Carleman Estimates with No Lower-Order Terms for General Riemann Wave Equations. Global Uniqueness and Observability in One Shot

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Abstract

This paper considers a fully general (Riemann) wave equation on a finite-dimensional Riemannian manifold, with energy level $(H^1 \times L_2)$-terms, under essentially minimal smoothness assumptions on the variable (in time and space) coefficients. The paper provides Carleman-type inequalities: first pointwise, for C^2-solutions, then in integral form for $H^{1,1}(Q)$-solutions. The aim of the present approach is to provide Carleman inequalities which do *not* contain lower-order terms, a distinguishing feature over most of the literature. Accordingly, global uniqueness results for over-determined problems as well as Continuous Observability/Uniform Stabilization inequalities follow in one shot, as a part of the same stream of arguments. Constants in the estimates are, therefore, generally explicit. The paper emphasizes the more challenging pure Neumann B.C. case. The paper is a generalization from the Euclidean to the Riemannian setting of [L-T-Z.1] in the more difficult case of purely Neumann B.C., and of [K-K.1] in the case of Dirichlet B.C. The approach is Riemann geometric, but different from —indeed, more flexible than—the one in [L-T-Y.1].

*Research partially supported by the National Science Foundation under grant DMS-9804056 and the Army Research Office under grant DAA4-96-1-0059.

†Research performed while visiting the Mathematics Department, University of Virginia, Charlottesville, VA 22904. Research partially supported by the NSF grant DMS-9804056, and by the National Science Foundation of China.

1 Introduction

Throughout this paper M is a finite-dimensional Riemannian manifold with metric $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and squared norm $|X|^2 = g(X,X)$, and Ω is an *open bounded*, connected, compact set of M with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. We let n denote the outward unit normal field along the boundary Γ. further, we denote by Δ the Laplace (Laplace-Beltrami) operator on the manifold M and by D the Levi-Civita connection on M [DoC.1], [Le.1], [T.1].

In this paper, we study the following wave equation with energy level terms on Ω:

$$w_{tt} = \Delta w + F(w) + f \quad \text{in } (0,T] \times \Omega \tag{1.1}$$

under the following standing assumptions: the energy-level differential term

$$F(w) \equiv \langle P(t,x), Dw \rangle + p_1(t,x) w_t + p_0(t,x) w, \tag{1.2a}$$

where p_0, p_1 are functions on $[0,T] \times \Omega$, and $P(t)$ is a vector field on M for $t > 0$, satisfies the following estimate: there exists a constant $C_T > 0$ such that

$$|F(w)|^2 \le C_T\{|Dw|^2 + w_t^2 + w^2\}, \quad x,t \in Q, \tag{1.2b}$$

where $Dw = \nabla_g w$ for the scalar function w. So Dw is a vector field, $Dw \in \mathcal{X}(M) =$ the set of all vector fields on M. Two vertical bars $|\cdot|$ denote the norm in the tensor space T_x or its completion $L^2(\Omega, T)$ [He.1]. Furthermore, we assume throughout that the forcing term f in (1.1) satisfies

$$f \in L_2(0,T;\Omega) \equiv L_2(Q); \quad \int_Q f^2 dQ < \infty, \tag{1.3}$$

where $dQ = d\Omega\, dt$, and $d\Omega$ is the volume element of the manifold M in its Riemann metric g.

Remark 1.1. Property (1.2b) is fulfilled if $P \in L_\infty(0,T;\Lambda)$ [He.1], $p_0, p_1 \in L_\infty(Q)$. In effect, one could relax the regularity assumption on the lower-order coefficient p_0 to read $p_0 \in L_p(Q)$ for $p = \dim \Omega + 1$, see [L-T-Z.1, Remark 1.1.1] by using a Sobolev embedding theorem. $\square$

Main assumptions. In addition to the standing assumptions (1.2b), (1.3) on the first-order operator F and the forcing term f, the following assumption is postulated throughout Section 9 of this paper.

(A.1) There exists a function $d : \overline{\Omega} \Rightarrow \mathbb{R}$ of class C^3 that is strictly convex in the metric g. This means that the Hessian $D^2 d$ (a 2-order tensor) satisfies: $D^2 d(X,X) > 0$, $\forall\, x \in \overline{\Omega}$, $\forall\, X \in M_x$. By translation and rescaling, we can always achieve (see Remark 1.2 below) that $d(x)$ satisfies the following conditions

$$D^2 d(X,X) \equiv \langle D_X(Dd), X \rangle_g \ge 2|X|_g^2, \quad \forall\, x \in \Omega,\ \forall X \in M_x, \quad \min_{\overline{\Omega}} d(x) \equiv m > 0, \tag{1.4}$$

where, as we said, $D^2 d$ is the Hessian of d (a 2-order tensor) and where M_x is the tangent space at $x \in \Omega$. In (1.4), we think of m as arbitrarily small, in order not to deteriorate the threshold time T_0 in (1.6b) below.

A working assumption throughout Section 9, to be later relaxed in Section 10 is that $d(x)$ has no critical point on $\overline{\Omega}$:

(A.2)

$$\inf_{x\in\Omega} |Dd| = p > 0, \text{ so that we can take: } \kappa = \inf_{\overline{\Omega}} \frac{|Dd|^2}{d} > 4 \tag{1.5}$$

by rescaling, see Remark 1.2 below. The above setting is kept throughout Chapter 9.

Pseudo-convex function. Having chosen, on the strength of assumption (A.1), a strictly convex potential function $d(x)$ satisfying condition (1.4), we next introduce the function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ of class C^3 by setting

$$\phi(x,t) = d(x) - c\left(t - \frac{T}{2}\right)^2, \quad 0 \le t \le T, \quad x \in \Omega, \tag{1.6a}$$

where $T > 0$ and $0 < c < 1$ are selected as follows. We define first T_0 by setting, as usual,

$$T_0^2 = 4 \max_{x\in\overline{\Omega}} d(x). \tag{1.6b}$$

Let $T > T_0$ be given. By (1.6b), there exists $\delta > 0$ such that

$$T^2 > 4 \max_{x\in\overline{\Omega}} d(x) + 4\delta. \tag{1.6c}$$

For this $\delta > 0$, there exists a contstant c, $0 < c < 1$, such that

$$cT^2 > 4 \max_{x\in\overline{\Omega}} d(x) + 4\delta. \tag{1.6d}$$

Henceforth, let $\phi(x,t)$ be defined by (1.6a) with T and c chosen as described above, unless otherwise explicitly noted. Such function $\phi(x,t)$ has the following properties:

(a) for the constant $\delta > 0$, fixed in (1.6c), we have via (1.6d) that:

$$\phi(x,0) = \phi(x,T) = d(x) - c\frac{T^2}{4} \le \max_{\overline{\Omega}} d(x) - c\frac{T^2}{4} \le -\delta, \text{ uniformly in } x \in \overline{\Omega}; \tag{1.7}$$

(b) there are t_0 and t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, such that

$$\min_{x\in\overline{\Omega}, t\in[t_0,t_1]} \phi(x,t) \ge \sigma, \quad 0 < \sigma < m, \tag{1.8}$$

since $\phi\left(x, \frac{T}{2}\right) = d(x) \ge m > 0$, under present choice, with m given by (1.4).

Remark 1.2. If we rescale the original function $d(x)$ ($d(x) \geq m > 0$), i.e., if we replace $d(x)$ by $d_{\text{new}}(x) = ad(x)$ for a constant $a > 1$, then:

$$\begin{cases} Dd_{\text{new}} = aDd;\ |Dd_{\text{new}}|^2 = a^2|Dd|^2;\ D^2d_{\text{new}} = aD^2d; \\ T^2_{0,\text{new}} = aT_0^2;\ T^2_{\text{new}} = aT^2,\ D^2d_{\text{new}}(Dd_{\text{new}}, Dd_{\text{new}}) = a^3D^2d(Dd, Dd); \\ \text{the coercivity lower bound 2 in (1.4) is replaced by } 2a > 2; \\ \kappa_{\text{new}} = \inf_{\overline{\Omega}} \frac{|Dd_{\text{new}}|^2}{d_{\text{new}}} = a\kappa = a \inf_{\overline{\Omega}} \frac{|Dd|^2}{d}. \end{cases}$$

Hence, condition (1.5) (right) can always be achieved by rescaling, if $d(x)$ has no critical points in $\overline{\Omega}$. The importance of condition $\kappa > 4$ in (1.5) will be seen in (1.21) below, in asserting that

$$f(c) \equiv (5 + 3c)\kappa - 4(1 + 7c) \geq 0 \text{ for all } c < 1 \text{ and } c \text{ near } 1. \tag{1.9}$$

Indeed, $f(c = 1) > 0$ by (1.5) (right), and hence (1.9) holds true. Thus, c in (1.6d) can be taken arbitrarily close to 1, as needed.

Consequences of assumptions (A.1), (A.2) and of the scaling condition in (1.5). Let $d(x)$ be the strictly convex (potential) function provided by assumptions (A.1), (A.2) and subject to the translation/scaling conditions in (1.4), (1.5). It then follows that by choosing the function

$$\alpha(x) \equiv \Delta d(x) - c - 1, \tag{1.10}$$

the following two properties (p_1), (p_2), hold true (see proof of (1.12) below, while (1.11) is immediate):

(p_1)

$$\Delta d(x) - 2c - \alpha(x) \equiv 1 - c > 0, \quad \forall\, x \in \overline{\Omega}; \tag{1.11}$$

(p_2)

$$\begin{aligned} &[2c + \Delta d - \alpha]|Dd|^2 + 2D^2d(Dd, Dd) - 4c^2(\Delta d + 6c - \alpha)\left(t - \frac{T}{2}\right)^2 \\ &\geq\ 4(1 + 7c)\phi^*(x, t), \quad \forall\, t, x \in Q = [0, T] \times \Omega, \end{aligned} \tag{1.12}$$

where $\phi^*(x, t)$ is the function defined by

$$\phi^*(x, t) = d(x) - c^2\left(t - \frac{T}{2}\right)^2, \quad x \in \Omega,\ 0 \leq t \leq T. \tag{1.13}$$

Since $0 < c < 1$, we note via (1.6a), (1.13) that

$$\phi^*(x, t) \geq \phi(x, t), \quad x \in \Omega,\ 0 \leq t \leq T. \tag{1.14}$$

Next, we define two sets, subsets of $\Omega \times [0,T]$:

$$Q(\sigma) \equiv \{(x,t):\ x \in \Omega,\ 0 \le t \le T,\ \phi(x,t) \ge \sigma > 0\}, \tag{1.15}$$

$$Q^*(\sigma^*) \equiv \{(x,t):\ x \in \Omega,\ 0 \le t \le T,\ \phi^*(x,t) \ge \sigma^* > 0\},\ 0 < \sigma^* < \sigma, \tag{1.16}$$

for a constant σ^* chosen to satisfy $0 < \sigma^* < \sigma$, with $\sigma < m$ defined in (1.8). Indeed, recalling (1.8), we readily have

$$[t_0, t_1] \times \Omega \subset Q(\sigma) \subset Q^*(\sigma^*) \subset [0,T] \times \Omega. \tag{1.17}$$

Proof of (1.12). By virtue of the choice (1.10) for α and assumption (1.5), we compute via (1.10), (1.5), (1.9), for all $(t,x) \in Q$:

$$[2c + \Delta d - \alpha]|Dd|^2 + 2D^2d(Dd,Dd) - 4c^2(\Delta d + 6c - \alpha)\left(t - \frac{T}{2}\right)^2$$

$$= (1+3c)|Dd|^2 + 2D^2d(Dd,Dd) - 4c^2(1+7c)\left(t - \frac{T}{2}\right)^2 \tag{1.18}$$

$$\text{(by (1.4))} \quad \ge (5+3c)|Dd|^2 - 4c^2(1+7c)\left(t - \frac{T}{2}\right)^2 \tag{1.19}$$

$$\text{(by (1.5))} \quad \ge (5+3c)\kappa d(x) - 4c^2(1+7c)\left(t - \frac{T}{2}\right)^2 \tag{1.20}$$

$$\text{(by (1.9))} \quad \ge 4(1+7c)\left[d(x) - c^2\left(t - \frac{T}{2}\right)^2\right] \equiv 4(1+7c)\phi^*(x,t), \tag{1.21}$$

recalling (1.13), and (1.12) is proved, as desired.

Critical property to absorb lower-order terms. We shall see in Section 4, Eqn. (4.20) below, that the following property, derived from (1.12), is critical in the present approach in absorbing lower-order terms in w: under assumptions (A.1), (A.2) and for the choice of α in (1.10) we have:

(p'_2)

$$[2c + \Delta d - \alpha]|Dd|^2 + 2D^2d(Dd,Dd) - 4c^2(\Delta d + 6c - \alpha)\left(t - \frac{T}{2}\right)^2$$

$$\text{(by (1.12))} \quad \ge 4(1+7c)\phi^*(x,t) \ge 4(1+7c)\sigma^* = \beta_1 > 0, \quad \forall\ (x,t) \in Q^*(\sigma^*), \tag{1.22}$$

recalling the definition of the set $Q^*(\sigma^*)$ in (1.16).

The term on the LHS of (1.22) is precisely the term that will arise as a coefficient in front of the lower-order term, see (4.5), (4.7), (4.8) below.

2 Main results of the paper. Literature

Throughout this paper, the 'energy' is given by

$$E(t) = \int_{\Omega} [w_t^2 + |Dw|^2 + w^2] d\Omega, \tag{2.1}$$

see (5.4). This is the quantity which occurs in the estimates.

Main results of the paper. We hereby summarize the main results of the present paper on the Riemann wave Eqn. (1.1), with the understanding that *the standing hypotheses (1.2) on F and (1.3) on f are in force throughout.*

(1) Results under assumptions (A.1) = (1.4) and (A.2) = (1.5). These include, in order of presentation:

(1a) Carleman-type estimates which contain no lower-order terms (l.o.t.) and are valid for $H^{2,2}(Q)$-solutions of Eqn. (1.1) [with no B.C.]. The first version thereof is Theorem 5.1 (Section 5), while the second, final version is Theorem 6.1 (Section 6). Their proof rests on the fundamental lemma of Section 3, which provides a *pointwise* Carleman-type estimate (at each (t, x)) for C^2-solutions of (1.1), with *controlled sign* on the coefficient of the l.o.t., at least in the suitable set in space and time of $\Omega \times [0, T]$. This set is $Q^*(\sigma^*)$ under assumptions (A.1) and (A.2).

(1b) Extension of Theorem 5.1 and Theorem 6.1 to $H^{1,1}(Q)$-solution of Eqn. (1.1), by a regularizing procedure. This is done in Section 7.

(1c) A Continuous Observability Inequality—*a-fortiori* a global uniqueness theorem—for $H^{1,1}(Q)$-solutions, as given in Theorem 8.1 (Section 8) in the case of pure homogeneous Dirichlet B.C.; and as given in Theorems 9.1 and 9.3 (Section 9) in the case of pure homogeneous Neumann B.C.

(2) Extension of the results of part (1) to a setting that removes assumption (A.2) = (1.5). This is done, as in [L-T-Z.1, Section 10], by considering Ω as the union of two overlapping subdomains Ω_1 and Ω_2: $\Omega = \Omega_1 \cup \Omega_2$, and assuming the existence of two smooth strictly convex functions d_1 and d_2 on Ω [as in (A.1) = (1.4)], with the additional feature that d_i has no critical point on Ω_i, thereby eliminating assumption (A.2), at least on each Ω_i. Accordingly, the main result of the present paper is Theorem 10.1.1. Only some key aspects of the proof of this theorem will be given explicitly in Section 10—those dealing with the elimination of the l.o.t. in the final Carleman estimates—as the proof follows rather faithfully the one given in [L-T-Z.1, Section 10].

Literature. To begin with, regarding the generality of model (1.1), we notice that such Riemann wave equation includes, in particular, a general second-order hyperbolic equation defined on an Euclidean bounded domain, with principal part coefficients $a_{ij}(x)$ variable in space, and coefficients of energy level terms variable in both time and space [L-T-Y.1-2]. The case with no energy lvel terms was dealt with in [Y.1], where the geometric method was introduced. Here in these aforementioned references, the transformation $\{g_{ij}(x)\} = \{a_{ij}(x)\}^{-1}$ (2 $n \times n$ positive symmetric matrices), gives the coefficients of the Riemann metric g. Indeed, one may more generally start with a general second-order elliptic operator on the manifold M, and change it into the Laplace-Beltrami operator as in (1.1), through a similar transformation based on the coefficients. Having assessed the generality of our model, we may next state our goal: the main goal of the present paper is then to obtain Carleman-type inequalities for the aforementioned general model (1.1), with the additional requirement that they do not contain lower-order terms (l.o.t.). This is a most desirable feat, which is in contrast with much of the large literature on this subject, from the canonical equations of the mid-80's [L-T.3-4], [Lio.1-2], to the very general geometric optics approach in [B-L-R.1], [Lit.1-2], to the earlier approaches to Carleman estimates on Euclidean setting [Ta.1], [L-T.5] to the Riemann geometric generalization in [L-T-Y.1-2]. The advantages of eliminating l.o.t. in the Carleman estimates are then several and substantial. To begin with, as a consequence, this approach permits one to obtain (i) global uniqueness results of over-determined problems, as well as the much stronger continuous observability/uniform stabilization inequalities in one shot, as part of the same stream of arguments. This is in contrast with much of the literature on the subject, where elimination of l.o.t. from the estimate would require appealing to, or invoking of, or assuming a global uniqueness result, not a ready-available task in the presence of non-smooth coefficients, particularly with $L_\infty(Q)$-coefficients in time and space at the energy level. By contrast the present approach establishes—rather than assumes—global uniqueness results, precisely in the form needed by the COI/US inequalities, which compare very favorably with the literature [Ho.1], [Ho.3], [I.1], [Lit.3], [Ta.3,5]. An additional bonus of the present approach is that the final COI/US inequalities are obtained with explicit constants, certainly so in the case of Dirichlet B.C., and also in the case of Neumann B.C., if one tolerates geometrical conditions on the controlled/observed portion Γ_1 of the boundary. Knowledge of explicit constants in these estimates plays a positive role in dealing with semi-linear problems [L-T.10], [Z.1], [Z.2]. Only a few recent references [Ta.2], [I-Y.1], [L-T-Z.1] have addressed successfully the issue of obtaining Carleman/COI/US estimates with no lower-order terms for control theoretic purposes. Both [I-Y.1] and its more general version [L-T-Z.1], done simultaneously and independently, refer, however, to the case of the *Euclidean* Laplacian, with particular emphasis on the more challenging purely Neumann case. Also [K-K.1], which refers to $H^{2,2}(Q)$-solutions and includes the Dirichlet, or Dirichlet/Neumann, but excludes the purely Neumann case, refers to the Euclidean Laplacian. All are inspired by a pointwise Carleman-type estimate [L-R-S.1, Lemma 1, p. 124]: Indeed, [K-K.1] and [I-Y.1] invoke such pointwise Carleman estimate, while [L-T-Z.1] provides a new version of it, more suitable for the purely Neumann B.C. case, in order to relax geometrical conditions. The present paper is a faithful extension of [L-T-Z.1] from the Euclidean to the Riemannian setting. In particular, the preliminary fundamental Lemma 3.1 of Section 3 is the present Riemannian counterpart of the

Lemma 3.1 in [L-T-Z.1] in the Euclidean setting, in turn related—as recalled above—to the [L-R-S.1, Lemma 1, p. 124]. One may give serveral classes of examples where all the necessary assumptions—particularly in the purely Neumann B.C. case—are satisfied. One such attractive class is given in Appendix B, which is the Riemannian counterpart of the result given in the Euclidean setting in [L-T-Z.1, Appendix A, Theorem A.4.1]. Further information on the literature is given in [L-T-Z.1, Section 2.3], to which we refer for lack of space.

3 A fundamental lemma

The following lemma is the key starting point of our analysis. It gives a pointwise estimate which is the counterpart, in the present Riemann metric, of [L-T-Z.1, Lemma 3.1]. Its proof is a parallel development, however, in the Riemann metric, of that given in [L-T-Z.1] in the Euclidean metric. To streamline the treatment and write the resulting computations in a more concise form, we introduce some notation as well as some operators. We recall from Section 1 that D denotes the Levi-Civita connection of (M,g) and that $\mathcal{X}(M)$ is the set of all vector fields on M. Here below, we let $f(t,x)$, $h(t,x) \in C^1(\mathbb{R} \times M)$ and $X \in \mathcal{X}(M)$. Then [G-P-V.1], [T.1]

$$\begin{gathered}\text{div } X = \textstyle\sum_{i=1}^N \left[D_{\frac{\partial}{\partial x_i}} X\right]_i, \text{ with the property: } \operatorname{div}(fX) = f \text{ div } X + X(f), \\ \text{satisfying Green's formula : } \textstyle\int_\Omega \text{div } X \, d\Omega = \int_\Gamma \langle X, n\rangle d\Gamma,\end{gathered} \tag{3.0}$$

is the divergence operator on the manifold (M,g). At this point, we set by definition

$$\hat{D}f \equiv (f_t, -Df); \quad \widehat{\text{div}}(h,X) \equiv h_t + \text{div } X; \tag{3.1}$$

$$(h,X)(f) \equiv hf_t + X(f); \quad \mathcal{A}w \equiv \widehat{\text{div}}\hat{D}w, \tag{3.2}$$

where the notation "(function, vector field)" means simply a pair, in that order, of the function and the vector field noted.

Lemma 3.0. As a consequence of the above definitions, we have the following identities:

(a)

$$\mathcal{A}w = \widehat{\text{div}}(w_t, -Dw) = w_{tt} - \operatorname{div}(Dw) = w_{tt} - \Delta w; \tag{3.3}$$

(b)

$$\begin{aligned} \widehat{\text{div}}[f(h,X)] &= \widehat{\text{div}}(fh, fX) = (fh)_t + \operatorname{div}(fX) && (3.4)\\ \text{(by (3.0))} \quad &= f(h_t + \text{div } X) + hf_t + X(f) && (3.5)\\ &= f\,\widehat{\text{div}}(h,X) + (h,X)(f); && (3.6)\end{aligned}$$

(c)

$$\begin{cases} (\hat{D}f)(h) &= (f_t, -Df)(h) = f_t h_t - Df(h), & (3.7a) \\ (\hat{D}h)(h) &= h_t^2 - |Dh|^2; & (3.7b) \end{cases}$$

(d)

$$\widehat{\text{div}} f(X) = f\,\widehat{\text{div}}\,X + X(f). \tag{3.8}$$

Proof. The steps in (3.3) use: definition (3.2) for $\mathcal{A}$; definition (3.1) for $\widehat{\text{div}}$; and div $Dw =$ div $\nabla w = \Delta w$ for the Laplace-Beltrami operator. Next, the steps from (3.4) through (3.6) use: (3.1) twice for $\widehat{\text{div}}$; (3.0); and (3.2) (left). Finally, (3.1) for $\hat{D}$ and (3.2) (left) prove (3.7). □

Below, to facilitate the comparison with [L-T-Z.1], we shall generally adhere to its notation.

Lemma 3.1. Let

$$w(t,x) \in C^2(\mathbb{R} \times M);\ \ell(t,x) \in C^2(\mathbb{R} \times M);\ \psi(t,x) \in C^2 \text{ in } t \text{ and } C^1 \text{ in } x \tag{3.9}$$

be three given functions. Set

$$\theta(t,x) = e^{\ell(t,x)};\ v(t,x) = \theta(t,x)w(t,x) = e^{\ell(t,x)}w(t,x), (t,x) \in Q = (0,T] \times \Omega; \tag{3.10a}$$

$$\theta_t = \ell_t\theta;\ D\theta = \theta D\ell;\ v_t = \ell_t v + \theta w_t;\ Dv = \theta Dw + vD\ell. \tag{3.10b}$$

Then, with reference to $\mathcal{A}w = w_{tt} - \Delta w$ in (3.3), the following pointwise inequality holds true:

$$\begin{aligned} &\theta^2(\mathcal{A}w)^2 + (\psi_t v^2)_t + 2\widehat{\text{div}}\left\{[(a+\psi)v^2 - v_t^2 + |Dv|^2]\hat{D}\ell + [2\hat{D}\ell(v) - \psi v]\hat{D}v\right\} \\ &\geq\ -8v_t D\ell_t(v) + 2[\Delta\ell + \ell_{tt} - \psi]v_t^2 + 2(\mathcal{A}\ell + \psi)|Dv|^2 \\ &\quad +2v\langle D\psi, Dv\rangle + 4D^2\ell(Dv,Dv) + \left\{2\widehat{\text{div}}[(a+\psi)\hat{D}\ell] + \psi_{tt} + 2a\psi\right\}v^2, \end{aligned} \tag{3.11}$$

where we have set (via (3.7b) with $h = \ell$)

$$a \equiv \hat{D}\ell(\ell) - \mathcal{A}\ell - \psi = \ell_t^2 - |D\ell|^2 - \ell_{tt} + \Delta\ell - \psi, \tag{3.12}$$

and $D^2\ell(\,\cdot\,,\,\cdot\,)$ is the Hessian of ℓ (a 2-order tensor).

Proof. Step 1. First, by (3.1) on $\hat{D}$ and (3.10a–b), we have

$$\hat{D}\theta = (\theta_t, -D\theta) = \theta(\ell_t, -D\ell) = \theta\hat{D}\ell; \tag{3.13}$$

$$\begin{aligned} \hat{D}v &= (v_t, -Dv) = (\ell_t v + \theta w_t, -\theta Dw - vD\ell) \\ &= \theta(w_t, -Dw) + v(\ell_t, -D\ell) = \theta\hat{D}w + v\hat{D}\ell. \end{aligned} \tag{3.14}$$

The goal of this Step 1 is to show the following result (counterpart of [L-T-Z.1, Eqn. (3.11)]):

$$\theta^2(\mathcal{A}w)^2 = \left\{\mathcal{A}v + [\hat{D}\ell(\ell) - \mathcal{A}\ell]v - 2\hat{D}v(\ell)\right\}^2 \tag{3.15}$$

$$= (I_1 + I_2 + I_3)^2 \tag{3.16}$$

$$\geq 2[I_1 I_2 + I_1 I_3 + I_2 I_3], \tag{3.17}$$

where we have set (as in [L-T-Z.1, Eqn. (3.10)]) via (3.12):

$$\begin{cases} I_1 \equiv \mathcal{A}v + [\hat{D}\ell(\ell) - \mathcal{A}\ell - \psi]v = \mathcal{A}v + av = v_{tt} - \Delta v + av; & (3.18) \\ I_2 \equiv -2\hat{D}v(\ell) = -2\hat{D}\ell(v), \ I_3 = \psi v. & (3.19) \end{cases}$$

Proof of (3.16). To this end, we shall first prove that

$$\mathcal{A}v = \theta\mathcal{A}w + v\mathcal{A}\ell + 2\hat{D}v(\ell) - v\hat{D}\ell(\ell). \tag{3.20}$$

Indeed, to establish (3.20), we first compute by (3.2) (right), (3.14), (3.8):

$$\mathcal{A}v = \widehat{\text{div}}\hat{D}v = \widehat{\text{div}}(\theta\hat{D}w) + \widehat{\text{div}}(v\hat{D}\ell) \tag{3.21}$$

$$= \theta\widehat{\text{div}}\,\hat{D}w + v\,\widehat{\text{div}}\,\hat{D}\ell + \hat{D}w(\theta) + \hat{D}\ell(v) \tag{3.22}$$

$$= \theta\mathcal{A}w + v\mathcal{A}\ell + \hat{D}w(\theta) + \hat{D}\ell(v). \tag{3.23}$$

Regarding the last two terms in (3.23), we obtain via (3.7a) applied repeatedly, (3.10a–b), and (3.7b)

$$\hat{D}w(\theta) = w_t\theta_t - Dw(\theta) = \theta(w_t\ell_t - Dw(\ell)) = \theta\hat{D}w(\ell) \tag{3.24}$$

$$\hat{D}\ell(v) = \ell_t v_t - \langle D\ell, Dv\rangle = \ell_t(\ell_t v + \theta w_t) - \langle D\ell, \theta Dw + vD\ell\rangle$$

$$= \theta[\ell_t w_t - Dw(\ell)] + v[\ell_t^2 - D\ell(\ell)] \tag{3.25}$$

$$= \theta\hat{D}w(\ell) + v\hat{D}\ell(\ell). \tag{3.26}$$

Substituting (3.24) and (3.26) into the last two terms of (3.23) yields

$$\mathcal{A}v = \theta\mathcal{A}w + v\mathcal{A}\ell + 2[\theta\hat{D}w(\ell) + v\hat{D}\ell] - v\hat{D}\ell(\ell) \tag{3.27}$$

$$\text{(by (3.7a))} \qquad = \theta\mathcal{A}w + v\mathcal{A}\ell + 2\hat{D}v(\ell) - v\hat{D}\ell(\ell), \tag{3.28}$$

and (3.28) proves (3.20), as claimed.

Finally, we use (3.23) in the form $\theta \mathcal{A} w = \mathcal{A} v + v[\hat{D}\ell(\ell) - \mathcal{A}\ell] - 2\hat{D}v(\ell)$, where $\hat{D}v(\ell) = \hat{D}\ell(v)$, to obtain (3.15), (3.16).

Step 2. With reference to (3.18), (3.19), the goal of this Step 2 is to show the following result (counterpart of [L-T-Z.1, Eqn. (3.12)],

$$\begin{aligned} I_1 I_2 &= \widehat{\text{div}}[(av^2 - v_t^2 + |Dv|^2)\hat{D}\ell + 2\hat{D}\ell(v)\hat{D}v] + (\widehat{\text{div}}(a\hat{D}\ell))v^2 \\ &- 4v_t D\ell_t(v) + (\ell_{tt} + \Delta\ell)v_t^2 + (\mathcal{A}\ell)|Dv|^2 + 2D^2\ell(Dv, Dv). \end{aligned} \tag{3.29}$$

Proof of (3.29). By recalling I_1, I_2 from (3.18), (3.19), we compute via (3.2) on $\mathcal{A}$:

$$I_1 I_2 = -2(\mathcal{A}v + av)\hat{D}v(\ell) = -2(\widehat{\text{div}}(\hat{D}v))\hat{D}v(\ell) - 2av\hat{D}v(\ell). \tag{3.30}$$

By (3.8) with $X = \hat{D}v$ and $f = \hat{D}v(\ell)$, we have

$$(\widehat{\text{div}}(\hat{D}v))\hat{D}v(\ell) = \widehat{\text{div}}(\hat{D}v(\ell)\hat{D}v) - \hat{D}v(\hat{D}v(\ell)). \tag{3.31}$$

Then (3.31) used in (3.30) along with $2v\hat{D}v(\ell) = \hat{D}\ell(v^2)$, yields with $\hat{D}v(\ell) = \hat{D}\ell(v)$:

$$\begin{aligned} I_1 I_2 &= -2\widehat{\text{div}}(\hat{D}v(\ell)\hat{D}v) + 2\hat{D}v(\hat{D}v(\ell)) - a\hat{D}\ell(v^2) && (3.32) \\ &= -\widehat{\text{div}}(2\hat{D}\ell(v)\hat{D}v + av^2\hat{D}\ell) + \widehat{\text{div}}(av^2\hat{D}\ell) \\ &\quad + 2\hat{D}v(\hat{D}v(\ell)) - a\hat{D}\ell(v^2) && (3.33) \\ \text{(by (3.8))} \quad &= -\widehat{\text{div}}(2\hat{D}\ell(v)\hat{D}v + av^2\hat{D}\ell) + v^2\widehat{\text{div}}(a\hat{D}\ell) \\ &\quad + a\hat{D}\ell(v^2) + 2\hat{D}v(\hat{D}v(\ell)) - a\hat{D}\ell(v^2), && (3.34) \end{aligned}$$

after applying, in the last step, (3.8) with $f = v^2$ and $X = a\hat{D}\ell$.

We next evaluate the term $2\hat{D}v(\hat{D}\ell(v))$ in (3.34). To this end, by (3.7a) with $f = v$, $h = \hat{D}\ell(v) = \hat{D}v(\ell)$, one first has

$$\begin{aligned} 2\hat{D}v(\hat{D}\ell(v)) &= 2v_t(\hat{D}\ell(v))_t - 2Dv(\hat{D}\ell(v)) && (3.35) \\ &= 2v_t[\hat{D}\ell_t(v) + \hat{D}\ell(v_t)] - 2Dv(\ell_t v_t - D\ell(v)) && (3.36) \\ &= 2v_t\hat{D}\ell_t(v) + 2v_t\hat{D}\ell(v_t) - 2\ell_t Dv(v_t) \\ &\quad - 2v_t Dv(\ell_t) + 2Dv(D\ell(v)) && (3.37) \\ \text{(by (3.7a))} \quad &= 2v_t[\ell_{tt}v_t - D\ell_t(v)] + 2v_t[\ell_t v_{tt} - D\ell(v_t)] && (3.38) \\ &\quad - 2v_t D\ell_t(v) - \ell_t(|Dv|^2)_t + 2Dv(D\ell(v)), && (3.39) \end{aligned}$$

where $D\ell_t(v) = Dv(\ell_t)$. By using a key property of the Levi-Civita connection [Le.1], we compute the last term of (3.39) as follows

$$
\begin{aligned}
Dv(D\ell(v)) &= Dv\langle D\ell, Dv\rangle = X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle && (3.40)\\
&= \langle D_{Dv}(D\ell), Dv\rangle + \langle D\ell, D_{Dv}(Dv)\rangle && (3.41)\\
\text{(by (1.4))} \quad &= D^2\ell(Dv, Dv) + \langle D\ell, D_{Dv}(Dv)\rangle, && (3.42)
\end{aligned}
$$

where, in the last step, we have used the definition in (1.4) of the Hessian $D^2\ell$. It is shown in Appendix A that

$$
\langle D\ell, D_{Dv}(Dv)\rangle = \frac{1}{2}\, D\ell(|Dv|^2). \tag{3.43}
$$

Using (3.43) in (3.42), we obtain

$$
2Dv(D\ell(v)) = 2D^2\ell(Dv, Dv) + D\ell(|Dv|^2). \tag{3.44}
$$

Substituting (3.44) for the last term in (3.39) yields

$$
\begin{aligned}
2\hat{D}v(\hat{D}\ell(v)) &= 2v_t^2\ell_{tt} - 4v_t D\ell_t(v) + \hat{D}\ell(v_t^2) - \ell_t(|Dv|^2)_t \\
&\quad + D\ell(|Dv|^2) + 2D^2\ell(Dv, Dv) && (3.45)\\
\text{(by (3.7a))} \quad &= 2v_t^2\ell_{tt} - 4v_t D\ell_t(v) + \hat{D}\ell(v_t^2 - |Dv|^2) + 2D^2\ell(Dv, Dv) && (3.46)
\end{aligned}
$$

by invoking, in the last step, (3.7a) with $f = \ell$ and $h = |Dv|^2$ on the fourth and fifth terms in (3.45). Next, recalling (3.8) with $f = v_t^2 - |Dv|^2$ and $X = \hat{D}\ell$, we obtain

$$
\begin{aligned}
\widehat{\text{div}}[(v_t^2 - |Dv|^2)\hat{D}\ell] &= (v_t^2 - |Dv|^2)\widehat{\text{div}}\hat{D}\ell + \hat{D}\ell(v_t^2 - |Dv|^2)\\
\text{(by (3.2))} \quad &= (v_t^2 - |Dv|^2)\mathcal{A}\ell + \hat{D}\ell(v_t^2 - |Dv|^2) && (3.47)\\
\text{(by (3.3))} \quad &= v_t^2(\ell_{tt} - \Delta\ell) - |Dv|^2\mathcal{A}\ell + \hat{D}\ell(v_t^2 - |Dv|^2). && (3.48)
\end{aligned}
$$

Substituting the last term in (3.48) for the third term in (3.46), we obtain

$$
\begin{aligned}
2\hat{D}v(\hat{D}\ell(v)) &= v_t^2(\ell_{tt} + \Delta\ell) + |Dv|^2\mathcal{A}\ell + \widehat{\text{div}}[(v_t^2 - |Dv|^2)\hat{D}\ell]\\
&\quad - 4v_t D\ell_t(v) + 2D^2\ell(Dv, Dv), && (3.49)
\end{aligned}
$$

which was our present objective. Substituting (3.49) into (3.34) then yields

$$
\begin{aligned}
I_1 I_2 &= -\widehat{\text{div}}[(av^2 - v_t^2 + |Dv|^2)\hat{D}\ell + 2\hat{D}\ell(v)\hat{D}v] + v^2\widehat{\text{div}}(a\hat{D}\ell)\\
&\quad - 4v_t D\ell_t(v) + (\ell_{tt} + \Delta\ell)v_t^2 + (\mathcal{A}\ell)|Dv|^2 + 2D^2\ell(Dv, Dv), && (3.50)
\end{aligned}
$$

which is precisely (3.29), as desired.

Step 3. The goal of this step is to show that

$$\begin{aligned} I_1 I_3 &= \widehat{\text{div}}(\psi v \hat{D} v) - \frac{1}{2}(v^2\psi_t)_t + \Big(\frac{1}{2}\psi_{tt} + a\psi\Big)v^2 \\ &+ v\langle Dv, D\psi\rangle + \psi(|Dv|^2 - v_t^2). \end{aligned} \tag{3.51}$$

Proof of (3.51). By (3.18), (3.19), we compute via (3.2) on $\mathcal{A}$:

$$\begin{aligned} I_1 I_3 &= (\mathcal{A}v + av)\psi v = (\widehat{\text{div}}\hat{D}v)\psi v + a\psi v^2 && (3.52) \\ \text{(by (3.8))} &= \widehat{\text{div}}(\psi v \hat{D}v) - \hat{D}v(\psi v) + a\psi v^2 && (3.53) \\ \text{(by (3.7a))} &= \widehat{\text{div}}(\psi v \hat{D}v) - [v_t(\psi v)_t - Dv(\psi v)] + a\psi v^2 && (3.54) \\ &= \widehat{\text{div}}(\psi v \hat{D}v) - [v_t(\psi_t v + \psi v_t) - (\psi Dv(v) + vDv(\psi))] + a\psi v^2 && (3.55) \\ &= \widehat{\text{div}}(\psi v \hat{D}v) - v(v_t\psi_t - Dv(\psi)) - \psi(v_t^2 - Dv(v)) + a\psi v^2. && (3.56) \end{aligned}$$

Using

$$[-vv_t\psi_t + a\psi v^2] = -\frac{1}{2}(v^2\psi_t)_t + \Big(\frac{1}{2}\psi_{tt} + a\psi\Big)v^2 \text{ and } Dv(v) = |Dv|^2, \tag{3.57}$$

in (3.56), we finally obtain

$$I_1 I_3 = \widehat{\text{div}}(\psi v \hat{D}v) - \frac{1}{2}(v^2\psi_t)_t + \Big(\frac{1}{2}\psi_{tt} + a\psi\Big)v^2 + vDv(\psi) - \psi(v_t^2 - |Dv|^2), \tag{3.58}$$

which is precisely (3.51), as desired.

Step 4. By (3.19), we readily have, via (3.8) with $f = v^2$, $X = \psi\hat{D}\ell$:

$$I_2 I_3 = -2\psi v \hat{D}\ell(v) = -\widehat{\text{div}}(\psi v^2 \hat{D}\ell) + v^2\widehat{\text{div}}(\psi\hat{D}\ell), \tag{3.59}$$

since $\hat{D}\ell(v^2) = 2v\hat{D}v(\ell)$, as noted below (3.31).

Step 5. Inserting (3.29), (3.51), (3.58) in (3.17) yields the desired inequality (3.11). The proof of Lemma 3.1 is complete. □

4 A basic pointwise inequality

We now make suitable choices of the functions $\ell(t,x)$ and $\psi(x)$ involved in Lemma 3.1, thus obtaining the perfect counterpart of [L-T-Z.1, Theorem 4.1].

Theorem 4.1. Let

$$w(t,x) \in C^2(\mathbb{R}_t \times \Omega);\ d(x) \in C^3(\Omega),\ \alpha(x) \in C^1(\Omega), \tag{4.1}$$

be three given functions [at this stage, w need not be a solution of Eqn. (1.1a), d need not be the function provided by the assumptions (A.1) and (A.2) and α need not be given by (1.10)]. If $\tau > 0$ is a parameter, we introduce the functions

$$\ell(t,x) = \tau \left[d(x) - c\left(t - \frac{T}{2}\right)^2 \right] = \tau\phi(t,x); \tag{4.2}$$

$$\psi(x) \equiv \tau\alpha(x); \qquad \theta(t,x) = e^{\ell(t,x)} = e^{\tau\phi(t,x)}, \tag{4.3}$$

where $\phi(t,x)$ is defined consistently with (1.6a), with constant $0 < c < 1$ selected as in (1.6d). Then, with the above choices, Lemma 3.1 specializes as follows:

$$\begin{gathered}\psi_t = 0;\ \ell_t = -2\tau c\left(t - \frac{T}{2}\right);\ \ell_{tt} = -2c\tau;\ D\ell = \tau Dd; \\ D\psi = \tau D\alpha;\ D\ell_t = 0;\ \Delta\ell = \tau\Delta d;\ \Delta\ell = \tau\Delta d,\end{gathered} \tag{4.4}$$

so that the pointwise estimate (3.11) becomes with $\mathcal{A}w = w_{tt} - \Delta w$ in (3.3), and $v = \theta w$:

$$\begin{aligned}&\theta^2(\mathcal{A}w)^2 + 2\widehat{\text{div}}\left\{[(a+\psi)v^2 - v_t^2 + |Dv|^2]\hat{D}\ell + [2\hat{D}\ell(v) - \psi v]\hat{D}v\right\} \\ &\quad\geq\ 2\tau(\Delta d - 2c - \alpha)v_t^2 + 2\tau\left(\alpha - 2c - \Delta d - \frac{\epsilon}{2\tau}\right)|Dv|^2 \\ &\qquad + 4\tau D^2 d(Dv,Dv) + \theta^2\tilde{B}w^2,\end{aligned} \tag{4.5}$$

where $\epsilon > 0$ and

$$\begin{aligned} a &= \tau^2\left[4c^2\left(t - \frac{T}{2}\right)^2 - |Dd|^2\right] + 2c\tau + \tau\Delta d - \psi \\ &= \tau^2\left[4c^2\left(t - \frac{T}{2}\right)^2 - |Dd|^2\right] + \mathcal{O}(\tau);\end{aligned} \tag{4.6}$$

$$\left\{\begin{aligned}\tilde{B} &= 2\beta(x,t)\tau^3 + \mathcal{O}(\tau^2); &(4.7)\\ \beta(x,t) &= (2c + \Delta d - \alpha)|Dd|^2 + 2D^2d(Dd,Dd) - 4c^2(\Delta d + 6c - \alpha)\left(t - \frac{T}{2}\right)^2. &(4.8)\end{aligned}\right.$$

Proof. We only need to compute the coefficients of all terms on the right-hand side of inequality (3.11), as well as of the term a in (3.12),using the choice of functions made in (4.2), (4.3). This task is a direct computation which yields (4.4) and (4.6) at once. Then, (4.3), (4.4), (3.3) give

$$\ell_{tt} + \Delta\ell - \psi = \tau(\Delta d - 2c - \alpha); \ \mathcal{A}\ell + \psi = \ell_{tt} - \Delta\ell + \psi = \tau(\alpha - 2c - \Delta d). \tag{4.9}$$

We then verify (4.7), (4.8) for the coefficient $\tilde{B}$ of $v^2 = \theta^2 w^2$ in (4.5), which is defined in (3.13). Recalling (3.8), (3.7a), (3.2), (3.3), we find since $\psi_t \equiv 0$ by (4.4):

$$\begin{aligned}\tilde{B} &\equiv 2\,\widehat{\mathrm{div}}[(\alpha+\psi)\hat{D}\ell] + \psi_{tt} + 2a\psi = 2(a+\psi)\widehat{\mathrm{div}}\hat{D}\ell + 2\hat{D}\ell(\alpha+\psi) + 2a\psi \quad (4.10)\\ &= \{2(a+\psi)(\ell_{tt} - \Delta\ell)\} + \{2(a_t + \psi_t)\ell_t - 2D\ell(a+\psi)\} + 2a\psi. \quad (4.11)\end{aligned}$$

By (4.6) on a, (4.4) on ℓ_{tt} and $\Delta\ell$, we find

$$(a+\psi)(\ell_{tt} - \Delta\ell) = \left\{\tau^2\left[4c^2\left(t - \frac{T}{2}\right)^2 - |Dd|^2\right] + \mathcal{O}(\tau) + \psi\right\}(-2c\tau - \tau\Delta d); \tag{4.12}$$

$$(a_t + \psi_t)\ell_t = \left\{\tau^2\left[8c^2\left(t - \frac{T}{2}\right)\right] + \mathcal{O}(\tau)\right\}\left\{-2\tau c\left(t - \frac{T}{2}\right)\right\}; \tag{4.13}$$

$$\begin{aligned}D\ell(a) &= \tau Dd(a) = \tau Dd\left\{\tau^2 4c^2\left(t - \frac{T}{2}\right)^2 - \tau^2|Dd|^2 + \mathcal{O}(\tau)\right\}\\ &= \tau^3 Dd(|Dd|^2) + \mathcal{O}(\tau), \quad (4.14)\end{aligned}$$

where by a property of the Levi-Civita connection [Le.1], and the definition of $D^2 d$ in (1.4), one has

$$Dd(|Dd|^2) = Dd\langle Dd, Dd\rangle = 2\langle D_{Dd}(Dd), Dd\rangle = 2D^2 d(Dd, Dd). \tag{4.15}$$

Inserting (4.12)–(4.15) along with $D\ell(\psi) = \tau Dd(\psi)$ into (4.11), we obtain

$$\begin{aligned}\tilde{B} &= 2\tau^3(2c + \Delta d - \alpha)|Dd|^2 + 4\tau^3 D^2 d(Dd, Dd)\\ &\quad + 8\tau^3 c^2(\alpha - 6c - \Delta d)\left(t - \frac{T}{2}\right)^2 + \mathcal{O}(\tau^2), \quad (4.16)\end{aligned}$$

and (4.7), (4.8) are proved, as desired.

We finally verify (4.5). To this end, we insert formulas (4.4), (4.9) (left) and (4.15) into the RHS of inequality (3.11), use $D\psi = \tau D\alpha$ from (4.4) as well as

$$|v\langle D\psi, Dv\rangle| \leq \epsilon|Dv|^2 + \frac{\tau}{4\epsilon}|D\alpha|^2 v^2, \tag{4.17}$$

and obtain (4.5), as desired. □

Inequality (4.5), as well as formula (4.7), (4.8) for $\tilde{B}$ are exactly the same as those obtain in [L-T-Z.1, Theorem 4.1, Eqns. (4.6) and (4.8)] in the Euclidean case. Thus, we next proceed following [L-T-Z.1]. Namely, the pointwise estimate of interest in Corollary 4.2 below is obtained for function $d(x) \in C^3(\overline{\Omega})$, $\alpha \in C^1(\overline{\Omega})$, such that the following three inequalities hold true:

(H.1)

$$\Delta d - 2c - \alpha \geq \rho > 0, \qquad \forall\, x \in \overline{\Omega}; \tag{4.18}$$

(H.2)

$$D^2 d(X, X) \geq 2|X|_g^2, \qquad \forall\, x \in \Omega,\ \forall\, X \in M_x; \tag{4.19}$$

(H.3)

$$[2c + \Delta d - \alpha]|Dd|^2 + 2D^2 d(Dd, Dd) - 4c^2(\Delta d + 6c - \alpha)\left(t - \frac{T}{2}\right)^2 \geq \tilde{\beta} > 0,\ \forall\, (t, x) \in Q^*(\sigma^*), \tag{4.20}$$

where $Q^*(\sigma^*)$ is the subset of $[0, T] \times \Omega$ defined in (1.16). As noted in Section 1, all these three inequalities hold true, in particular, in the case of our interest under assumptions (A.1) and (A.2) (subject to the translation/rescaling conditions in (1.4), (1.5)) and the choice $\alpha(x) = \Delta d(x) - c - 1$ in (1.10), as to obtain properties $(\mathrm{p}_1) = (1.11)$, which is (H.1) = (4.18) with $\rho = 1 - c > 0$—as well as $(\mathrm{p}_2') = (1.22)$—which is (H.3) = (4.20) with $\tilde{\beta} = 4(1 + 7c)\sigma^* > 0$. This is the content of the next corollary.

Corollary 4.2. With $0 < c < 1$ chosen in (1.6d), let $d(x) \in C^3(\overline{\Omega})$, and $\alpha(x) \in C^1(\overline{\Omega})$ be two functions such that inequalities (H.1) = (4.18), (H.2) = (4.19), (H.3) = (4.20) hold true. As seen in Section 1: (1.11)–(1.22), this is the case, in particular if $d(x)$ is the strictly convex function provided by assumption (A.1) and (A.2) and suitably rescaled as to achieve the conditions in (1.4), (1.5), and then $\alpha(x)$ is chosen to be (see (1.10)):

$$\alpha(x) \equiv \Delta d(x) - c - 1, \tag{4.21}$$

in which case $\rho = 1 - c > 0$ and $\tilde{\beta} = 4(1 + 7c)\sigma^* > 0$, see (1.22). Let $w \in C^2(\mathbb{R}_t \times \Omega)$. Then, with such choices in (4.2), (4.3) for $\ell(t, x)$ and $\psi(x)$, respectively, Theorem 4.1 specializes to the following results for all $T > T_0$: for all $\epsilon > 0$ suitably small and all $\tau \geq 1$, the following inequality holds true:

$$\theta^2(\mathcal{A}w)^2 + 2\widehat{\operatorname{div}}\left\{[(a + \psi)v^2 - v_t^2 + |Dv|^2]\hat{D}\ell + [2\hat{D}\ell(v) - \psi v]\hat{D}v\right\} \geq \tau\epsilon\rho\theta^2[w_t^2 + |Dw|^2] + B\theta^2 w^2,\ \forall\, t \in [0, T],\ \forall\, x \in \overline{\Omega}, \tag{4.22}$$

where, recalling $\tilde{B}$ from (4.7), (4.8), we have for $0 < \epsilon < \min\{2\rho, 1\}$:

$$\begin{cases} B = \tilde{B} - 2\epsilon\rho\tau^3(\phi_t^2 + |D\phi|^2) \geq \tilde{B} - 2\epsilon\rho\tau^3 r & (4.23\text{a}) \\ r = \max\limits_{\overline{Q}}(\phi_t^2 + |D\phi|^2), \quad |D\phi| = |Dd| & (4.23\text{b}) \end{cases}$$

Thus, for $\epsilon > 0$ suitably small, the constant $\beta = \beta_\epsilon \equiv \tilde{\beta} - \epsilon\rho r$ is positive via (4.20), and recalling (4.7), (4.8), (4.20), we obtain from (4.23):

$$\begin{cases} Bw^2 \geq [2\tau^3\beta + \mathcal{O}(\tau^2)]w^2, \qquad \forall\ (t,x) \in Q^*(\sigma^*); & (4.24) \\ \beta = \beta_\epsilon = [\tilde{\beta} - \epsilon\rho r] > 0,\ B = \mathcal{O}(\tau^3), \quad \forall\ (x,t) \in [0,T] \times \Omega, & (4.25) \end{cases}$$

where $Q^*(\sigma^*)$ is the subset of Q defined by (1.16).

Proof. First, preliminarily, we note that by (4.21), taking $0 < \epsilon < \min\{2\rho, 1\}$, we obtain for $2\tau \geq 1$:

$$\alpha - 2c - \Delta d - \frac{\epsilon}{2\tau} = -1 - 3c - \frac{\epsilon}{2\tau} \geq -3 - c, \text{ for } 0 < \epsilon < [2\rho, 1],\ 2\tau \geq 1. \tag{4.26}$$

Moreover, the formulas

$$v_t = \tau\phi_t v + \theta w_t; \quad Dv = \tau v Dd + \theta Dw,\ v = \theta w, \tag{4.27}$$

obtained by $v = \theta w$ and (3.10) imply the inequalities (see [L-T-Z.1, (4.28)]

$$2v_t^2 \geq \theta^2 w_t^2 - 2\tau^2\phi_t^2 v^2; \quad 2|Dv|^2 \geq \theta^2|Dw|^2 - 2\tau^2|Dd|^2 v^2,\ v^2 = \theta^2 w^2. \tag{4.28}$$

After these preliminaries, we return to the RHS of (4.5), use here (4.21), (4.26), (1.4) and (4.28) and obtain $0 < \epsilon < 1$:

$$\begin{aligned} \text{RHS of (4.5)} \quad &\equiv \quad 2\tau(\Delta d - 2c - \alpha)v_t^2 + 2\tau\left(\alpha - 2c - \Delta d - \frac{\epsilon}{2\tau}\right)|Dv|^2 \\ &\quad + 4\tau D^2 d(Dv, Dv) + \theta^2\tilde{B}w^2 & (4.29) \\ \text{(by (4.21), (4.26), (1.4))} \quad &\geq \quad 2\tau\rho v_t^2 + 2\tau\rho|Dv|^2 + \theta^2\tilde{B}w^2 & (4.30) \\ \text{(by (4.28))} \quad &\geq \quad \tau\rho\left\{[\theta^2 w_t^2 - 2\tau^2\phi_t^2\theta^2 w^2] + [\theta^2|Dw|^2\right. \\ &\quad - \left. 2\tau^2|Dd|^2\theta^2 w^2]\right\} + \tilde{B}\theta^2 w^2 & (4.31) \\ &\geq \quad \tau\epsilon\rho\theta^2\{w_t^2 + |Dw|^2\} \\ &\quad + [\tilde{B} - 2\epsilon\rho\tau^3(\phi_t^2 + |Dd|^2)]\theta^2 w^2, & (4.32) \end{aligned}$$

and (4.32) establishes (4.22) and (4.23a) (left). Then, (4.23a) right follows via (4.23d). Finally, (4.7) and (4.23a) show (4.24), (4.25), except for the positive sign of $\tilde{\beta}$. We finally show that $\tilde{\beta} > 0$ in the set $Q^*(\sigma^*)$ defined in (1.16) and this will yield $\beta > 0$ in $Q^*(\sigma^*)$ for $\epsilon > 0$ sufficiently small. By definition (4.8), we compute, via (4.21), (1.4a), (1.4b), (1.6a) [same computations as in (1.18)–(1.22), repeated for convenience]

$$\begin{aligned}
\tilde{\beta}(x,t) &= [2c+\Delta d-\alpha]|Dd|^2+2D^2d(Dd,Dd) \\
&\quad -4c^2[\Delta d+6c-\alpha]\left(t-\frac{T}{2}\right)^2 && (4.33) \\
\text{(by (4.21))} &= [2c+(1+c)]|Dd|^2+2D^2d(Dd,Dd) \\
&\quad -4c^2[(1+c)+6c]\left(t-\frac{T}{2}\right)^2 && (4.34) \\
\text{(by (1.4))} &\geq (5+3c)|Dd|^2-(1+7c)4c^2\left(t-\frac{T}{2}\right)^2 && (4.35) \\
\text{(by (1.5))} &\geq (5+3c)\kappa\, d(x)-(1+7c)4c^2\left(t-\frac{T}{2}\right)^2 && (4.36) \\
&\geq 4c(1+7c)\left[d(x)-c^2\left(t-\frac{T}{2}\right)^2\right] && (4.37) \\
\text{(by (1.6a))} &\geq 4c(1+7c)\phi^*(x,t)\geq 4c(1+7c)\sigma^*>0,\ \forall\ (t,x)\in Q^*(\sigma^*), && (4.38)
\end{aligned}$$

where the set $Q^*(\sigma^*)$ is the subset of $Q=[0,T\ \times\Omega$ defined by (1.16). Thus, Corollary 4.2 is proved. $\square$

We complete this section by providing one Green identity and one notational simplification, to be invoked in the subsequent Section 5.

Claim 1. For any function $f\in C^1(\mathbb{R}_t\times M)$ and any vector field $X\in\mathcal{X}(M)$, we have the following Green formula via (3.1) and Green formula for div X in (3.0),

$$\begin{aligned}
\int_Q \widehat{\text{div}}(f,X)dQ &= \int_\Omega\int_0^T f_t dt\, dQ+\int_0^T\int_\Omega \text{div}\ X\, d\Omega\, dt && (4.39) \\
&= \left[\int_\Omega f\, d\Omega\right]_0^T+\int_\Sigma\langle X,n\rangle\, d\Sigma. && (4.40)
\end{aligned}$$

Claim 2. With reference to the second term on the left-hand side of inequality

(4.22), set, via (3.1) for ℓ defined in (4.2), so that $D\ell = \tau Dd$:

$$f_1 \equiv (a+\psi)v^2 - v_t^2 + |Dv|^2 \tag{4.41}$$

$$\begin{aligned} f_2 &\equiv 2\hat{D}\ell(v) - \psi v = 2\ell_t v_t - 2\tau Dd(v) - \psi v && (4.42a) \\ &= 2\ell_t v_t - 2\tau \langle Dv, Dd\rangle - \psi v. && (4.42b) \end{aligned}$$

Then, we have by (4.31), (4.32),

$$\begin{aligned} &2\widehat{\text{div}}\left\{[(a+\psi)v^2 - v_t^2 + |Dv|^2]\hat{D}\ell + [2\hat{D}\ell(v) - \psi v]\hat{D}v\right\} \\ &= 2\widehat{\text{div}}\left\{f_1\hat{D}\ell + f_2\hat{D}v\right\} = 2\widehat{\text{div}}\left\{f_1(\ell_t, -D\ell) + f_2(v_t, -Dv)\right\} \\ &= 2\widehat{\text{div}}\left(f_1\ell_t + f_2 v_t, -f_1 D\ell - f_2 Dv\right). \end{aligned} \tag{4.43}$$

Hence, by (4.33) and (4.30),

$$\begin{aligned} &2\int_Q \widehat{\text{div}}\left\{[(a+\psi)v^2 - v_t^2 + |Dv|^2]\hat{D}\ell + [2\hat{D}\ell(v) - \psi v]\hat{D}v\right\} dQ \\ &= 2\int_Q \widehat{\text{div}}(f_1\ell_t + f_2 v_t, -f_1 D\ell - f_2 Dv) dQ \\ \text{(by (4.30))} \quad &= 2\left[\int_\Omega (f_1\ell_t + f_2 v_t) d\Omega\right]_0^T - 2\int_\Sigma \langle f_1 D\ell + f_2 Dv, n\rangle d\Sigma. \end{aligned} \tag{4.44}$$

5 Carleman estimates for smooth solutions of Eqn. (1.1a). First version

The next key result yields a Carleman-type estimate.

Theorem 5.1. With $0 < c < 1$ chosen in (1.6a), let $d(x) \in C^3(\overline{\Omega})$, $\alpha(x) \in C^1(\overline{\Omega})$ be two functions such that inequalities (H.1) = (4.18), (H.2) = (4.19), (H.3) = (4.20) hold true. As seen in Section 1: (1.11)–(1.22), this is the case, in particular, if $d(x)$ is the strictly convex function provided by assumptions (A.1) and (A.2), and then $\alpha(x) = \Delta d(x) - c - 1$, as in (4.21) =(1.11). Let $\phi(x,t)$ be the pseudo-convex function defined by (1.6). Let $w \in C^2(\mathbb{R}_t \times \Omega)$ be a solution of Eqn. (1.1) [and no B.C.] under the standing assumptions (1.2) for $F(w)$ and (1.3) for f. Then the following one-parameter family of estimates holds true, with $\rho > 0$ (see (4.18)): $\rho = 1 - c$ under (A.1), (A.2)), and $\beta > 0$ (see (4.25)):

(i) for all $\tau > 0$ sufficiently large and any $\epsilon > 0$ small,

$$BT|_\Sigma + 2\int_0^T \int_\Omega e^{2\tau\phi} f^2 dQ + C_{1,T} e^{2\tau\sigma} \int_0^T \int_\Omega w^2 dQ$$

$$\geq \; [\tau\epsilon\rho - 2C_T]\int_0^T\int_\Omega e^{2\tau\phi}[w_t^2 + |Dw|^2]dQ$$

$$+ \; (2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T)\int_{Q(\sigma)} e^{2\tau\phi}w^2 dx\, dt - C_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)], \tag{5.1}$$

where $Q(\sigma)$ is the subset of $[0,T]\times\Omega$ defined by (1.15), and where we recall from (4.25) (left) that β depends on ϵ.

(ii) for all $\tau > 0$ sufficiently large and any $\epsilon > 0$ small,

$$BT|_\Sigma + 2\int_0^T\int_\Omega e^{\tau\phi}f^2 dQ \;\geq\; [\tau\epsilon\rho - 2C_T]e^{2\tau\sigma}\int_{t_0}^{t_1}[w_t^2 + |Dw|^2]d\Omega\, dt$$

$$- \; C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt - C_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)]. \tag{5.2}$$

Here, $\delta > 0$, $\sigma > 0$, and $\sigma > -\delta$ are the constants in (1.6c) (1.8), while C_T is a positive constant depending on T and d. Moreover, the boundary terms $BT|_\Sigma$, $\Sigma = [0,T]\times\Gamma$, are defined by

$$BT|_\Sigma = -2\int_\Sigma [f_1\langle D\ell, n\rangle + f_2\langle Dv, n\rangle]d\Sigma, \tag{5.3}$$

with f_1 and f_2 defined by (4.31), (4.32), for which an explicit formula in terms of w will be provided in Proposition 5.2 below. Moreover, as in (2.1), we have set

$$E(t) = \int_\Omega [w_t^2 + |Dw|^2 + w^2]d\Omega. \tag{5.4}$$

(iii) The above inequality may be extended to all solutions

$$w \in H^{2,2}(Q) \equiv L_2(0,T;H^2(\Omega)) \cap H^2(0,T;L_2(\Omega)).$$

Proof. Step 1. (i) By (3.3), we rewrite Eqn. (1.1) as

$$w_{tt} - \Delta w = \mathcal{A}w = F(w) + f. \tag{5.5}$$

Next, with $w \in C^2(\mathbb{R}_t\times\Omega)$, we return to inequality (4.22) of Corollary 4.2 and integrate it over $Q = (0,T]\times\Omega$. Recalling Claim 2 above, Eqn. (4.31), (4.32), (4.34), we obtain

$$\int_Q \theta^2(\mathcal{A}w)^2 dQ + 2\int_Q \widehat{\operatorname{div}}\left\{[(a+\psi)v^2 - v_t^2 + |Dv|^2]\hat{D}\ell + [2\hat{D}\ell(v) - \psi v]\hat{D}v\right\} dQ$$

$$
\begin{aligned}
&= \int_Q \theta^2(\mathcal{A}w)^2 dQ + 2\left[\int_\Omega (f_1\ell_t + f_2 v_t) d\Omega\right]_0^T \\
&\quad - 2\int_\Sigma f_1\langle D\ell, n\rangle d\Sigma - 2\int_\Sigma f_2\langle Dv, n\rangle d\Sigma \qquad (5.6) \\
&\geq \tau\epsilon\rho \int_Q \theta^2[w_t^2 + |Dw|^2] dQ + \int_Q \theta^2 B w^2 dQ. \qquad (5.7)
\end{aligned}
$$

We now analyze and estimate the terms in (5.7). First, since $Q \equiv Q^*(\sigma^*) \cup [Q^*(\sigma^*)]^c$, where $[\quad]^c$ denotes complement in Q, we obtain via (4.24),

$$
\begin{aligned}
\int_Q \theta^2 B w^2 dQ &= \int_{Q^*(\sigma^*)} \theta^2 B w^2 dx\, dt + \int_{[Q^*(\sigma^*)]^c} \theta^2 B w^2 dx\, dt \\
&\geq [2\tau^3\beta + \mathcal{O}(\tau^2] \int_{Q^*(\sigma^*)} \theta^2 w^2 dx\, dt + \int_{[Q^*(\sigma^*)]^c} \theta^2 B w^2 dx\, dt. \qquad (5.8)
\end{aligned}
$$

Next, via (5.5) and (1.2b), we obtain

$$
\begin{aligned}
\int_Q \theta^2(\mathcal{A}w)^2 dQ &= \int_Q \theta^2[F(w) + f]^2 dQ \\
&\leq 2C_T \int_Q \theta^2[w_t^2 + |Dw|^2 + w^2] dQ + 2\int_Q \theta^2 f^2 dQ. \qquad (5.9)
\end{aligned}
$$

Inserting (5.8), (5.9) in (5.7), we obtain via (5.3),

$$
\begin{aligned}
& BT|_\Sigma + 2\int_Q \theta^2 f^2 dQ - \int_{[Q^*(\sigma^*)]^c} \theta^2 B w^2 dx\, dt \\
&\geq [\tau\epsilon\rho - 2C_T] \int_Q \theta^2[w_t^2 + |Dw|^2] dQ \\
&\quad + [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q^*(\sigma^*)} \theta^2 w^2 dx\, dt - 2C_T \int_Q \theta^2 w^2 dQ \\
&\quad - 2\left[\int_\Omega (f_1\ell_t + f_2 v_t) d\Omega\right]_0^T. \qquad (5.10)
\end{aligned}
$$

Step 2. By (4.25), we have $B = \mathcal{O}(\tau^3)$ on $(0, T] \times \Omega$. Moreover, we have that $\phi \leq \sigma^*$ on $[Q^*(\sigma^*)]^c$ by (1.14) and the very definition (1.16). Finally, we have chosen

$0 < \sigma^* < \sigma$ below (1.16) and $[Q^*(\sigma^*)]^c$ is a subset of $[0,T] \times \Omega$. Hence, we obtain

$$
\begin{aligned}
-\int_{[Q^*(\sigma^*)]^c} e^{2\tau\phi} B w^2 dx\, dt &= \mathcal{O}(\tau^3) e^{2\tau\sigma^*} \int_{[Q^*(\sigma^*)]^c} w^2 dx\, dt \\
&\leq C_T e^{2\tau\sigma} \int_Q w^2 dx\, dt. \qquad (5.11)
\end{aligned}
$$

Step 3. Similarly, for $Q = Q(\sigma) \cup [Q(\sigma)]^c$ with $\phi \leq \sigma$ on $[Q(\sigma)]^c$, we obtain

$$
\begin{aligned}
-2C_T \int_Q \theta^2 w^2 dQ &= -2C_T \int_Q e^{2\tau\phi} w^2 dQ \\
&= -2C_T \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt - 2C_T \int_{[Q(\sigma)]^c} e^{2\tau\phi} w^2 dx\, dt \\
&\geq -2C_T \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt - 2C_T e^{2\tau\sigma} \int_{[Q(\sigma)]^c} w^2 dx\, dt. \qquad (5.12)
\end{aligned}
$$

Moreover, since $Q^*(\sigma^*) \supset Q(\sigma)$, see (1.17), we have via (5.12) for the two right terms of (5.10):

$$
\begin{aligned}
&[2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q^*(\sigma^*)} e^{2\tau\phi} w^2 dx\, dt - 2C_T \int_Q e^{2\tau\phi} w^2 dQ \\
&\geq [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt - 2C_T \int_Q e^{2\tau\phi} w^2 dQ \qquad (5.13) \\
\text{(by (5.12))} \quad &\geq [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt \\
&\quad - 2C_T e^{2\tau\sigma} \int_{[Q(\sigma)]^c} w^2 dx\, dt, \qquad (5.14)
\end{aligned}
$$

where in the last step we have invoked (5.12).

Finally, since $[Q(\sigma)]^c$ is a subset of $[0,T] \times \Omega$, we obtain from (5.14):

$$
\begin{aligned}
&[2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q^*(\sigma^*)} e^{2\tau\phi} w^2 dx\, dt - 2C_T \int_Q e^{2\tau\phi} w^2 dQ \\
&\geq [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt - 2C_T e^{2\tau\sigma} \int_Q w^2 dx\, dt. \qquad (5.15)
\end{aligned}
$$

We return to (5.10): on its left side we use the identity in (5.11), while on its right side we use inequality (5.15). We thus obtain

$$BT|_\Sigma + 2\int_Q e^{2\tau\phi} f^2 dQ + \mathcal{O}(\tau^3) e^{2\tau\sigma^*} \int_{[Q^*(\sigma^*)]^c} w^2 dx\, dt + 2C_T e^{2\tau\sigma} \int_{[Q(\sigma)]^c} w^2 dx\, dt$$

$$\geq \; [\tau\epsilon\rho - 2C_T] \int_0^T \int_\Omega e^{2\tau\phi} [w_t^2 + |Dw|^2] dQ$$

$$+ \; [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt - 2\left[\int_\Omega (f_1 \ell_t + f_2 v_t) d\Omega\right]_0^T . \quad (5.16)$$

On the other hand, if this time we use the inequality in (5.11), as well as the fact that $[Q(\sigma)]^c$ is a subset of Q, we obtain

$$BT|_\Sigma + 2\int_Q e^{2\tau\phi} f^2 dQ + C_{1T} e^{2\tau\sigma} \int_Q w^2 dQ \geq \text{LHS of (5.16)} \geq \text{RHS of (5.16)}. \quad (5.17)$$

Step 4. We shall show in Step 5 below that

$$|f_1 \ell_t + f_2 v_t| \leq C_T \tau^3 \theta^2 [w_t^2 + |Dw|^2 + w^2]. \quad (5.18)$$

Once (5.17) is proved, it follows recalling (5.4) for $E(t)$ and (1.7) that

$$2\left|\left[\int_\Omega (f_1 \ell_t + f_2 v_t) d\Omega\right]_0^T\right| \;\leq\; C_T \tau^3 \left[\int_\Omega e^{2\tau\phi} [w_t^2 + |Dw|^2 + w^2] d\Omega\right]_0^T$$

$$\text{(by (1.7))} \qquad \leq \; C_T \tau^3 e^{-2\tau\delta} [E(T) + E(0)], \quad (5.19)$$

where the first inequality is due to (5.17).

Inserting (5.19) into the right side of (5.17) and invoking the RHS of (5.16) yields the estimate

$$BT|_\Sigma + 2\int_Q e^{2\tau\phi} f^2 dQ + C_{1T} e^{2\tau\sigma} \int_Q w^2 dQ$$

$$\geq \; [\tau\epsilon\rho - 2C_T] \int_0^T \int_\Omega e^{2\tau\phi} [w_t^2 + |Dw|^2] dQ$$

$$+ \; [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\, dt$$

$$- \; C_T \tau^3 e^{-2\tau\delta} [E(T) + E(0)], \quad (5.20)$$

which is precisely the sought-after estimate (5.1).

Step 4. It remains to show estimate (5.18). We compute

$$|f_1\ell_t + f_2 v_t| \leq |f_1\ell_t| + |f_2 v_t|, \tag{5.21}$$

where, recalling (4.41) for f_1 and (4.42) for f_2, we shall show below that

$$|f_1\ell_t| \leq C_T[\tau^3 v^2 + \tau v_t^2 + \tau|Dv|^2]; \tag{5.22}$$

$$|f_2 v_t| \leq C_T\tau[v^2 + v_t^2 + |Dv|^2]. \tag{5.23}$$

Taking momentarily (5.22) and (5.23) for granted, we next return to the relationships (4.27),

$$v_t = \tau\phi_t\theta w + \theta w_t; \quad Dv = \tau v Dd + \theta Dw, \tag{5.24}$$

and obtain with $v = \theta w$,

$$\begin{cases} v_t^2 \leq 2[\theta^2 w_t^2 + \tau^2\phi_t^2\theta^2 w^2] \leq C_T\theta^2[w_t^2 + \tau^2 w^2]; & (5.25) \\ |Dv|^2 \leq 2[\theta^2|Dw|^2 + \tau^2\theta^2 w^2|Dd|^2] \leq C_T\theta^2[|Dw|^2 + \tau^2 w^2]. & (5.26) \end{cases}$$

Using $v = \theta w$, (5.25), (5.26) in (5.22) and (5.23), respectively, we readily obtain

$$|f_1\ell_t| \leq C_T\tau^3\theta^2[w^2 + w_t^2 + |Dw|^2]; \tag{5.27}$$

$$|f_2 v_t| \leq C_T\tau^3\theta^2[w^2 + w_t^2 + |Dw|^2]. \tag{5.28}$$

Using (5.27), (5.28) into (5.21) yields

$$|f_1\ell_t + f_2 v_t| \leq C_T\tau^3\theta^2[w^2 + w_t^2 + |Dw|^2], \tag{5.29}$$

which is precisely the sought-after inequality (5.15). It remains to show (5.22) and (5.23).

Proof of (5.22). By (4.41) on f_1, we compute

$$\begin{aligned} |f_1\ell_t| &= |[(a+\psi)v^2 - v_t^2 + |Dv|^2]\ell_t| \\ &\leq C_T[\tau^2 v^2 + v_t^2 + |Dv|^2]\tau, \end{aligned} \tag{5.30}$$

since $a = \mathcal{O}(\tau^2)$ by (4.6), $\psi = \tau\alpha$ by (4.3), and $\ell_t = \mathcal{O}(\tau)$ by (4.4).
Then (5.28) proves (5.22), as desired.

Proof of (5.23). By (4.42) on f_2 and $\psi = \tau\alpha$ via (4.3), we compute

$$\begin{aligned} |f_2 v_t| &= |2\ell_t v_t^2 - 2\tau v_t\langle Dd, Dv\rangle - \tau\alpha\, v v_t| \\ &\leq C_T\tau[v^2 + v_t^2 + |Dv|^2], \end{aligned} \tag{5.31}$$

since $\ell_t = \mathcal{O}(\tau)$ by (4.4) and (5.31) proves (5.23), as desired.

The proof of part (i) of Theorem 5.1—that is, estimate (5.1)—is complete.

(ii) We take τ sufficiently large so that, since $\beta > 0$ by assumption, see (4.25), we then have that the term $[2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T]$ is positive and we then drop the corresponding lower-order *interior* term involving w^2 in (5.1). Moreover, we invoke the critical property (1.8) for ϕ on the first integral term on the right side of (5.1). Finally, we majorize $\int_\Omega w^2(t)d\Omega$ by $E(t)$, see (5.4), thus obtaining the term $e^{2\tau\sigma}\int_0^T E(t)dt$ on the RHS of (5.2). This way, (5.1) readily yields (5.2). □

To complement Theorem 5.1, we next express the boundary terms $BT|_\Sigma$ in (5.3) explicitly in terms of w, not v.

Proposition 5.2. With reference to Theorem 5.1, in particular (5.3), we have the following explicit expression for the boundary terms $BT|_\Sigma$:

$$BT|_\Sigma = -2\int_\Sigma [f_1\langle D\ell, n\rangle + f_2\langle Dv, n\rangle]d\Sigma \quad (5.32)$$

$$= -2\tau\int_\Sigma (f_1 + \theta w f_2)\langle Dd, n\rangle]d\Sigma - 2\int_\Sigma \theta f_2\langle Dw, n\rangle d\Sigma, \quad (5.33)$$

where $\langle Dd, n\rangle = \frac{\partial d}{\partial n}$ and $\langle Dw, n\rangle = \frac{\partial w}{\partial n}$ and where

$$\begin{aligned} f_1 + \theta w f_2 &= \theta^2\{[|Dw|^2 - w_t^2] \\ &\quad + [a - \tau^2(|Dd|^2 + \phi_t^2) + 2\tau\ell_t\phi_t]w^2 + 2[\ell_t - \tau\phi_t]ww_t\}; \end{aligned} \quad (5.34)$$

$$\theta f_2 = 2\theta^2\{\ell_t[w_t + \tau\phi_t w] - \tau\langle Dw, Dd\rangle - \tau^2 w|Dd|^2\} - \tau\theta\alpha w. \quad (5.35)$$

Proof. To pass from the definition (5.32) = (5.3) to (5.33), we merely use $D\ell = \tau Dd$ by (4.4) and $Dv = \theta Dw + \tau\theta w\, Dd$ by (4.27). Next, recalling the definition (4.41), (4.42) for f_1 and f_2, along with the formulas in (4.27) for v_t and Dv where $\psi = \tau\alpha$, we readily obtain

$$\begin{aligned} f_1 &= (a + \psi)v^2 - v_t^2 + |Dv|^2 \\ &= \theta^2\{(a + \psi)w^2 - (w_t + \tau\phi_t w)^2 + |Dw + \tau w\, Dd|^2; \end{aligned} \quad (5.36)$$

$$\begin{aligned} f_2 &= 2\ell_t v_t - 2\tau\langle Dv, Dd\rangle - \tau\alpha v \\ &= 2\theta\{\ell_t[w_t + \tau\phi_t w] - \tau\langle Dw, Dd\rangle - \tau^2 w|Dd|^2\} - \tau\theta\alpha w. \end{aligned} \quad (5.37)$$

Then (5.36) and (5.37) yield

$$\begin{aligned} f_1 + \theta w f_2 &= \theta^2\{(a + \psi)w^2 - (w_t + \tau\phi_t w)^2 + |Dw + \tau w\, Dd|^2\} \\ &\quad + 2\theta^2\{w\ell_t[w_t + \tau\phi_t w] - \tau w\langle Dw, Dd\rangle - \tau^2 w^2|Dd|^2\} - \tau\theta\alpha w^2, \end{aligned} \quad (5.38)$$

from which (5.34) follows using $\psi = \tau\alpha$ and

$$|Dw + \tau Dd|^2 - 2\tau w\langle Dw, Dd\rangle - 2\tau^2 w^2|Dd|^2 = |Dw|^2 - \tau^2 w^2 |Dd|^2. \tag{5.39}$$

Proposition 5.2 is proved. $\square$

6 Carleman estimate for smooth solutions of Eqn. (1.1a). Second version

A preliminary equivalence. Let $u \in H^1(\Omega)$ and let $\tilde{\Gamma}_1$ be any (fixed) portion of the boundary Γ with positive measure. Then the following inequality holds true: there exist positive constants $0 < \tilde{k}_1 < \tilde{k}_2 < \infty$, independent of u, such that

$$\tilde{k}_1 \int_\Omega [u^2 + |Du|^2] d\Omega \leq \int_\Omega |Du|^2 d\Omega + \int_{\tilde{\Gamma}_1} u^2 d\Gamma \leq \tilde{k}_2 \int_\Omega [u^2 + |Du|^2] d\Omega, \tag{6.1}$$

see [L-T-Z.1, Eqn. (6.1)]. Next, for $w \in C^2(\mathbb{R}_t \times \Omega)$, or even $w \in H^{2,2}(Q)$, we introduce

$$\mathcal{E}(t) \equiv \int_\Omega [w_t + |Dw|^2] d\Omega + \int_{\Gamma_1} w^2 d\Gamma_1, \tag{6.2}$$

with $\Gamma_1 = \Gamma \backslash \Gamma_0$, where Γ_0 will be the uncontrolled/unobserved portion of the boundary to be defined in Section 8: in (8.2) in the Dirichlet case and in (9.2) in the Neumann case. Recalling $E(t)$ from (2.1) = (5.4), we then see that (6.1) yields the equivalence

$$k_1 E(t) \leq \mathcal{E}(t) \leq k_2 E(t), \tag{6.3}$$

for some positive constants $k_1 > 0$, $k_2 > 0$.

We can state the main result of the present section.

Theorem 6.1. With $0 < c < 1$ chosen in (1.6d), let $d(x) \in C^3(\overline{\Omega})$, $\alpha(x) \in C^1(\overline{\Omega})$ be two functions such that inequalities (H.1) = (4.18), (H.2) = (4.19), (H.3) = (4.20) hold true. As seen in Section 1; (1.11)–(1.22), this is the case, in particular, if $d(x)$ is the strictly convex function provided by assumptions (A.1) and (A.2) and suitably scaled as to achieve conditions (1.4b), and then $\alpha(x) = \Delta d(x) - c - 1$, as in (1.10) = (4.21). Let $\phi(t,x)$ be the pseudo-convex function defined by (1.6) and define $\theta(t,x) = \exp(\tau\phi(t,x))$ as in (4.3). Finally, let $w \in H^{2,2}(Q)$ be a solution of Eqn. (1.1a) [and no B.C.], subject to the standing assumptions (1.2) on $F(w)$ and (1.3) on f. Then, the following one-parameter family of estimates hold true, for all τ sufficiently large, and any $\epsilon > 0$ small as in (4.22):

$$\overline{BT}|_\Sigma + \int_0^T \int_\Omega e^{2\tau\phi} f^2 dQ + \text{const}_\phi \int_0^T \int_\Omega f^2 dQ$$

$$\geq \Big\{\Big[\frac{k_1}{2}(\epsilon\tau\rho - 2C_T)(t_1 - t_2)e^{-C_T T}$$

$$- \frac{C_{1,T}k_2}{2k_1} T e^{C_T T}\Big] e^{2\tau\sigma} - C_T\tau^3 e^{-2\tau\delta}\Big\}[E(0) + E(T)] \tag{6.4}$$

$$\geq k_\phi[E(0) + E(T)], \text{ for a constant } k_\phi > 0, \tag{6.5}$$

since $\sigma > -\delta$, see (1.8). Here, the boundary terms $\overline{BT}|_\Sigma$ are given in terms of the boundary terms $BT|_\Sigma$ in (5.3) by

$$\overline{BT}|_\Sigma = BT|_\Sigma + C_T(\tau+1)e^{2\tau\sigma}\Big[\int_0^T\int_\Gamma \Big|w_t\frac{\partial w}{\partial n}\Big| d\Sigma$$

$$+ \int_0^T\int_{\Gamma_1} |ww_t| d\Sigma_1 + \int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt\Big]. \tag{6.6}$$

Proof. Step 1. We return to estimate (5.2) of Theorem 5.1 (ii), add the term $(\tau\epsilon\rho - 2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt$ to both sides, recall (6.2) for $\mathcal{E}(t)$ and obtain

$$BT|_\Sigma + (\epsilon\tau\rho - 2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt + 2\int_0^T\int_\Omega e^{2\tau\phi} f^2 dQ$$

$$\geq (\tau\epsilon\rho - 2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\mathcal{E}(t)dt$$

$$- C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt - C_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)]. \tag{6.7}$$

The remaining part of the proof follows closely [L-T-Z.1, Proof of Theorem 6.1], [L-T.4] and is included for completeness.

Step 2. In a standard way, multiplying Eqn. (1.1) by w_t and integrating over Ω yields, after an application of the first Green's identity

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega [w_t^2 + |Dw|^2]d\Omega + \int_{\Gamma_1} w^2 d\Gamma_1\right)$$

$$= \int_\Gamma \frac{\partial w}{\partial n} w_t d\Gamma + \int_{\Gamma_1} ww_t d\Gamma_1 + \int_\Omega [F(w) + f]w_t d\Omega. \tag{6.8}$$

Notice that on both sides of (6.8) we have added the term $\frac{1}{2}\frac{\partial}{\partial t}\int_{\Gamma_1} w^2 d\Gamma_1 = \int_{\Gamma_1} ww_t d\Gamma_1$. Recalling $\mathcal{E}(t)$ in (6.2), we integrate (6.8) over (s,t) and obtain

$$\mathcal{E}(t) = \mathcal{E}(s) + 2\int_s^t \Big[\int_\Gamma \frac{\partial w}{\partial n} w_t d\Gamma + \int_{\Gamma_1} ww_t d\Gamma_1\Big] dr + 2\int_s^t [F(w) + f]w_t d\Omega\, dr. \tag{6.9}$$

We apply Schwarz inequality on $[F(w)+f]w_t$, recall estimate (1.2) for $F(w)$, invoke the left side $E(t) \leq \frac{1}{k_1}\mathcal{E}(t)$ of equivalence (6.3), and obtain

$$\mathcal{E}(t) \;\leq\; [\mathcal{E}(s)+N(T)] + C_T\int_s^t \mathcal{E}(r)dr; \tag{6.10}$$

$$\mathcal{E}(s) \;\leq\; [\mathcal{E}(t)+N(T)] + C_T\int_s^t \mathcal{E}(r)dr, \tag{6.11}$$

(C_T includes the constant $\frac{1}{k_1}$ of equivalence), where we have set

$$N(T) = \int_0^T\int_\Omega f^2 dQ + 2\int_0^T\int_\Gamma \left|\frac{\partial w}{\partial n} w_t\right| d\Sigma + 2\int_0^T\int_{\Gamma_1} |ww_t| d\Sigma_1. \tag{6.12}$$

Gronwall's inequality applied on (6.10), (6.11) then yields for $0 \leq s \leq t \leq T$,

$$\mathcal{E}(t) \leq [\mathcal{E}(s)+N(T)]e^{C_T(t-s)};\; \mathcal{E}(s) \leq [\mathcal{E}(t)+N(T)]e^{C_T(t-s)}. \tag{6.13}$$

Set $t=T$ and $s=t$ in the first (left) inequality of (6.13); and set $s=0$ in the second (right) inequality of (6.13), to obtain

$$\mathcal{E}(T) \leq [\mathcal{E}(t)+N(T)]e^{C_T T}; \quad \mathcal{E}(0) \leq [\mathcal{E}(t)+N(T)]e^{C_T T}. \tag{6.14}$$

Summing up these two inequalites in (6.14) yields for $0 \leq t \leq T$,

$$\mathcal{E}(t) \;\geq\; \frac{\mathcal{E}(T)+\mathcal{E}(0)}{2}\, e^{-C_T T} - N(T) \tag{6.15}$$

$$\geq\; \frac{k_1}{2}[E(T)+E(0)]e^{-C_T T} - N(T), \tag{6.16}$$

after recalling the left side of the equivalence in (6.3). Similarly, summing up the left inequality of (6.13) for $s=0$ and the right inequality of (6.13) for $s=t$ and $t=T$, and using the equivalence (6.3) yields for $0 \leq t \leq T$,

$$E(t) \leq \frac{1}{k_1}\left[\frac{k_2(E(0)+E(T))}{2} + N(T)\right] e^{C_T T}, \tag{6.17}$$

and hence, by (6.17),

$$-C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt \;\geq\; -\frac{C_{1,T}k_2}{2k_1}\, Te^{C_T T}e^{2\tau\sigma}[E(0)+E(T)]$$

$$-\frac{C_{1,T}T}{k_1}\, e^{C_T T}e^{2\tau\sigma}N(T). \tag{6.18}$$

Step 3. We insert (6.16) into the first integral on the right side of (6.7) and use (6.18) and readily obtain (6.4), (6.6), by invoking (6.12) for $N(T)$.

Finally, we recall the critical relation $\sigma > 0$, $\delta > 0$, $\sigma > -\delta$ from (1.7), (1.18), so that $[\epsilon\tau e^{2\tau\sigma} - \tau^3 e^{-2\tau\delta}]$ is positive for all τ large enough. Thus, (6.4) yields (6.5). □

7 Extension of estimates to finite energy solutions

So far our estimates have been stated and proved for $C^2(\mathbb{R}_t \times \Omega)$-solutions, hence $H^{2,2}(Q)$-solutions (Theorem 5.1(iii), Theorem 6.1) of Eqn. (1.1) with $f \in L_2(Q)$ as in (1.3). In this section, we point out that it is possible to extend all our previous estimates to finite energy solutions of Eqn. (1.1) in the following class

$$\begin{cases} w \in H^{1,1}(Q) \equiv L_2(0,T;H^1(\Omega)) \cap H^1(0,T;L_2(\Omega)); \\ w_t, \dfrac{\partial w}{\partial n} \in L_2(0,T;L_2(\Gamma)). \end{cases} \tag{7.1}$$

Thus, the present section is the counterpart of Section 8 in [L-T-Z.1], from the Euclidean to the Riemannian setting. As in this reference, we notice that in order to achieve this goal, it suffices to extend the validity of estimate (5.1) of Theorem 5.1(i) from $H^{2,2}(Q)$-solutions to finite energy solutions defined by the class in (7.1). Here, the main difficulty is the fact that finite energy solutions subject to homogeneous Neumann B.C. do not produce (in dimension ≥ 2) H^1-traces on the boundary [L-T.6]. [By contrast, finite energy solutions subject to homogeneous Dirichlet B.C. do produce H^1-traces: the $\frac{\partial w}{\partial n}$ in $L_2(\Sigma)$ is dominated by the $H^1(\Omega) \times L_2(\Omega)$-energy of the initial data by the $H^1(\Omega) \times L_2(\Omega)$-energy of the initial data and the $L_1(0,T;L_2(\Omega))$-norm of f, thanks to sharp results on the regularity of general second-order hyperbolic equations with Dirichlet B.C. given in [L-L-T.1], [L-T.1], which hold true also in the Riemannian case. To overcome this difficulty, [L-T-Z.1, Section 8] employed a regularizing procedure inspired by [La-Ta.1]. The same proof of [L-T-Z.1, Theorem 8.2] works in the Riemannian setting and yields:

Theorem 7.1. Let $f \in L_2(Q)$. Let $w \in H^{2,2}(Q)$ be a solution of Eqn. (1.1) for which inequality (5.1) of Theorem 5.1(i) holds true, at least as guaranteed by Theorem 5.1(iii). Let u be a solution of Eqn. (1.1) in the class defined by (7.1). Then, estimate (5.1) is satisfied by such solution u as well. Accordingly, estimate (5.2) of Theorem 5.1(ii), as well as estimate (6.5) of Theorem 6.1, can be extended from $H^{2,2}(Q)$-solutions of Eqn. (1.1) to finite energy solutions in the class (7.1). □

8 Continuous observability inequality—*a fortiori*, a global uniqueness theorem—with pure homogeneous Dirichlet B.C. on Σ

In this section we consider the following problem

$$\begin{cases} w_{tt} = \Delta w + F(w) \quad \text{in } (0,T] \times \Omega \equiv Q; & (8.1a) \\ w|_\Sigma \equiv 0 \quad \text{in } (0,T] \times \Gamma \equiv \Sigma, & (8.1b) \end{cases}$$

where $F(w)$ satisfies assumption (1.2) and we define

$$\Gamma_{0,D} \equiv \left\{ x \in \Gamma : \langle Dd(x), n(x) \rangle \equiv \frac{\partial d(x)}{\partial n} \leq 0 \right\}; \quad \Gamma_{1,D} = \Gamma \setminus \Gamma_{0,D}. \tag{8.2}$$

The following corollary of Theorem 6.1 provides a continuous observability inequality in the Dirichlet B.C. case; *a-fortiori*, a global uniqueness result.

Theorem 8.1. Assume hypotheses (A.1) and (A.2): thus there exists a strictly convex function $d(x)$ (rescaled as in (1.5)), which along with the choice $\alpha(x) = \Delta d(x) - c - 1$ in (1.10) = (4.21), where $0 < c < 1$ as in (1.6d), satisfies properties (p_1) = (1.11), (p_2') = (1.22) so that inequalities (H.1) = (4.18), (H.2) = (4.19), (H.3) = (4.20) hold true. Let $\Gamma_{0,D}$, $\Gamma_{1,D}$ be defined by (8.2), and let $T > T_0$ in problem (8.1), where T_0 is defined in (1.6b). Let $w \in H^{1,1}(Q) = L_2(0,T; H^1(\Omega)) \cap H^1(0,T; L_2(\Omega))$ be a solution of problem (8.1a-b). Then: (i) the following inequality holds true:

$$\int_0^T \int_{\Gamma_{1,D}} \left(\frac{\partial w}{\partial n} \right)^2 d\Gamma_1 \, dt \geq k_\phi [E(0) + E(T)]; \tag{8.3}$$

(ii) *a fortiori*, if $w \in H^{1,1}(Q)$ is a solution of problem (8.1a-b) satisfying, in addition, $\frac{\partial w}{\partial n}\big|_{\Sigma_1} \equiv 0$, then $w \equiv 0$ in Q; in fact, in $\mathbb{R}_t \times \Omega$.

Proof. (i) We first prove the continuous observability inequality (8.3) for a solution $w \in H^{2,2}(Q)$ of problem (8.1a-b). In this case, then, such solution w satisfies Theorem 6.1, that is estimate (6.5), under present assumptions. Moreover, by (6.6) and (5.32)–(5.35), we obtain

$$w \equiv 0 \text{ on } \Sigma \Rightarrow \overline{BT}|_\Sigma = BT|_\Sigma, \text{ where } [f_1 + \theta w f_2]\Big|_\Sigma \equiv \theta^2 |Dw|^2;$$
$$\theta f_2|_\Sigma = -2\tau\theta^2 \langle Dw, Dd \rangle, \tag{8.4}$$

so that

$$\overline{BT}\Big|_\Sigma = BT\Big|_\Sigma = -2\tau \int_\Sigma \theta^2 |Dw|^2 \langle Dw, n \rangle d\Sigma + 4\tau \int_\Sigma \theta^2 \langle Dw, Dd \rangle \langle Dw, n \rangle d\Sigma. \tag{8.5}$$

But $w \equiv 0$ on Σ implies also $Dw = \langle Dw, n \rangle n$, hence

$$|Dw|^2 = |\langle Dw, n \rangle|^2, \text{ as well as } \langle Dw, Dd \rangle = \langle Dw, n \rangle \langle Dd, n \rangle, \tag{8.6}$$

so that (8.5) becomes

$$\begin{aligned} \overline{BT}\Big|_\Sigma &= BT\Big|_\Sigma = 2\tau \int_\Sigma \theta^2 |\langle Dw, n \rangle|^2 \langle Dd, n \rangle d\Sigma \\ &= 2\tau \int_\Sigma \theta^2 \left(\frac{\partial w}{\partial n} \right)^2 \langle Dd, n \rangle d\Sigma. \end{aligned} \tag{8.7}$$

Finally, since $\langle Dd, n\rangle \leq 0$ on $\Gamma_{0,D}$ by definition (8.2), estimate (6.5) of Theorem 6.1 becomes by (8.7),

$$
\begin{aligned}
2\tau \int_0^T \int_{\Gamma_{1,d}} \theta^2 \left(\frac{\partial w}{\partial n}\right)^2 \langle Dd, n\rangle d\Gamma_1\, dt \ &\geq \ 2\tau \int_0^T \int_\Gamma \theta^2 \left(\frac{\partial w}{\partial n}\right)^2 \langle Dd, n\rangle d\Gamma\, dt \\
&= \ \overline{BT}|_\Sigma \geq k_\phi [E(T) + E(0)],
\end{aligned}
\tag{8.8}
$$

and (8.8) shows (8.3), as desired, at least for $H^{2,2}(Q)$-solutions w. A density argument based on the regularity trace inequality [L-T.1], [L-L-T.1] extends then the validity of (8.3) to $H^{1,1}(Q)$-solutions w.

(ii) If a solution $w \in H^{1,1}(Q)$ of (8.1a-b) satisfies also $\frac{\partial w}{\partial n} = 0$ on Σ_1, then (8.3) implies $E(0) = 0$. Since problem (8.1a-b) is well posed—forward and backward—it follows then that $w \equiv 0$ on $\mathbb{R}_t \times \Omega$. □

9 Global uniqueness and continuous observability with pure homogeneous Neumann B.C. A global uniqueness theorem

In this subsection we consider the following overdetermined problem

$$
\begin{cases}
w_{tt} = \Delta w + F(w) & \text{in } (0,T] \times \Omega = Q; \quad (9.1a)\\
\left.\dfrac{\partial w}{\partial n}\right|_\Sigma = 0 & \text{in } (0,T] \times \Gamma = \Sigma; \quad (9.1b)\\
w|_{\Sigma_{1,N}} = 0 & \text{in } (0,T] \times \Gamma_{1,N} = \Sigma_{1,N}, \quad (9.1c)
\end{cases}
$$

where $F(w)$ satisfies assumption (1.2), and we define

$$
\Gamma_{0,N} \equiv \left\{ x \in \Gamma : \langle Dd(x), n(x)\rangle = \frac{\partial d(x)}{\partial n} = 0 \right\}, \Gamma_{1,N} = \Gamma \setminus \Gamma_{0,N}. \tag{9.2}
$$

The following corollary of Theorem 6.1 provides a global uniqueness result.

Theorem 9.1. Assume hypotheses (A.1) and (A.2): thus, there exists a strictly convex function $d(x)$ (rescaled as in (1.5)), which along with the choice $\alpha(x) = \Delta d(x) - c - 1$ in (1.10) = (4.21), where $0 < c < 1$ as in (1.6d), satisfies properties (p_1) = (1.11), (p_2') = (1.22), so that inequalities (H.1) = (4.18), (H.2) = (4.19), (H.3) = (4.20) hold true. Let $\Gamma_{0,N}$, $\Gamma_{1,N}$ be defined by (9.2), and let $T > T_0$ in problem (9.1), where T_0 is defined by (1.6b). Let $w \in H^{1,1}(Q) = L_2(0,T; H^1(\Omega)) \cap H^1(0,T; L_2(\Omega))$ be a solution of problem (9.1a-b-c). Then: $w \equiv 0$ in Q; in fact, in $\mathbb{R}_t \times \Omega$.

Proof. We first prove the desired uniqueness for a solution w of problem (9.1a-b-c) of class $H^{2,2}(Q)$. In this case, such solution satisfies Theorem 6.1, that is estimate (6.5).

Our present goal is to show that, for such solution, the boundary terms $\overline{BT}|_\Sigma$ given by (6.6) do vanish: $\overline{BT}|_\Sigma = 0$. In fact, by (9.1b), we see via (6.6) that $\overline{BT}|_\Sigma = BT|_\Sigma$, the latter being given by (5.32)–(5.35). Indeed, these same identities show at once that because of (9.1a-b-c) (9.2), we then have

$$\overline{BT}|_\Sigma = BT|_\Sigma = -2\tau \int_{\Sigma_1} (f_1 + \theta w f_2)\langle Dd, n\rangle d\Sigma_1 \tag{9.3a}$$

$$\text{on } \Sigma_1 : \ (f_1 + \theta w f_2)|_{\Sigma_1} = f_1|_{\Sigma_1} = \theta^2 |Dw|^2. \tag{9.3b}$$

Finally, on $\Sigma_1 :\ w = 0$ and $\frac{\partial w}{\partial n} = \langle Dw, n\rangle$ imply $Dw = \langle Dw, n\rangle n + \langle Dw, \bar{\tau}\rangle \bar{\tau} = 0$, where $\bar{\tau}$ is a unit tangential vector. Thus $Dw = 0$ on Σ, used in (9.3a-b) yields $\overline{BT}|_\Sigma = 0$, as desired. But such $H^{2,2}(Q)$-solution satisfies Theorem 6.1, hence inequality (6.5) holds true and yields $E(0) = 0$. Then $w \equiv 0$ on $\mathbb{R}_t \times \Omega$, since problem (9.1a-b) is well-posed forward and backward.

Extension now of the result to $H^{1,1}(Q)$-solutions employs critically Theorem 7.1. □

Continuous observability without geometrical conditions on Γ_1. Non-explicit constant. Key to the elimination of geometrical conditions on the (controlled or observed) portion Γ_1 of the boundary Γ, is the following result from [L-T.4, Section 7.2].

Lemma 9.2. Let w be a solution of Eqn. (1.1) in the class (7.1). Given $\epsilon > 0$ $\epsilon_0 > 0$ arbitrary, given $T > 0$, there exists a constant $C = C_{\epsilon,\epsilon_0,T} > 0$, such that

$$\int_\epsilon^{T-\epsilon} \int_{\tilde{\Gamma}_1} |\nabla_{\tan} w|^2 d\tilde{\Gamma}_1 dt \ \leq \ C\Big\{ \int_0^T \int_{\tilde{\Gamma}_1} w_t^2 d\tilde{\Gamma}_1 dt + \int_0^T \int_\Gamma \Big(\frac{\partial w}{\partial \nu}\Big)^2 d\Gamma \, dt$$

$$+ \ \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|f\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q)} \Big\}. \tag{9.4}$$

Using Lemma 9.2, we shall establish as in [L-T-Z.1, Section 9] the sought-after continuous observability inequality

Theorem 9.3. Assume hypotheses (A.1) and (A.2). Let $w \in H^{1,1}(Q)$ be a solution of problem (9.1a-b). Then, the following continuous observability inequality holds true for $T > T_0$, with T_0 given by (1.6b): there exists a constant $C_T > 0$ such that

$$\int_0^T \int_{\Gamma_1} [w^2 + w_t^2] d\Sigma_1 \geq C_T E(0). \tag{9.5}$$

Proof. As in [L-T-Z.1, Section 9]. [The constant C_T is not explicit in this case, as elimination of the l.o.t. in (9.4) requires a compactness/uniqueness contradiction argument, which employs Theorem 9.1 and loses control of the constant in (9.5), see [L-T-Z.1, Proof of Theorem 9.2].] □

10 Replacement of assumption (A.2) = (1.5) by virtue of two vector fields

Orientation. This section is the counterpart of the presently relevant part of [L-T-Z.1, Section 10] from the Euclidean to the Riemannian setting. The goal is to dispense with the working assumption (A.2) = (1.5). This is done by writing Ω as the overlapping union of two 'nice' subdomains Ω_1 and Ω_2, in correspondence of two functions d_i, $i = 1, 2$, each strictly convex in Ω and thus satisfying (1.4), where now, however, each d_i has *no critical point* on Ω_i, $i = 1, 2$. This way, two radial vector fields are employed.

10.1 Basic setting using two conservative vector fields as in (1.4). Statement of main results

Postulated setting. We divide the original open bounded set Ω into two overlapping subdomains Ω_1 and Ω_2: $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$, chosen (in infinitely many ways) as to fulfill all the conditions, in particular, (a), (b), (c) below (after Section 1).

We assume that there exist two functions $d_i : \overline{\Omega} \Rightarrow \mathbb{R}$, of class C^3, $i = 1, 2$, which are strictly convex in the Riemann metric, such that the following preliminary conditions (a) and (b) are satisfied:

(a)

$$D^2 d_i(X, X) \equiv \langle D_X(Dd_i), X\rangle_g \geq 2|X|_g^2, \quad \forall\, x \in \Omega,\ \forall\, X \in M_x, \tag{10.1.1}$$

$$\min_{\overline{\Omega}} d_i(x) \geq m > 0, \quad i = 1, 2; \tag{10.1.2}$$

(b)

$$\inf_{\overline{\Omega}_i} |Dd_i| \geq p > 0, \text{ so that we can take } \kappa_i = \inf_{\overline{\Omega}_i} \frac{|Dd_i|}{d_i} > 4, \tag{10.1.3}$$

by rescaling, see Remark 1.2. Next, following Section 1, we define the function for $i = 1, 2$:

$$\phi_i(x, t) = d_i(x) - c\Big(t - \frac{T}{2}\Big)^2, \quad x \in \Omega,\ 0 \leq t \leq T, \tag{10.1.4}$$

with $0 < c < 1$ and T selected as follow: $T > T_{0,i}$, $i = 1, 2$, where

$$T_{0,i}^2 = 4 \max_{\overline{\Omega}} d_i(x), \text{ so that } cT^2 > 4 \max_{\overline{\Omega}} d_i(x) + 4\delta, \tag{10.1.5}$$

for some $\delta > 0$ suitably small and kept fixed henceforth. Such functions ϕ_i have thus the following properties:

(i)

$$\phi_i(x, 0) = \phi_i(x, T) = d_i(x) - c\frac{T^2}{4} \leq -\delta, \text{ uniformly in } \overline{\Omega}; \tag{10.1.6}$$

(ii) there are t_0 and t_1, with $0 < t_0 < \frac{T}{2} < t_2 < T$, such that

$$\min_{x\in\overline{\Omega},t\in[t_0,t_1]} \phi_i(x,t) \geq \sigma, \quad 0 < \sigma < m. \tag{10.1.7}$$

Next, we define the function

$$\phi_i^*(x,t) = d_i(x) - c^2\Big(t - \frac{T}{2}\Big)^2, \quad x \in \Omega,\ 0 \leq t \leq T, \tag{10.1.8}$$

so that, since $0 < c < 1$, we have by (10.1.4) and (10.1.8):

$$\phi_i^*(x,t) \geq \phi_i(x,t), \quad x \in \Omega,\ 0 \leq t \leq T,\ i = 1,2. \tag{10.1.9}$$

Furthermore, we define the sets (subsets of $\Omega \times [0,T]$):

$$Q_i(\sigma) \ \equiv\ \{(x,t):\ x \in \Omega,\ 0 \leq t \leq T,\ \phi_i(x,t) \geq \sigma > 0\}; \tag{10.1.10}$$

$$Q_i^*(\sigma^*) \ \equiv\ \{(x,t):\ x \in \Omega,\ 0 \leq t \leq T,\ \phi_i^*(x,t) \geq \sigma^* > 0\}, 0 < \sigma^* < \sigma, \tag{10.1.11}$$

with constant σ^* selected as to satisfy $0 < \sigma^* < \sigma < m$, see (10.1.2) and (10.1.7). By recalling also (10.1.7) and (10.1.9), we obtain

$$\begin{cases} \Omega \times [t_0,t_1] \subset Q_i(\sigma) \subset Q_i^*(\sigma^*) \subset \Omega \times [0,T], \\ \text{by (10.1.6), at } t = 0 \text{ and } t = T\text{: no point of } \Omega \text{ belongs to } Q_i^*(\sigma^*). \end{cases} \tag{10.1.12}$$

(c) We finally require (assume) that

$$\begin{cases} \Omega_i \supset \text{orthogonal projections of } Q_i^*(\sigma^*) \text{ onto } \Omega, \\ \text{so that: for any } (x,t) \in Q_i^*(\sigma^*) \Rightarrow x \in \Omega_i, \\ \text{and hence } |Dd_i(x)| \geq p > 0 \text{ by (10.1.3).} \end{cases} \tag{10.1.13}$$

Consequences of above setting. With reference to the above setting, we define functions $\alpha_i(x) \in C^1(\overline{\Omega})$, by setting

$$\alpha_i(x) \equiv \Delta d_i(x) - c - 1, \qquad x \in \Omega, \tag{10.1.14}$$

so that the following properties $(p_{1;i}), (p_{2;i})$ hold true ((10.1.15) is immediate, while (10.1.16) is proved below).

$(p_{1,i})$

$$\Delta d_i(x) - 2c - \alpha_i(x) \equiv 1 - c > 0, \quad \forall\, x \in \overline{\Omega}; \tag{10.1.15}$$

$(p_{2;i})$

$$\beta_i(x,t) \equiv [2c+\Delta d_i - \alpha_i]|Dd_i|^2 + 2D^2 d_i(Dd_i, Dd_i)$$
$$- \ 4c^2(\Delta d_i + 6c - \alpha_i)\Big(t - \frac{T}{2}\Big)^2 \tag{10.1.16}$$
$$\geq 4(1+7c)\phi_i^*(x,t), \quad \forall\, x \in \Omega_i, \ \forall\, t \in [0,T];$$
$$\text{in particular in } Q_i^*(\sigma^*) \text{ by (10.1.13).} \tag{10.1.17}$$

Proof of (10.1.17). See proof of (1.12): The steps up to Eqn. (1.19) continue to hold true, this time by virtue of (10.1.14), (10.1.1), for $i = 1,2$:

$$\beta_i(x,t) \geq (5+3c)|Dd_i|^2 - 4c^2(1+7c)\Big(t - \frac{T}{2}\Big)^2, \ \forall\, x,t \in Q \equiv [0,T] \times \Omega. \tag{10.1.18}$$

The next step from (1.19) to (1.20) requires, however, assumption (1.5), whose present counterpart is assumption (10.1.3), which is valid, however, only on $\overline{\Omega}_i$. Hence, likewise (10.1.18) implies

$$\beta_i(x,t) \geq (5+3c)\kappa_i d_i(x) - 4c^2(1+7c)\Big(t - \frac{T}{2}\Big)^2 \tag{10.1.19}$$
$$\geq 4(1+7c)\Big[d_i(x) - c^2\Big(t - \frac{T}{2}\Big)^2\Big]$$
$$\equiv 4(1+7c)\phi_i^*(x,t), \ \forall\, x \in \Omega_i, \ t \in [0,T], \tag{10.1.20}$$

this time only for $x \in \Omega_i$; and (10.1.20) establishes the first statement in (10.1.17), as desired. To claim, finally, that (10.1.17) holds true, in particular, for $(x,t) \in Q_i^*(\sigma^*)$, we invoke property (10.1.13).

Critical property to absorb lower-order terms. As a consequence of (10.1.17), we obtain

$(p'_{2;i})$

$$\beta_i(x,t) \equiv [2c+\Delta d_i - \alpha_i]|Dd_i|^2 + 2D^2 d_i(Dd_i, Dd_i)$$
$$- \ 4c^2(\Delta d_i + 6c - \alpha_i)\Big(t - \frac{T}{2}\Big)^2 \tag{10.1.21}$$
$$(\text{by (10.1.17)}) \ \geq 4(1+7c)\phi_i^*(x,t) \geq 4(1+7c)\sigma^* \equiv \tilde{\beta} > 0, \quad \forall\, (x,t) \in Q_i^*, \tag{10.1.22}$$

recalling the definition of the set $Q_i^*(\sigma^*)$ in (10.1.11). The term on the LHS of (10.1.21) is precisely the term that arises as a coefficient in front of the lower-order term, see (10.2.6)–(10.2.9) below.

The *main result of the present paper* is the following Theorem 10.1.1 which extends all the previous results of Sections 1 through 9 to the setting of the present Section 10, where assumption (A.2) = (1.5) is removed.

Theorem 10.1.1. Let the setting of Section 10.1 based on assumptions (a) = (10.1.1), (b) = (10.1.3), and (c) = (10.1.13) be in force. In particular, T is given by $T > T_{0,i}$, see (10.1.5). Then:

(a) The Carleman estimates of Theorem 5.1, Eqn. (5.1), as well as of Theorem 6.1, Eqn. (6.5), continue to hold true for $H^{1,1}(Q)$-solutions.

(b) The following continuous observability inequality holds true for $H^{1,1}(Q)$-solutions of the Dirichlet problem (8.1a-b)

$$\int_0^T \int_{\Gamma_{1,D}} \left(\frac{\partial w}{\partial n}\right)^2 d\Gamma_1 dt \geq k_\phi [E(T) + E(0)], \tag{10.1.23}$$

with $\Gamma_{1,D} = \Gamma \setminus \Gamma_{0,D}$ and $\Gamma_{0,D}$ defined by (8.2). *A-fortiori*, this implies a global uniqueness theorem for $H^{1,1}(Q)$-solutions, if, in addition, $\frac{\partial w}{\partial n}\big|_{\Sigma_1} \equiv 0$; i.e., it follows that, then, $w \equiv 0$.

(c) The following continuous observability inequality holds true for $H^{1,1}(Q)$-solutions of the Neumann problem (9.1a-b)

$$\int_0^T \int_{\Gamma_{1,N}} [w_t^2 + w^2] d\Gamma_1 dt \geq k_\phi [E(0) + E(T)], \tag{10.1.24}$$

with $\Gamma_{1,N} = \Gamma \setminus \Gamma_{0,N}$ and $\Gamma_{0,N}$ defined by (9.2). A class of triples $\{\Omega, \Gamma_{0,N}, \Gamma_{1,N}\}$ satisfying all these conditions is given in Appendix B. *A-fortiori*, this implies a global uniqueness theorem for $H^{1,1}(Q)$-solutions, if, in addition, $w|_{\Sigma_1} \equiv 0$; i.e., it follows that, then, $w \equiv 0$. □

10.2 Cut-off functions $\chi_i(t,x)$ and corresponding sub-problems for $w_i = \chi_i w$

Cut-off functions χ_i Let $\chi_i(t,x)$ be a smooth cut-off function. At this stage it is not important to specify how it is constructed. Eventually, in the case of purely Neumann B.C. associated with Eqn. (1.1), $\chi_i(t,x)$ will be the complicated function constructed in [L-T-Z.1, Section 10.2], which has the important feature, among others, to be *only* time-dependent (but not space-dependent) on a *small interior layer of the boundary* Γ. This latter goal is dictated by the Neumann B.C. and would not be necessary when dealing with Dirichlet B.C. At any rate, we only assume here that such cut-off functions fulfill the requirement:

$$|\chi_i| \leq \text{const}; \text{ and } \chi_i(t,x) \equiv 1 \text{ on } Q_i(\sigma), \tag{10.2.1}$$

which is one of the properties satisfied by the cut-off functions in [L-T-Z.1, Section 10.2].

Dynamical system for $w_i \equiv \chi_i w$. Let $w \in C^2(\mathbb{R}_t \times M)$ be a solution of Eqn. (1.1). We introduce new variables on $[0,T]\times\Omega$:

$$w_i(t,x) \equiv \chi_i(t,x)w(t,x);\ \ f_i(t,x) \equiv \chi_i(t,x)f(t,x),\ \ i=1,2. \tag{10.2.2}$$

[The f_i here should not be confused with the f_i in Sections 4,5.] We then see that each term w_i satisfies the following problem

$$\begin{cases} w_{i,tt} - \Delta w_i = F(w_i) + f_i + \mathcal{K}_i w, \quad i=2,3, & (10.2.3a) \\ w_i(0,\,\cdot\,) = w_{i,0};\ \ w_{i,t}(0,\,\cdot\,) = w_{i,1}; & (10.2.3b) \end{cases}$$

$$\mathcal{K}_i \equiv [D_t^2 - \Delta - F, \chi_i] = \text{commutator active only on (supp } \chi_i); \tag{10.2.3c}$$

$$w_{i,0} = \chi_i(0,\,\cdot\,)w(0,\,\cdot\,);\ \ w_{i,1} = \chi_{i,t}(0,\,\cdot\,)w(0,\,\cdot\,) + \chi_i(0,\,\cdot\,)w_t(0,\,\cdot\,). \tag{10.2.3d}$$

In (10.2.3c), $D_t = \frac{d}{dt}$, while [,] denotes the corresponding commutator of order 1 in time and space. Accordingly, with χ_i smooth, we then obtain via (10.2.3a) and (1.2b) on F:

$$\begin{aligned}(\mathcal{A}w_i)^2 \ \equiv\ & (w_{i,tt} - \Delta w_i)^2 \le C_T\{|w_{i,t}^2 + |Dw_i|^2 + w_i^2 \\ & + f_i^2 + [w_t^2 + |Dw|^2 + w^2](\text{supp } \chi_i)\},\ \ (t,x)\in[0,T]\times\Omega.\end{aligned} \tag{10.2.4}$$

Preliminary estimate: Counterpart of Corollary 4.2. As constructed above each problem w_i in (10.2.3), $i=1,2$, satisfies the setting of Section 1. As a result, each problem (10.2.3) satisfies the counterpart of Theorem 4.1/Corollary 4.2, Eqns. (4.22)–(4.25); in particular, we recall (4.7), (4.8) for $\tilde{B}$ and $\beta(x,t)$. We take this result as our present starting point.

Proposition 10.2.1. Let $w \in C^2(\mathbb{R}_t \times \Omega)$ be a solution of (1.1). Let the setting of Section 10.1 based on assumptions (a) = (10.1.1), (b) = (10.1.3), and (c) = (10.1.13) be in force. Then, each problem (10.2.3), $i=1,2$, satisfies the following pointwise inequality for $\epsilon > 0$ small:

$$\theta_i^2(\mathcal{A}w_i)^2 + 2\widehat{\text{div}}\left\{[(a_i+\psi_i)v_i^2 - v_{i,t}^2 + |Dv_i|^2]\hat{D}\ell_i + [2\hat{D}\ell_i(v_i) - \psi_i v_i]\hat{D}v_i\right\} \tag{10.2.5}$$

$$\ge\ \tau\epsilon\rho\theta_i^2[w_{i,t}^2 + |Dw_i|^2] + B_i\theta_i^2 w_i^2,\ \ \forall\, t\in[0,T],\ \forall\, x\in\Omega, \tag{10.2.6}$$

see (4.22), where $\rho = 1-c > 0$ is a constant, and where for $i=1,2$ and (10.1.22), we have [recalling (4.7), (4.8), (4.20), (4.23)–(4.25)]

$$B_i \ =\ \tilde{B}_i - 2\epsilon\rho\tau^3(\phi_{i,t}^2 + |D\phi_i|^2) \ge \tilde{B}_i - 2\epsilon\rho\tau^3 r \tag{10.2.7}$$

$$\ge\ [2\tau^3\beta + \mathcal{O}(\tau^2)],\quad \forall\,(x,t)\in \text{ set } Q_i^*(\sigma^*); \tag{10.2.8}$$

$$\beta \equiv \beta_\epsilon \equiv \tilde{\beta} - \epsilon\rho r > 0, \quad r = \max_i \max_{\overline{Q}} (\phi_{i,t}^2 + |D\phi_i|^2), \ |D\phi_i| = |Dd_i|; \tag{10.2.9}$$

$$B_i = \mathcal{O}(\tau^3), \quad \forall\ (x,t) \in [0,T] \times \Omega, \tag{10.2.10}$$

where the set $Q_i^*(\sigma^*)$ is defined in (10.1.11), $\tilde{\beta} > 0$ is defined in (10.1.22), and $\beta > 0$ is a constant depending on $\epsilon > 0$. Above, all the quantities with a subscript "i", correspond to the quantities in Sections 3–4, without subscript. Thus, $\theta_i = e^{\tau\phi_i}$, ϕ_i as in (10.1.4); $\ell_i = \tau d_i$, $\psi_i = \tau\alpha_i$, with d_i and α_i given by (10.1.2), (10.1.14). Thus, $\tilde{B}_i$ on the left of (10.2.7) is the counterpart of (4.7) via (4.8), while the estimate in (10.2.18) is a consequence of (10.1.22). □

10.3 Carleman estimate for the w_i-problem

Building up on Proposition 10.2.1, we obtain the counterpart of Theorem 5.1 (Carleman estimate, first version) for the w_i-problems.

Proposition 10.3.1. Let $w \in C^2(\mathbb{R}_t \times \Omega)$ be a solution of Eqn. (1.1) [and no B.C.]. Let the setting of Section 10.1 based on assumptions (a) = (10.1.1), (b) = (10.1.3), and (c) = (10.1.13) be in force. Let w_i, f_i be as in (10.2.2). Let $E(t)$ be defined by (5.4). Then, for $\epsilon > 0$ small, as in (10.2.9), and for all τ sufficiently large (with $\epsilon\tau$ large with respect to C_T in (1.2b), the following family of estimates holds true:

$$\begin{aligned} &(BT)_{w_i}\Big|_\Sigma + C_{1,T} e^{2\tau\sigma} \int_0^T E(t)dt + C_{2,T} \int_0^T f_i^2 d\Omega\, dt \\ &\geq\ [\tau\epsilon\rho - 2C_T] \int_0^T \int_\Omega e^{2\tau\phi_i} [w_{i,t}^2 + |Dw_i|^2] d\Omega\, dt \\ &+\ [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q_i(\sigma)} e^{2\tau\phi_i} w^2 dx\, dt - C_T \tau^3 e^{-2\tau\delta} [E(0) + E(T)], \end{aligned} \tag{10.3.1}$$

where $Q_i(\sigma)$ is the subset of $\Omega \times [0,T]$ defined by (10.1.10); moreover, the constants $\sigma > 0$, $\delta > 0$ are defined in (10.1.7), (10.1.5), while the critical constant $\beta > 0$ is defined by (10.2.9) by use of $\tilde{\beta}$ in (10.1.22). We do not specify the boundary terms here (counterpart of (5.3)): they could be made more specific by further specializing the cut-off functions χ_i (see [L-T-Z.1, Eqn. (10.3.3] for χ_i suitably defined to handle the case of Neumann B.C.).

(ii) The above inequality (10.3.1) can be extended to $H^{2,2}(Q)$-solutions.

Remark 10.3.1. For τ sufficiently large as to make $[2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] > 0$ (see $\beta > 0$ in (10.2.9)), we reach one of our goals and drop the integral term involving w^2, accordingly, from inequality (10.3.1). □

Proof. The proof follows closely the one of [L-T-Z.1, Proposition 10.3.1], and thus only a central part dealing directly with the lower-order term will be given here explicitly.

Step 1. To begin with, the first step is the counterpart of Eqns. (5.6), (5.7) for the w-problem. We likewise now return to Proposition 10.2.1, estimates (10.2.6) and integrate it over $Q = (0,T] \times \Omega$. We obtain

$$\int_Q \theta_i^2(\mathcal{A}w_i)^2 dQ + 2\int_Q \widehat{\text{div}}\Big\{[(a_i+\psi_i)v_i^2 - v_{i,t}^2 + |Dv_i|^2]\hat{D}\ell_i + [2\hat{D}\ell_i(v_i) - \psi_i v_i]\hat{D}v_i\Big\} dQ$$

$$\geq \tau\epsilon\rho \int_Q \theta_i^2[w_{i,t}^2 + |Dw_i|^2]dQ + \int_Q \theta_i^2 B_i w_i^2 dQ. \qquad (10.3.2)$$

This is the exact counterpart of the estimate in [L-T-Z.1, Eqn. (10.3.4)].

Step 2. Precisely as in Step 2 of Theorem 10.3.1 in [L-T-Z.1], one proves that

$$\begin{aligned}\int_Q \theta_i^2(\mathcal{A}w_i)^2 dQ &= \int_0^T\int_\Omega \theta_i^2(w_{i,tt} - \Delta w_i)^2 d\Omega\, dt \\ &\leq C_T\Big\{\int_0^T\int_\Omega e^{2\tau\phi_i}[w_{i,t}^2 + |Dw_i|^2]d\Omega\, dt \\ &+ \int_{Q_i(\sigma)} e^{2\tau\phi_i}w_i^2 dx\, dt + e^{2\tau\sigma}\int_0^T E(t)dt + \int_0^T\int_\Omega f_i^2 d\Omega\, dt\Big\}, \quad (10.3.3)\end{aligned}$$

where the set $Q_i(\sigma)$ is defined by (1.1.10). The proof of (10.3.3) *uses both properties of* χ_i in (10.2.1) and (10.2.4).

Step 3. We shall focus on the terms in (10.3.2). [What we do here corresponds to Step 5 in the proof of Proposition 10.3.1 in [L-T-Z.1].] We proceed as in the proof of Theorem 5.1, below Eqn. (5.7). We split $\Omega \times [0,T] = Q_i^*(\sigma^*) \cup [Q_i^*(\sigma^*)]^c$, where $[\quad]^c$ is the complement in $\Omega \times [0,T]$, and $Q_i^*(\sigma^*)$ is the set defined in (10.1.11). Because of property (10.1.13) which was assumed in the selection of Ω_i, we have that for any $(x,t) \in Q_i^*(\sigma^*)$, then the space coordinate $x \in \Omega_i$, and hence $|Dd_i(x)| \geq p > 0$ by (10.1.3). As a result of property (10.1.13), we obtained the validity of estimate (10.1.17) on all of $Q_i^*(\sigma^*)$, hence of estimate (10.1.22) on all of $Q_i^*(\sigma^*)$, finally of estimate (10.2.8) on all of $Q_i^*(\sigma^*)$. Employing (10.2.8) we then estimate the last term in (10.3.2):

$$\begin{aligned}\int_0^T\int_\Omega e^{2\tau\phi_i}B_i w_i^2 d\Omega\, dt &= \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i}B_i w_i^2 dx\, dt + \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i}B_i w_i^2 dx\, dt \quad (10.3.4) \\ \text{(by (10.2.8))} \qquad &\geq [2\tau^3\beta + \mathcal{O}(\tau^2)]\int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i}w_i^2 dx\, dt\end{aligned}$$

$$+ \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} B_i w_i^2 dx\, dt \tag{10.3.5}$$

[Eqn. (10.3.5) is the counterpart of (5.8)]. As to the last term in (10.3.5), we proceed as in Step 2 in the proof of Theorem 5.1: first, by (10.2.10), we have $B_i = \mathcal{O}(\tau^3)$ on all of $\Omega \times [0, T]$; second, we have $\phi_i \leq \phi_i^* \leq \sigma^*$ on $[Q_i^*(\sigma^*)]^c$ by (10.1.9) and the very definition of $Q_i^*(\sigma^*)$ in (10.1.11). Finally, since we have chosen $0 < \sigma^* < \sigma$ in (10.1.11), we then obtain

$$\begin{aligned} - \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} B_i w_i^2 dx\, dt &\leq \mathcal{O}(\tau^3) \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} w_i^2 dx\, dt \qquad (10.3.6) \\ &\leq \mathcal{O}(\tau^3) e^{2\tau\sigma^*} \int_{[Q_i^*(\sigma^*)]^c} w_i^2 dx\, dt \\ &\leq \mathcal{O}(\tau^3) e^{2\tau\sigma} \int_Q w_i^2 dx\, dt \leq \mathcal{O}(\tau^3) e^{2\tau\sigma} \int_Q w^2 dQ, \qquad (10.3.7) \end{aligned}$$

majorizing w_i^2 by (const w^2), see (10.2.1), and majorizing $[Q_i^*(\sigma^*)]^c$ by $\Omega \times [0, T]$. Eqn. (10.3.7) is the counterpart of (5.11), and of [L-T-Z., Eqn. (10.3.22)].

Step 4. Next, we substitute (10.3.5) for the last term on the right side of (10.3.2), move the last term of (10.3.5) on $[Q_i^*(\sigma^*)]^c$ to the left side of (10.3.2), and apply for it estimate (10.3.7). Finally, on the first integral term on the left side of (10.3.2), we invoke estimate (10.3.3). We thus obtain

$$\begin{aligned} &\mathcal{O}(\tau^3) e^{2\tau\sigma} \int_Q w^2 dQ + C_T \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt + C_T e^{2\tau\sigma} \int_0^T E(t) dt + C_T \int_0^T \int_\Omega f_i^2 d\Omega\, dt \\ &+ 2 \int_Q \widehat{\text{div}} \Big\{ [(a_i + \psi_i) v_i^2 - v_{i,t}^2 + |Dv_i|^2] \hat{D}\ell_i \\ &+ [2\hat{D}\ell_i(v_i) - \psi_i v_i] \hat{D} v_i \Big\} dQ \\ &\geq [\tau\epsilon\rho - C_T] \int_Q e^{2\tau\phi_i} [w_{i,t}^2 + |Dw_i|^2] dQ \\ &+ [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i} w_i^2 dx\, dt. \qquad (10.3.8) \end{aligned}$$

Regarding the integral terms on $Q_i^*(\sigma^*)$, $Q_i(\sigma)$, we see that since $Q_i^*(\sigma^*) \supset Q_i(\sigma)$ (see (10.1.12)), we have that for all τ sufficiently large, as in Step 3 in the proof of

Theorem 5.1:

$$[2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i} w_i^2 dx\, dt - C_T \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt$$

$$\geq \ [2\tau^3\beta + \mathcal{O}(\tau^2) - C_T] \int_{Q_i(\sigma)} e^{2\tau\phi_i} w^2 dx\, dt, \tag{10.3.9}$$

since on $Q_i(\sigma)$ we have $\chi_i \equiv 1$ by (10.2.1), hence $w_i \equiv w$ by (10.2.2). Thus, using (10.3.9) in (10.3.8) and recalling $E(t)$ in (5.4), we readily obtain the desired estimate (10.3.1), after handling the term $\int_Q \widehat{\text{div}}\{\quad\}dQ$, which we do not do here, since we have deliberately left undefined the cut-off functions χ_i. For the critical case of Neumann B.C., this is done in [L-T-Z.1, Section 10.2], which admits here a faithful counterpart. $\square$

Final remark. The remaining Sections 10.4–10.7 of [L-T-Z.1] admit a faithful counterpart from the Euclidean to the Riemannian setting and will not be repeated. This therefore leads to Theorem 10.1.1.

Appendix A: Proof of (3.43)

In this appendix we prove identity (3.43), that is

$$\langle D\ell, D_{Dv}(Dv)\rangle = \frac{1}{2} D\ell(|Dv|^2). \tag{A.0}$$

Step 1. Let M be a manifold. We denote by $\{E_1, E_2, \ldots, E_n\}$ a *field normal* at $x \in M$. This means that:

(a) $E_1, \ldots, E_n$ are vector fields on the tangent space M_x at x;

(b) on M_x, we have that $E_1, \ldots, E_n$ are mutually orthogonal; that is $D_{E_j} E_i = 0$, $\forall\ i, j$;

(c) $E_1, \ldots, E_n$ form an orthonormal basis on the tangent space M_x. Thus, for any function v, the vector field $Dv = \nabla_g v$ on M_x is given by

$$Dv = \sum_{i=1}^{n} \langle Dv, E_i\rangle E_i; \quad \langle Dv, E_i\rangle = E_i(v), \ |Dv|^2 = \sum_{i=1}^{n} (E_i(v))^2. \tag{A.1}$$

Step 2. We shall first show that

$$D_{Dv} Dv = \frac{1}{2} \sum_{i=1}^{n} E_i(|Dv|^2) E_i \text{ at } x. \tag{A.2}$$

Indeed, by the axioms of a connection, if Y is any vector field, we obtain by (A.1),

$$D_{Dv}Y = D_{\sum_{i=1}^n E_j(v)E_j}Y = \sum_{j=1}^n E_j(v)D_{E_j}Y. \tag{A.3}$$

Next, we specialize to $Y = Dv$ given by (A.1) and compute

$$D_{E_j}Y = D_{E_j}Dv = D_{E_j}\left(\sum_{i=1}^n E_i(v)E_i\right) = \sum_{i=1}^n D_{E_j}(E_i(v))E_i + \sum_{i=1}^n E_i(v)D_{E_j}E_i, \tag{A.4}$$

where vanishing takes place by orthogonality. Inserting (A.4) in (A.3), with $D_X f = X(f)$, we arrive at

$$\begin{aligned} D_{Dv}Dv &= \sum_{j=1}^n E_j(v)\sum_{i=1}^n E_jE_i(v)E_i = \sum_{i,j=1}^n E_j(v)E_jE_i(v)E_i \\ &= \sum_{i,j=1}^n E_j(v)E_iE_j(v)E_i, \end{aligned} \tag{A.5}$$

changing the order of the indices. Next, we notice that

$$E_i[(E_j(v))^2] = D_{E_i}(E_j(v))^2 = 2E_j(v)E_iE_j(v), \tag{A.6}$$

which substituted into (A.5), yields, recalling (A.1):

$$\begin{aligned} D_{Dv}Dv &= \frac{1}{2}\sum_{i,j=1}^n E_i[(E_j(v))^2]E_i = \frac{1}{2}\sum_{i=1}^n E_i\left[\sum_{j=1}^n (E_j(v))^2\right]E_i \\ \text{(by (A.1))} \quad &= \frac{1}{2}\sum_{i=1}^n E_i(|Dv|^2)E_i, \end{aligned} \tag{A.7}$$

and (A.2) is proved.

Step 3. We finally establish (A.0). Using (A.2), we obtain

$$\begin{aligned} \langle D\ell, D_{Dv}Dv\rangle &= \frac{1}{2}\left\langle D\ell, \sum_{i=1}^n E_i(|Dv|^2)E_i\right\rangle = \frac{1}{2}\sum_{i=1}^n \langle D\ell, E_i(|Dv|^2)E_i\rangle \\ &= \frac{1}{2}\sum_{i=1}^n \langle D\ell, X\rangle = \frac{1}{2}\sum_{i=1}^n X(\ell) \\ &= \frac{1}{2}\sum_{i=1}^n E_i(|Dv|^2)E_i(\ell) = \frac{1}{2}\sum_{i=1}^n E_i(\ell)E_i(|Dv|^2). \end{aligned} \tag{A.8}$$

On the other hand, by (A.1) we can write

$$D\ell = \sum_{i=1}^{n} E_i(\ell)E_i, \text{ hence } D\ell(|Dv|^2) = \sum_{i=1}^{n} E_i(\ell)E_i(|Dv|^2). \tag{A.9}$$

Combining (A.8) and (A.9) yields (A.0), as desired. □

Appendix B

In this appendix we provide an attractive class of subsets Ω in a Riemannian manifold M, for which it is possible to claim the existence of a smooth function d defined around the portion Γ_0 of $\partial\Omega$, such that: (i) d is strictly convex on Γ_0, and moreover, (ii) its normal derivative vanishes on Γ_0. The present appendix, in particular Theorem B.1 below and its proof, are all the close counterpart of [L-T-Z.1, appendix 4, in particular Theorem A.4.1] as generalized from the Euclidean setting to the Riemannian setting.

Let $\{M, g\}$ be an n-dimensional Riemannian manifold, with Levi-Civita connection D. Let $\Omega \subset M$ be an open, connected, compact subset of M, with boundary $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. The portion Γ_0 of $\partial\Omega$ is defined as follows. Let $\ell : M \to \mathbb{R}$ be a function of class C^2. Then, we define

$$\begin{cases} \Gamma_0 = \{x \in \partial\Omega : \ \ell(x) = 0\}, \\ \text{with the further provision that } D\ell (= \nabla_g \ell) \neq 0 \text{ on } \Gamma_0. \end{cases} \tag{B.0}$$

Theorem B.1. In the above setting, assume that

(i)

$$D^2\ell(X, X)(x) \geq 0, \qquad \forall\ x \in \Gamma_0,\ \forall\ X \in M_x \tag{B.1}$$

('convexity of ℓ near Γ_0');

(ii) There exists a function $d_0 : \overline{\Omega} \to \mathbb{R}$ of class C^2, such that

(ii$_1$)

$$D^2 d_0(X, X)(x) \geq \rho_0 |X|_g^2, \quad \forall\ x \in \Gamma_0,\ \forall\ X \in M_x, \tag{B.2}$$

for some constant $\rho_0 > 0$ (strict convexity of d_0 near Γ_0);

(ii$_2$)

$$\left.\frac{\partial d_0}{\partial n}\right|_{\Gamma_0} = \langle Dd_0, n\rangle_g \leq 0 \quad \text{ on } \Gamma_0, \tag{B.3}$$

where n is the unit outward normal field to Γ_0 on M which points in the same direction as $\nabla_g \ell = D\ell$. Then: there exists a function $d : \overline{\Omega} \to \mathbb{R}$ of class C^2 [which is explicitly constructed in a layer (collar) of Γ_0, the critical set], such that it satisfies the following two conditions:

(a)

$$\left.\frac{\partial d}{\partial n}\right|_{\Gamma_0} = \langle Dd, n\rangle_g = 0 \quad \text{ on } \Gamma_0; \tag{B.4}$$

(b)

$$D^2 d(X,X)(x) \geq (\rho_0 - \epsilon)|X|_g^2, \quad \forall\, x \in \Gamma_0, \; \forall\, X \in M_x, \tag{B.5}$$

where $\epsilon > 0$ is arbitrarily small.

Definition of d. The function $d(x)$ is explicitly constructed *near* Γ_0*, within* Ω, as a perturbation of the original function d_0 assumed in (ii) above, as follows:

$$\begin{cases} d(x) \;\equiv\; d_0(x) + z(x) \qquad x \text{ near } \Gamma_0 \text{ in } \Omega; & \text{(B.6)} \\ z(x) \;\equiv\; -\dfrac{\partial d_0}{\partial n}(\ell k) + \lambda \ell^2, \quad k \equiv \dfrac{1}{|D\ell|_g}, & \text{(B.7)} \end{cases}$$

where λ is a sufficiently large parameter, to be selected below in the proof, while

$$\frac{\partial d_0}{\partial n} \text{ denotes an extension of } \left.\frac{\partial d_0}{\partial n}\right|_{\Gamma_0} \text{ from the set } \Gamma_0 \text{ (defined by } \ell(x) = 0\text{) to a layer (collar) of } \Gamma_0 \text{, within } \Omega, \tag{B.8a}$$

which is defined by

$$\frac{\partial d_0}{\partial n} \equiv \langle Dd_0, n\rangle_g = \left\langle Dd_0, \frac{D\ell}{|D\ell|_g}\right\rangle_g = \langle Dd_0, kD\ell\rangle_g. \tag{B.8b}$$

$n \equiv \frac{D\ell}{|D\ell|_g}$ (consistently with the statement below (B.3).

Proof. (a) First we establish property (B.4). To this end, we perform direct computations and obtain:

$$\begin{cases} \left.\dfrac{\partial(\ell k)}{\partial n}\right|_{\Gamma_0} = \left[\cancel{\ell\dfrac{\partial k}{\partial n}} + k\,\dfrac{\partial \ell}{\partial n}\right]_{\Gamma_0} = \frac{1}{|D\ell|_g}\langle D\ell, n\rangle_g \equiv 1 \text{ on } \Gamma_0; & \text{(B.9)} \\ \left.\dfrac{\partial(\ell^2)}{\partial n}\right|_{\Gamma_0} = 2\ell\left.\dfrac{\partial \ell}{\partial n}\right|_{\Gamma_0} = 0, & \text{(B.10)} \end{cases}$$

since $\ell = 0$ on Γ_0, and recalling n from (B.3) and k from (B.7). Returning to $z(x)$ given by (B.7), we take its normal derivative at Γ_0, and obtain by virtue of (B.9), (B.10):

$$\begin{aligned} \left.\frac{\partial z}{\partial n}\right|_{\Gamma_0} &= \left[-\left(\frac{\partial^2 d_0}{\partial n^2}\right)(\ell k) - \frac{\partial d_0}{\partial n}\,\frac{\partial(\ell k)}{\partial n} + \lambda\,\frac{\partial(\ell^2)}{\partial n}\right]_{\Gamma_0} \\ &= -\left.\frac{\partial d_0}{\partial n}\right|_{\Gamma_0}. \end{aligned} \tag{B.11}$$

Finally, returning to (B.6), and using (B.11), we obtain

$$\left.\frac{\partial d}{\partial n}\right|_{\Gamma_0} = \left[\frac{\partial d_0}{\partial n} + \frac{\partial z}{\partial n}\right]_{\Gamma_0} = \left[\frac{\partial d_0}{\partial n} - \frac{\partial d_0}{\partial n}\right]_{\Gamma_0} = 0, \tag{B.12}$$

and thus (B.4) is established, as desired.

(b) We now prove property (B.5) for a suitable large λ. For convenience, we set

$$\text{near } \Gamma_0: \ p \equiv -\frac{\partial d_0}{\partial n} k, \ \text{so that } z = p\ell + \lambda \ell^2, \tag{B.13}$$

recalling (B.8) and (B.7) for z. Next, from (B.6), we compute preliminarily

$$D^2 d(X,X) = D^2 d_0(X,X) + D^2 z(X,X). \tag{B.14}$$

Step 1. Lemma B.2. The following identities hold true:
(b_1)

$$\begin{aligned} \text{near } \Gamma_0: \ D^2 z(X,X) &= pD^2\ell(X,X) + \ell D^2 p(X,X) + 2\lambda \ell D^2 \ell(X,X) \\ &+ 2\lambda |\langle D\ell, X\rangle_g|^2 + 2\langle D\ell, X\rangle_g \langle Dp, X\rangle_g; \end{aligned} \tag{B.15}$$

(b_2)

$$\begin{aligned} \text{on } \Gamma_0: \ D^2 z(X,X) &= pD^2\ell(X,X) + 2\lambda |\langle D\ell, X\rangle_g|^2 \\ &+ 2\langle D\ell, X\rangle_g \langle Dp, X\rangle_g. \end{aligned} \tag{B.16}$$

Proof. Identity (B.16) in (b_2) follows at once from identity (B.15) in (b_1) by setting $\ell = 0$ (see Eqn. (B.0) of Γ_0).

Proof of (b_1). By definition of the Hessian tensor:

$$D^2 z(X,X) = \langle D_X(Dz), X\rangle_g, \tag{B.17}$$

where, recalling $z = p\ell + \lambda\ell^2$ (near Γ_0) by (B.13), we find

$$\left\{ \begin{aligned} Dz &= pD\ell + \ell Dp + 2\lambda \ell D\ell, && \text{(B.18)} \\ D_X(Dz) &= X(p)D\ell + pD_X(D\ell) + X(\ell)Dp + \ell D_X(Dp) \\ &+ 2\lambda[X(\ell)D\ell + \ell D_X(D\ell)]. && \text{(B.19)} \end{aligned} \right.$$

Substituting (B.19) in (B.17) yields

$$\begin{aligned} D^2 z(X,X) &= X(p)\langle D\ell, X\rangle_g + p\langle D_X(D\ell), X\rangle_g \\ &+ X(\ell)\langle Dp, X\rangle_g + \ell\langle D_X(Dp), X\rangle_g \\ &+ 2\lambda X(\ell)\langle D\ell, X\rangle_g + 2\lambda\ell\langle D_X(D\ell), X\rangle_g. \end{aligned} \tag{B.20}$$

Recalling again the definition of Hessian tensor for ℓ and p, and recalling likewise that $X(f) = \langle Df, X\rangle_g$, $f = \ell, p$, we recognize at once that (B.20) is precisely (B.15), as desired. $\square$

Step 2. Let $\epsilon_0 > 0$ arbitrary. We estimate the last two terms in (B.16)

$$\begin{aligned} & 2\lambda|\langle D\ell, X\rangle_g|^2 + 2\langle D\ell, X\rangle_g \langle Dp, X\rangle_g \\ & \qquad \geq \left[2\lambda - \frac{1}{\epsilon_0}\right] |\langle D\ell, X\rangle_g|^2 - \epsilon_0 |\langle Dp, X\rangle_g|^2. \end{aligned} \tag{B.21}$$

Step 3. Regarding the first term in (B.16), we obtain

$$\text{on } \Gamma_0 : \quad pD^2\ell(X, X) \geq 0, \tag{B.22}$$

since, as $k > 0$ by (B.7), we have

$$\begin{cases} p = -\left.\dfrac{\partial d_0}{\partial n}\right|_{\Gamma_0} k \geq 0 & \text{on } \Gamma_0, \text{ by assumption (B.3);} \\ D^2\ell(X, X)(\Gamma_0) \geq 0 & \text{on } \Gamma_0, \text{ by assumption (B.1).} \end{cases} \tag{B.23}$$

Thus, dropping the first term in (B.16) by virtue of (B.22) and invoking (B.21) for the remaining two terms, we obtain from (B.16)

$$\text{on } \Gamma_0 : \; D^2 z(X, X) \geq \left[2\lambda - \frac{1}{\epsilon_0}\right] |\langle D\ell, X\rangle_g|^2 - \epsilon_0 |\langle Dp, X\rangle_g|^2 \tag{B.24}$$

$$\geq -\epsilon |X|_g^2, \tag{B.25}$$

where, in the last step, we have chosen

$$2\lambda - \frac{1}{\epsilon_0} > 0 \text{ and } \epsilon \equiv \epsilon_0 \max_{\Gamma_0} |Dp|_g^2. \tag{B.26}$$

Finally, returning to (B.14) and invoking (B.25) and assumption (B.2), we obtain

$$\text{on } \Gamma_0 : \; D^2 d(X, X) \geq D^2 d_0(X, X) - \epsilon |X|_g^2 \tag{B.27}$$

$$\geq [\rho_0 - \epsilon] |X|_g^2, \quad \forall\, x \in \Gamma_0, \; \forall\, X \in M_x, \tag{B.28}$$

and (B.28) establishes (B.2), as desired. The proof of Theorem B.1 is complete. $\square$

Remark B.1. The same proof works if we interchange the signs in assumptions (B.1) and (B.3), that is, if we assume instead

(i′)

$$D^2\ell(X,X)(x) \leq 0,\ \forall x \in \Gamma_0,\ \forall\ X \in M_x \tag{B.29}$$

(‘concavity of ℓ near Γ_0’)

(ii′)

$$\left.\frac{\partial d_0}{\partial n}\right|_{\Gamma_0} = \langle Dd_0, n\rangle_g \geq 0 \text{ on } \Gamma_0. \tag{B.30}$$

Indeed, (B.29) and (B.30) still produce inequality (B.22), recalling (B.23). □

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