \quad (2.4) \right.$$

} with $H(\cdot)$ identified below in (2.9).

Proof of (2.4) [see I.L. + R.T. p530] Divide a function $u \in L_2(\Sigma_1)$ in two parts:

$$\Sigma_A \equiv \{ u \in L_2(\Sigma_1) : |u| \geq N \text{ a.e.} \}; \quad \Sigma_B = \Sigma_1 - \Sigma_A \qquad (2.5)$$

First, from hypothesis (H.1)(iii) on g; $M_2 s \leq g(s) \leq M_1 s$, $|s| > N$, and get

$$\int_{\Sigma_A} [y_t^2 + g^2(y_t)] \, d\Sigma \leq \int_{\Sigma_A} \left[y_t M_2^{-1} g(y_t) + g(y_t) M_1 y_t \right] d\Sigma_A$$

$$= (M_1 + M_2^{-1}) \int_{\Sigma_A} \overbrace{y_t \, g(y_t)}^{+} \, d\Sigma_A$$

$$\leq (M_1 + M_2^{-1}) \int_{\Sigma_1} g(y_t) \, y_t \, d\Sigma_1 \qquad (2.6)$$

Next, from the properties of the constructed function $h(s)$ in (1.4), we estimate

$$\int_{\Sigma_B} [y_t^2 + g^2(y_t)] \, d\Sigma_B \leq \int_{\Sigma_B} h(y_t \, g(y_t)) \, d\Sigma_B$$

Jensen inequality, since $h(\cdot)$ is concave

$$\leq (\text{meas}\, \Sigma_1) \, h\left(\frac{1}{\text{meas}\, \Sigma_1} \int_{\Sigma_1} y_t \, g(y_t) \, d\Sigma_1 \right)$$

(by (1.5))

$$= (\text{meas}\, \Sigma_1) \, \tilde{h}\left(\int_{\Sigma_1} y_t \, g(y_t) \, d\Sigma_1 \right) \qquad (2.7)$$

Then, (2.6) and (2.7) imply

$$\int_{\Sigma} \left[y_t^2 + g^2(y_t)\right] d\Sigma = \int_{\Sigma_A} \left[y_t^2 + g^2(y_t)\right] d\Sigma_A + \int_{\Sigma_B} \left[y_t^2 + g^2(y_t)\right] d\Sigma_B$$

((2.6) + (2.7))

$$\leq \underbrace{\left[(M_1 + M_2^{-1}) + (\text{meas}\,\Sigma_1)\,\tilde{h}\right]}_{H} \int_{\Sigma_1} g(y_t)\, y_t \, d\Sigma_1 \qquad (2.8)$$

$$H = \Big[\underbrace{(M_1 + M_2^{-1})}_{\text{linear}} I + (\text{meas})\, \underbrace{\tilde{h}}_{\text{concave increasing}}\Big] \qquad (2.9)$$

Thus Claim 1 has been established □

<u>Step 3</u> Step 1 + Step 2 imply $\Rightarrow$

$$E(T) + p(E(T)) \leq E(0) \qquad (2.10)$$

$p(\cdot)$ the strictly increasing positive function in (1.6)

<u>Proof of (2.10)</u> Combine (2.3) and (2.4) to obtain (2.2') with $H(\cdot)$ defined in (2.9). Thus we get

$$T\,E(T) \leq C_T(E(0))\,(\text{meas}\,\Sigma_1) \left[\underbrace{\frac{M_1 + M_2^{-1}}{\text{meas}\,\Sigma_1}}_{c} + \tilde{h}\right] \int_{\Sigma_1} g(y_t)\, y_t \, d\Sigma_1 \qquad (2.11)$$

or redefining C_T:

$$\frac{E(T)}{C_T(E(0))(\mathrm{meas}\,\Sigma_1)} \leq \Big[\underbrace{\frac{M_1 + M_2^{-1}}{\mathrm{meas}\,\Sigma_1}}_{c} + \tilde{h}\Big] \int_{\Sigma_1} g(y_t)\,y_t \, d\Sigma_1 \qquad (2.12)$$

Define

$$K = \frac{1}{C_T(E(0))(\mathrm{meas}\,\Sigma_1)} \; ; \qquad c = \frac{M_1 + M_2^{-1}}{\mathrm{meas}\,\Sigma_1} \qquad (2.13)$$

Thus rewrite (2.12) as

$$K\,E(T) \leq \underbrace{[c\,I + \tilde{h}]}_{\text{invertible}} \int_{\Sigma_1} g(y_t)\,y_t\, d\Sigma_1 \qquad (2.14)$$

$$\Downarrow$$

$$\underbrace{[c\,I + \tilde{h}]^{-1}(K\,E(T))}_{\substack{\parallel \\ p(E(T))}} \leq \int_{\Sigma_1} g(y_t)\,y_t\, d\Sigma_1 \qquad (2.15)$$

by (1.6)

$$p(E(T)) \leq \int_{\Sigma_1} g(y_t)\,y_t\, d\Sigma_1 \overset{(2.1)\ \text{step 1}}{=} E(0) - E(T) \qquad (2.16)$$

hence

$$E(T) + p(E(T)) \leq E(0) \qquad (2.17)$$

and (2.10), step 3, is proved □

Step 4 We now "march" in steps of T (can take $T=1$): We generalize (2.10) to

$$E(mT) + p(E(mT)) \leq E((m-1)T), \quad m = 1, 2, 3, \ldots \quad (2.18)$$

with the same $p(\cdot)$ independent of m.

This is so, since the step "m" corresponding to (2.2') is now

$$\int_{(m-1)T}^{mT} E(t)\,dt \leq \underbrace{c_T(E((m-1)T))}_{\leq c_T(E(0))} \, H\left(\int_{(m-1)T}^{mT} \int_{\Gamma_1} g(y_t)\, y_t \, d\Sigma_1 \right) \quad (2.19)$$

$\leq c_T(E(0))$ by dissipativity (2.1), step 1

this constant determines p, as seen in step 3, independent of m.

Thus, the computations of step 3 applied to (2.19) yield (2.18), as desired.

Step 5 Set for convenience

$$s_m = E(mT), \quad m = 0, 1 \ldots \quad (2.20)$$

and changing $m \to m+1$ we re-write (2.18) via (2.20) as

$$\boxed{s_{m+1} + p(s_{m+1}) = [I + p]\, s_{m+1} \leq s_m} \quad (2.21)$$

$m = 0, 1, \ldots$

Since $[I+p]$ is invertible, as p is strictly increasing we have the explicit recurrence relation

$$s_{m+1} \le [I+p]^{-1} s_m, \qquad m=0,1,2.. \quad \text{, hence} \qquad (2.22)$$

$$s_{m+1} - s_m \le [I+p]^{-1} s_m - s_m, \quad \text{or}$$

$$[s_{m+1} - s_m] + \underbrace{[I - [I+p]^{-1}]}_{\substack{\shortparallel \\ q}} s_m \le 0 \qquad (2.23)$$

q defined in (1.7), $q(0)=0$, increasing positive

$$s_{m+1} - s_m + q(s_m) \le 0 \qquad m=0,1,2.. \quad s_0 = E(0) \qquad (2.23')$$

This discrete version motivates the introduction of the ODE in (1.12) i.e.

$$S_t(t) + p(S(t)) = 0 \qquad S(0) = E(0) \qquad (2.24)$$

non-linear contraction

which has a global solution $S(t)$, and moreover if $p(x)>0$ for $x>0$ as in our case $\Rightarrow$

$$\lim_{t\to\infty} S(t) = 0 \qquad (2.25)$$

Step 6 The final step is a comparison result between the discrete version: $s_{m+1} + p(s_{m+1}) \le s_m$ in (2.21) and the continuous version (2.24).

<u>Lemma</u> We have

$$s_m \leq S(m) \;, \quad \text{or} \quad E(mT) \leq S(m) \to 0 \text{ as } m \to \infty \qquad (2.26)$$

solution of (2.24)

Thus the Energy goes to zero at discrete values $T, 2T, \ldots$ and since it is decreasing $\frac{dE}{dt} \leq 0$, then we obtain

$$E(t) \to 0 \quad \text{as } t \to \infty$$

The proof of (2.26) is by induction (Lemma 3.3 in I.L. + D.T. p 531) using that $S(t)$ is a non-linear contraction,

$S(t) \leq S(\tau)$, $t \geq \tau$:

3. Comments on Step 2

It remains to comment on Step 2, in particular:

how do we prove (2.3), re-written as:

$$\int_0^T E(t)\,dt \le c_T(E(0)) \int_{\Sigma_1} \left[y_t^2 + g^2(y_t)\right] d\Sigma_1 \quad ? \quad (3.0)$$

Sketch of proof of (2.3) = (3.0): step a Actually, one first proves the following energy estimate

$$\int_0^T E(t)\,dt \le c_T(E(0)) \int_{\Sigma_1} \left[y_t^2 + g^2(y_t)\right] d\Sigma_1 + \text{l.o.t.}, \quad (3.1)$$

$$\text{l.o.t} = C(\varepsilon)\left[\int_{\Sigma_1} y^2\,d\Sigma_1 + \int_Q y^2\,dQ\right] \quad (3.2)$$

which is the desired estimate (3.0) polluted by lower order terms as in (3.2)

step b Next, one absorbs the lower order terms by proving that [This is Lemma 3.1 p 526 in I.L. + D.T.'s paper]:

For $T > T_0$ = sufficiently large, then

$$\text{l.o.t} = \int_{\Sigma_1} y^2\,d\Sigma_1 + \int_Q y^2\,dQ \le C_T(E(0)) \left[\int_{\Sigma_1} \left[y_t^2 + g^2(y_t)\right] d\Sigma_1\right. \quad (3.3)$$

The proof uses a suitable (more technical) version of the compactness / uniqueness argument. It runs by contradiction. The uniqueness refers to an over-determined linear wave equation with $L_{\infty}(Q)$-potential and over-determined (2) boundary conditions. One may invoke the uniqueness result of Ruiz (~ early 90's), or else the (more general) result of I.L + R.T. + X.Z (2000). The proof of this step is delicate. Thus, after this step, estimate (3.1) implies the desired estimate (3.0)

Step c. How, then, do we prove estimate (3.1)??

The qualitative answer is:

- by using the usual multiplier methods (mid-80's) of the corresponding linear model: this is Proposition 3.1 in I.L + D.T. p 522. It gives the 'usual' estimate obtained by: $(x - x_0)\cdot\nabla$(solution) and (solution) as the 2 (usual) multipliers

.. by using the dissipative identity (2.1');

... by using Sobolev embeddings:

$$\begin{cases} H^1(\Omega) \subset L_{\frac{2n}{n-2}}(\Omega), \quad H^{1/2}(\Gamma) \subset L_{\frac{2n-2}{n-2}}(\Gamma), \quad n > 2 \end{cases} \qquad (3.4a)$$

(3.4b) LS23

$$\left\{ H^1(\Omega) \subset L_p(\Omega), \quad H^{1/2}(\Gamma) \subset L_p(\Gamma), \quad 1 \le p < \infty, \ n = 2 \right.$$

plus appropriate interpolation inequalities (e.g. for L_p-spaces if f_1 is present as in (1.2d)

.... It is worth mentioning .... the reason why only the geometrical condition (H.3):

$(x - x_0)\cdot\nu(x) \le 0$ in Γ_0 on the uncontrolled portion of the boundary is sufficient (unlike much, if not all, prior literature which assumed a geometrical condition: $(x-x_0)\cdot\nu(x) > 0$ in Γ_1 = the controlled portion of the boundary).

The reason is the following result (from I. Lasiecka + R. Triggiani AMO, 92, Ref [14] in L.L + D.T.'s paper):

$$\int_\alpha^{T-\alpha} \int_{\Gamma_1} \underbrace{|\nabla_\tau u|^2}_{\text{tangential gradient}} d\Gamma_1\, dt \le C_{\rho,\alpha} \left[\int_{\Sigma_1} \left|\frac{\partial u}{\partial \nu}\right|^2 + u_t^2 \right] d\Sigma_1$$

$$+ C_T \underbrace{|u|^2_{L^2(0,T;\, H^{1/2+\rho}(\Omega))}}, \ l.o.t.$$

$$\begin{cases} u_{tt} - \Delta u = f \\ u_t, \ \dfrac{\partial u}{\partial \nu} \in L_2(\Sigma_1) \\ u|_{\Sigma_0} \equiv 0 \end{cases} \qquad + \int_Q |f|^2\, dQ \qquad (3.5)$$

$0 < \alpha$ = small, ρ = arbitrary > 0

All the above, then, leads to the following estimate [(3.15) p 526 in I.L. + D.T.]:

$$\left[1 - \varepsilon\, E(0)\right]^{2(k-1)} \int_0^T E(t)\,dt \le C(E(0)) \left\{ \int_{\Sigma_1} \left[g^2(y_t) + y_t^2 \right] d\Sigma_1 \right.$$

$k \ge 1$

$$\left. + \ l.o.t \quad + E(T) \right\} \qquad (3.6)$$

$$l.o.t = C(\varepsilon)\left[\int_{\Sigma_1} y^2\, d\Sigma_1 + \int_Q y^2\, dQ \right] \qquad (3.7)$$

Then (3.6) yields by virtue of the dissipation (2.1')

$$\left[1 - \varepsilon\, E(0)\right]^{2(k-1)}\, T\, \underline{E(T)} \le C(E(0)) \int_{\Sigma_1} \left[g^2(y_t) + y_t^2 \right] d\Sigma_1$$

$$+ \, C(E(0))\, \underline{E(T)} + l.o.t \qquad (3.8)$$

Take ε small and T large to get

$$E(T) = \mathcal{O}\left\{ \int_{\Sigma_1} g^2(y_t) + y_t^2 \ d\Sigma_1 + l.o.t \right\} \qquad (3.9)$$

which substituted into the RHS of (3.6) yields estimate (3.1), as desired!