Inverse/Observability Estimates for Second-Order Hyperbolic Equations with Variable Coefficents*

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Abstract

We consider a general second-order hyperbolic equation defined on an open bounded domain $\Omega \subset \mathbb{R}^n$ with variable coeffecients in both the elliptic principal part and in the first-order terms as well. At first, no boundary conditions (B.C.) are imposed. Our main result (Theorem 3.5) is a reconstruction, or inverse, estimate for solutions w: under checkable conditions on the coefficients of the principal part, the $H^1(\Omega) \times L_2(\Omega)$—energy at time $t = T$, or at time $t = 0$, is dominated by the $L_2(\Sigma)$-norms of the boundary traces $\frac{\partial w}{\partial \nu_{\mathcal{A}}}$ and w_t, modulo an interior lower-order term. Once homogeneous B.C. are imposed, our results yield–under a uniqueness theorem, needed to absorb the lower order term–continuous observability estimates for both the Dirichlet and Neumann case, with an explicit, sharp observability time; hence, by duality, exact controllability results. Moreover, no artificial geometrical conditions are imposed on the controlled part of the boundary in the Neumann case. In contrast with existing literature, the first step of our method employs a Riemann geometry approach to reduce the original variable coefficient principal part problem in $\Omega \subset \mathbb{R}^n$ to a problem on an appropriate Riemann manifold (determined by the coefficients of the principal part), where the principal part is the Laplacian. In our second step, we employ explicit Carleman estimates at the differential level to take care of the variable first-order (energy level) terms. In our third step, we employ micro-local analysis yielding a sharp trace estimate, to remove artificial geometrical conditions on the controlled part of the boundary, in the Neumann case.

Keywords: Hyperbolic equation, inverse/observability estimates, exact controllability, Riemannian manifold, Carleman estimates.
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1 Introduction. Dual Problem: Continuous Observability Inequalities. Literature

Standing assumptions. (H.1): Let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain with boundary $\Gamma = \partial\Omega$ of class C^2. Let Γ_0 and Γ_1 be open disjoint subsets of Γ with $\Gamma = \Gamma_0 \cup \Gamma_1$. Let

$$\mathcal{A}w \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial w}{\partial x_j}\right), \quad x = [x_1, \dots, x_n] \tag{1.1}$$

be a second-order differential operator, with real coefficients $a_{ij} = a_{ji}$ of class C^1, satisfying the uniform ellipticity condition:

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a\sum_{i=1}^{n}\xi_i^2, \quad x \in \Omega, \tag{1.2a}$$

for some positive constant $a > 0$. Assume further that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0, \quad \forall\, x \in \mathbb{R}^n,\ \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n,\ \xi \neq 0. \tag{1.2b}$$

(H.2): Let $F_1(w)$ be a linear, first-order differential operator in all variables $\{t, x_1, \dots, x_n\}$ on w with $L_\infty(Q)$-coefficients, thus satisfying the following pointwise estimate: there exists a constant $C_T > 0$ such that

$$|F_1(w)|^2 \leq C_T[w_t^2 + |\nabla w|^2 + w^2], \forall\, t, x \in Q, \tag{1.3}$$

where $Q = (0, T] \times \Omega$ and $w(t, x) \in C^1(Q)$. Let $(0, T] \times \Gamma_i \equiv \Sigma_i,\ i = 0, 1; (0, T] \times \Gamma \equiv \Sigma$. [Lower regularity than L_∞ can be assumed for the zero order term, depending on the dimension n via Sobolev embedding, but we shall not insist on this detail.]

Dirichlet control. We consider the Dirichlet mixed second-order hyperbolic problem in the unkown $w(t, x)$ and its dual homogeneous problem in $\psi(t, x)$:

$$\begin{cases} w_{tt} + \mathcal{A}w = F_1(w) & \text{in } Q; \\ w(0,\ \cdot\ ) = w_0,\ w_t(0,\ \cdot\ ) = w_1 & \text{in } \Omega; \\ w|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \\ w|_{\Sigma_1} = u & \text{in } \Sigma_1; \end{cases} \qquad \begin{cases} \psi_{tt} + \mathcal{A}\psi = F(\psi) & \text{in } Q; \\ \psi(T,\ \cdot\ ) = \psi_0,\ \psi_t(T,\ \cdot\ ) = \psi_1 & \text{in } \Omega; \\ \psi|_{\Sigma} \equiv 0 & \text{in } \Sigma, \end{cases} \tag{1.4}$$

with control function $u \in L_2(0, T; L_2(\Gamma_1))$ in the Dirichlet B.C., where $F(\psi)$ is a suitable first-order differential operator, depending on the original operator F_1, and satisfying the same pointwise bound such as (1.3) for F_1.

Continuous observability inequality in the Dirichlet case. As our *first goal*, we seek to establish—under a suitable additional assumption—the following *a-priori* inequality for the homogeneous Dirichelt ψ-problem (1.4): there exists a constant $T_0 > 0$, depending upon the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ and the coefficients a_{ij}, such that for all $T > T_0$, there is a constant $c_T > 0$ for which

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}}\right)^2 d\Sigma_1 \geq c_T \|\{\psi_0, \psi_1\}\|^2_{H_0^1(\Omega)\times L_2(\Omega)}. \tag{1.5}$$

In (1.5), $\frac{\partial w}{\partial \nu_{\mathcal{A}}} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial w}{\partial x_j} \nu_i$ is the co-normal derivative, where $\nu = [\nu_1, \ldots, \nu_n]$ is the unit outward normal on Γ. Eqn. (1.5) is the *continuous observability inequality* for the ψ-problem (1.4) in the established terminology of [2]. As is well-known, e.g., [10], [13], [27], inequality (1.5) for the ψ-problem (1.4) is, by duality or transposition, *equivalent* to the exact controllability property of the non-homogeneous w-problem (1.4) at time T, on the space $L_2(\Omega) \times H^{-1}(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-controls; in other words, such exact controllability is the property that the map L_T:

$$\begin{cases} \{u, w_0 = 0, w_1 = 0\} \to L_T u \equiv \{w(T, \,\cdot\,), w_t(T, \,\cdot\,)\} \text{ is surjective} \\ \text{from } L_2(0,T;L_2(\Gamma_1)) \text{ onto } L_2(\Omega) \times H^{-1}(\Omega), \end{cases} \tag{1.6a}$$

with $\{w(T, \,\cdot\,), w_t(T, \,\cdot\,)\}$ solution of the w-problem (1.4) at $t = T$; while inequality (1.5) is a restatement [27] of the following standard [24, p. 235] inequality from below of the corresponding adjoint:

$$\|L_T^* z\|_{L_2(0,T;L_2(\Gamma_1))} \geq c_T \|z\|_{L_2(\Omega)\times H^{-1}(\Omega)}, \tag{1.6b}$$

which is well known to be equivalent to the surjectivity property (1.6a).

Remark 1.1. The converse (trace regularity) of inequality (1.5) always holds true, for any $T > 0$ [12], [11], [19]. $\square$

Neumann control. Here we let $\Gamma_0 \neq \emptyset$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, and consider the Neumann mixed second-order hyperbolic problem in the unknown $w(t,x)$ and is dual homogeneous version in $\psi(t,x)$:

$$\begin{cases} w_{tt} + \mathcal{A}w = F_1(w); \\ w(0, \,\cdot\,) = w_0, \; w_t(0, \,\cdot\,) = w_1; \\ w|_{\Sigma_0} \equiv 0; \\ \left.\dfrac{\partial w}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma_1} = u; \end{cases} \quad \begin{cases} \psi_{tt} + \mathcal{A}\psi = F(\psi) & \text{in } Q; \\ \psi(T, \,\cdot\,) = \psi_0, \; \psi_t(T, \,\cdot\,) = \psi_1 & \text{in } \Omega; \\ \psi|_{\Sigma_0} \equiv 0 & \text{in } \Sigma_0; \\ \left[\dfrac{\partial \psi}{\partial \nu_{\mathcal{A}}} + \beta\psi\right]_{\Sigma_1} \equiv 0 & \text{in } \Sigma_1, \end{cases} \tag{1.7}$$

with control function $u \in L_2(0,T;L_2(\Gamma_1)) \equiv L_2(\Sigma_1)$ in the Neuman B.C. where F is a suitable first-order differential operator depending on F_1, and satisfying the same pointwise estimate such as (1.3) for F_1 and β is a suitable function, depending on F_1.

Continuous observability inequality in the Neumann case. As our *second goal* we seek to establish—under a suitable additional assumption—the following *a-priori* inequality for the homogeneous Neumann ψ-problem (1.7): there exists a constant $T_0 > 0$, depending upon the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ and the coefficients a_{ij}, such that for all $T > T_0$, there is a constant $c_T > 0$ for which

$$\int_0^T \int_{\Gamma_1} \psi_t^2 \, d\Sigma_1 \geq c_T \|\{\psi_0, \psi_1\}\|^2_{H^1_{\Gamma_0}(\Omega)\times L_2(\Omega)}, \tag{1.8}$$

where $H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega) : f|_{\Gamma_0} = 0\}$, whenever the left-hand side is finite. This is the *continuous observability inequality* for the ψ-problem (1.7) [2]. Again, by duality or transposition, inequality (1.8) is *equivalent* (see e.g., [10], [14], [15]) to the exact controllability property of the non-homogeneous w-problem (1.7) at time T, on the space $H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$, within the class of $L_2(0,T;L_2(\Gamma_1))$-controls; in other words, such exact controllability is the property that the map L_T:

$$\begin{cases} \{u, w_0 = 0, w_1 = 0\} \to L_T u \equiv \{w(T, \,\cdot\,), w_t(T, \,\cdot\,)\} \text{ is surjective} \\ \text{from } L_2(0,T;L_2(\Gamma_1)) \text{ onto } H^1_{\Gamma_0} \times L_2(\Omega), \end{cases} \tag{1.9a}$$

with $\{w(T,\ \cdot\ ), w_t(T,\ \cdot\ )\}$ solution of the w-problem (1.7) at $t = T$, while inequality (1.8) is a restatement [14] of the following standard [24, p. 235] inequality from below of the corresponding adjoint:

$$\|L_T^* z\|_{L_2(0,T;L_2(\Gamma_1))} \geq c_T \|z\|_{H^1_{\Gamma_0} \times L_2(\Omega)}, \tag{1.9b}$$

which is well known to be equivalent to the surjectivity property (1.9a) [14].

Remark 1.2. The converse of inequality (1.8) is false for dim $\Omega \geq 2$, and is true for dim $\Omega = 1$ [16], for any $T > 0$. $\square$

Literature. Our results are more general than just continuous observability estimates, or—by duality—exact controllability statements. The latter are generally obtained in the literature through the former [2], on the basis of the standard Functional Analysis result [24, p. 235] quoted before. One exception is the approach pursued by W. Littman, who seeks exact controllability results directly, without passing through continuous observability inequalities [20], [21], [22].

A detailed analysis of the various methods used in the literature to establish continuous observability inequalities, such as (1.5) and (1.8), along with a description of their virtues and shortcomings was already given in our previous announcement [18] of the present paper. They are:

(i) (By now classical) differential multipliers—$h \cdot \nabla\psi, \psi$ div h—used after [11], in [13], [5], [14], [19], [27] in 1986, where $h(x)$ is a coercive vector field. They have been successful in proving the continuous observability inequalities (1.5) and (1.8) in the case where $\mathcal{A} = -\Delta$ (or in the case of *constant* coefficients a_{ij} of the principal part). However, these original differential multipliers tolerate additional terms only below the energy level; i.e., a *zero-order* operator F is fine, but a truly *first-order* operator F causes the method to fail.

(ii) Pseudo-differential multipliers, micro-local analysis, propagation of singularities initiated in [20] and culminated into the general treatment in [1]. However, it is not an easy matter to verify in applications and examples the (sharp) sufficient condition that all rays of geometric optics hit the effective controlled part $\Sigma_1 = (0,T] \times \Gamma_1$ of the lateral boundary Σ of the cylinder Q at a non-diffractive point. Moreover, the method uses C^∞ data and Γ, at least at present. Extension to other non-hyperbolic models, such as plate-like problems, seems a serious issue.

(iii) General pseudo-differential multipliers derived from pseudo-convex functions [6] for general evolution equations [25]. These techniques with pseudo-differential Carleman multipliers proposed in [25], which in prior literature [6] were applied to solutions with compact support (thereby not accounting for boundary traces which are instead critical for continuous observability equalities) are unifying across several evolution equations. However, they require the existence of a pseudo-convex function, a property which essentially can be verified mostly if not exclusively in the case of constant coefficients a_{ij} of the principal part $\mathcal{A}$. Moreover, at least in [25], the control is taken to be active on the entire boundary Γ.

(iv) Subsequent specific, corresponding differential multipliers versions (much more flexible than classical differential multipliers in (i)), tuned to second-order hyperbolic equations [17], [8], [3]. In the specific concrete analysis of differential Carleman multipliers tuned to second-order hyperbolic equations, the drawback of the existence of pseudo-convex function remains, of course, while now a more detailed analysis—this time at the differential rather than pseudo-differential level—allow the control to act on a suitable part of the boundary. These differential Carleman multipliers can be viewed as a non-trivial generalization of the original differential multipliers $h \cdot \nabla\psi$, ψ div h in (i), over which they possess an added flexibility via the parameter τ below, which allows to handle also those first-order terms F as in (1.3), that original multipliers could not deal with.

The general, technical, sharp approach in [1] which follows the dynamics along bi-characteristics—the carriers of energy—did not seal the problem. Other approaches, mentioned above, made connections with other ideas in the P.D.E.'s area, such as pseudo-convex functions and injected new enlightening into the continuous observability inequalities. Even this further development does not seal the problem, and the infusion of other ideas is possible. In this scenario, recently Riemann geometric methods were introduced and combined with classical differential multipliers as in (i), to establish continuous observability inequalities, such as (1.5) and (1.8) [28]. This method has the virtue to allow variable coefficients $a_{ij}(x)$ of the principal part $\mathcal{A}$, subject to certain verifiable assumptions. However, in its original form [28], this approach also cannot handle genuine first-order energy level terms F. The reason will be explained in Remark 4.2.1 below. Moreover, the treatment in [28] required unnecessary geometrical conditions.

Contribution of the present paper. In this paper we present a successful combination of three key ingredients which allow to establish the validity of the continuous observability inequalities (1.5) and (1.8) in the case of (a) variable coefficients $a_{ij}(x)$ of the principal part $\mathcal{A}$, subject to verifiable conditions, and (b) genuine first-order, energy level terms F_1, and (c) with no artificial geometric conditions in the Neumann case. These three ingredients are: (1) the Riemann geometric approach to [28]; (2) the Carleman differential multipliers used in [17], which now replace the original classical differential multipliers of [28], though in the Riemann metric; (3) the pseudo-differential approach in [15] which led to an L_2-estimate of the tangential derivative (gradient) of the solution w in terms of L_2-boundary estimates of w_t and $\frac{\partial w}{\partial \nu_{\mathcal{A}}}$, modulo lower-order terms; see Lemma 6.2 below.

It is ingredient (2) that permits to add a bonafide first-order operator F_1 as in (1.3), to the result of [28]. Further, it is ingredient (3) that permits the elimination of geometrical conditions present in [28] in the Neumann case.

The present approach provides the optimal time for the validity of the continuous observability inequalities (1.5) and (1.8), as is the case with pseudo-convex functions.

Our new main differential multiplier is (see (4.2.2) below)

$$e^{\tau\phi(x,t)}[\langle \nabla_g \phi, \nabla_g w \rangle_g - \phi_t w_t] \tag{1.10}$$

in the Riemann metric $(\mathbb{R}^n, g)$ below, where ϕ is the pseudo-convex function defined in (3.6a) below. Additional multipliers in the proof below are

$$w \left[\text{div}_0 e^{\tau\phi} \nabla_g \phi - \frac{d}{dt} \left(e^{\tau\phi} \phi_t \right) \right]; \; w e^{\tau\phi} \tag{1.11}$$

see Lemma 4.2.2 with $m = \mu$ defined in (4.2.14).

2 Riemannian Metric Generated by the Principal Part $\mathcal{A}$

Recalling the coefficients $a_{ij} = a_{ji}$ of $\mathcal{A}$, let $A(x)$ and $G(x)$ be, respectively, the coefficient matrix and its inverse

$$A(x) = (a_{ij}(x)); \quad G(x) = [A(x)]^{-1} = (g_{ij}(x)), \quad i,j = 1, \dots, n; \; x \in \mathbb{R}^n. \tag{2.1}$$

Both $A(x)$ and $G(x)$ are $n \times n$ matrices. $A(x)$ is positive definite for any $x \in \mathbb{R}^n$ by assumption (1.2b).

Riemannian metric. Let $\mathbb{R}^n$ have the usual topology and $x = [x_1, x_2, \dots, x_n]$ be the natural coordinate system. For each $x \in \mathbb{R}^n$, define the inner product and the norm on the tangent space $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$g(X,Y) = \langle X, Y \rangle_g = \sum_{ij=1}^{n} g_{ij}(x) \alpha_i \beta_j, \tag{2.2}$$

$$|X|_g = \langle X, X \rangle_g^{\frac{1}{2}}, \quad \forall \; X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{2.3}$$

It is easily checked from (1.2b) that $(\mathbb{R}^n, g)$ is a Riemannian manifold with the Riemannian metric g. We shall denote $g = \sum_{ij=1}^n g_{ij} dx_i dx_j$. (If $A(x) \equiv I$, i.e., $\mathcal{A} = -\Delta$, then $G(x) \equiv I$, and g is the Euclidean $\mathbb{R}^n$-metric.)

Euclidean metric. For each $x \in \mathbb{R}^n$, denote by

$$X \cdot Y = \sum_{i=1}^n \alpha_i \beta_i, \ |X|_0 = (X \cdot Y)^{\frac{1}{2}}, \ \forall \ X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \ Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x, \tag{2.4}$$

the Euclidean metric on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, and with reference to (2.1), set

$$A(x)X = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) \alpha_j \right) \frac{\partial}{\partial x_i}, \quad \forall \ X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x. \tag{2.5}$$

Thus, recalling the co-normal derivative defined below (1.5), we have

$$\frac{\partial w}{\partial \nu_{\mathcal{A}}} = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial w}{\partial x_j} \right) \nu_i = (A(x) \nabla_0 w) \cdot \nu. \tag{2.6}$$

In (2.6), and hereafter, we denote by a sub "0" entities in the Euclidean metric. Thus, for $f \in C^1(\Omega)$ and $X = \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i}$ a vector field on $\mathbb{R}^n$,

$$\nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}, \ \text{ and } \ \text{div}_0(X) = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i} \tag{2.7}$$

denote gradient of f and divergence of X in the Euclidean metric.

Further relationships. If $f \in C^1(\bar{\Omega})$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g, via the Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X \rangle_g, \tag{2.8}$$

where X is any vector field on the manifold $(\mathbb{R}^n, g)$. The following lemma provides further relationships [28, Lem. 2.1].

Lemma 2.1. *Let $x = [x_1, x_2, \ldots, x_n]$ be the natural coordinate system in $\mathbb{R}^n$. Let $f, h \in C^1(\bar{\Omega})$. Finally, let H, X be vector fields. Then, with reference to the above notation, we have*

(a)

$$\langle H(x), A(x)X(x) \rangle_g = H(x) \cdot X(x), \quad x \in \mathbb{R}^n; \tag{2.9}$$

(b)

$$\nabla_g f(x) = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial f}{\partial x_j} \right) \frac{\partial}{\partial x_i} = A(x) \nabla_0 f, \quad x \in \mathbb{R}^n; \tag{2.10}$$

(c) If $X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$, then by (2.8) and (2.10),

$$X(f) = \langle \nabla_g f, X \rangle_g = \langle A \nabla_0 f, X \rangle_g = \nabla_0 f \cdot X = \sum_{i=1}^n \xi_i \frac{\partial f}{\partial x_i}; \tag{2.11}$$

(d) By (2.6) and (2.10),

$$\frac{\partial w}{\partial \nu_{\mathcal{A}}} = (A(x)\nabla_0 w)\cdot\nu = \nabla_g w\cdot\nu; \tag{2.12}$$

(e) by (2.8), (2.10), (2.9),

$$\langle\nabla_g f,\nabla_g h\rangle_g = \nabla_g f(h) = \langle A(x)\nabla_0 f,\nabla_g h\rangle_g = \nabla_0 f\cdot\nabla_g h = \nabla_0 f\cdot A(x)\nabla_0 h, \quad x\in\mathbb{R}^n; \tag{2.13}$$

(f) If H is a vector field in $(\mathbb{R}^n, g)$ (see e.g., (2.16) below),

$$\begin{aligned}\langle\nabla_g f,\nabla_g(H(f))\rangle_g &= DH(\nabla_g f,\nabla_g f)+\frac{1}{2}\mathrm{div}_0(|\nabla_g f|_g^2 H)(x)\\ &\quad -\frac{1}{2}|\nabla_g f|_g^2(x)\mathrm{div}_0(H)(x), \quad x\in\mathbb{R}^n,\end{aligned} \tag{2.14}$$

where DH is the covariant differential discussed below;

(g) by (1.1), (2.7), (2.10),

$$\mathcal{A}w = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n a_{ij}(x)\frac{\partial w}{\partial x_j}\right) = -\mathrm{div}_0(A(x)\nabla_0 w) = -\mathrm{div}_0(\nabla_g w), \quad w\in C^2(\Omega). \tag{2.15}$$

Covariant differential. Denote the Levi-Civita connection in the Riemannian metric g by D. Let

$$H = \sum_{k=1}^n h_k\frac{\partial}{\partial x_k}; \qquad X = \sum_{k=1}^n \xi_k\frac{\partial}{\partial x_k}, \tag{2.16}$$

be vector fields on $(\mathbb{R}^n, g)$. The covariant differential DH of H determines a bilinear form on $\mathbb{R}^n_x\times\mathbb{R}^n_x$, for each $x\in\mathbb{R}^n$, defined by

$$DH(Y,X) = \langle D_X H, Y\rangle_g, \quad \forall\, X,Y\in\mathbb{R}^n_x, \tag{2.17}$$

where $D_X H$ is the covariant derivative of H with respect to X. This is computed as follows, in the notation of (2.16), (2.11), by using the axioms of a connection,

$$\begin{aligned} D_X H &= \sum_{k=1}^n D_X\left(h_k\frac{\partial}{\partial x_k}\right) = \sum_{k=1}^n X(h_k)\frac{\partial}{\partial x_k} + \sum_{k=1}^n h_k D_X\left(\frac{\partial}{\partial x_k}\right)\\ &= \sum_{k=1}^n X(h_k)\frac{\partial}{\partial x_k} + \sum_{k,i=1}^n h_k\xi_i D_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_k}\right),\end{aligned} \tag{2.18}$$

where by definition, see (2.11),

$$X(h_k) = \langle\nabla_g h_k, X\rangle_g = X\cdot\nabla_0 h_k = \sum_{i=1}^n \xi_i\frac{\partial h_k}{\partial x_i}; \quad D_{\partial/\partial x_i}\left(\frac{\partial}{\partial x_k}\right) = \sum_{\ell=1}^n \Gamma^\ell_{ik}\frac{\partial}{\partial x_\ell}, \tag{2.19}$$

Γ^ℓ_{ik} being the connection coefficients (Christoffel symbols) of the connection D,

$$\Gamma^\ell_{ik} = \frac{1}{2}\sum_{p=1}^n a_{\ell p}\left(\frac{\partial g_{kp}}{\partial x_i}+\frac{\partial g_{ip}}{\partial x_k}-\frac{\partial g_{ik}}{\partial x_p}\right), \quad (g_{ij}) = (a_{ij})^{-1}. \tag{2.20}$$

Inserting (2.20) into (2.19), and then (2.19) into (2.18) yields

$$D_X H = \sum_{k=1}^n X(h_k)\frac{\partial}{\partial x_k} + \sum_{\ell=1}^n \left(\sum_{k,i=1}^n h_k \xi_i \Gamma_{ik}^\ell\right) \frac{\partial}{\partial x_\ell} = \sum_{\ell=1}^n \left[X(h_\ell) + \sum_{k,i=1}^n h_k \xi_i \Gamma_{ik}^\ell\right] \frac{\partial}{\partial x_\ell}. \tag{2.21}$$

Finally, inserting (2.21) into (2.17), we obtain by (2.2), (2.16), and (2.19) for $X(h_\ell)$:

$$DH(X,X) = \langle D_X H, X\rangle_g = \sum_{\ell,j=1}^n \left[X(h_\ell) + \sum_{k,i=1}^n h_k \xi_i \Gamma_{ik}^\ell\right] \xi_j g_{\ell j} \tag{2.22}$$

$$(\text{by (2.19)}) \quad = \sum_{i,j=1}^n \left[\sum_{\ell=1}^n \frac{\partial h_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^n h_k g_{\ell j} \Gamma_{ik}^\ell\right] \xi_i \xi_j. \tag{2.23}$$

Thus, in $\mathbb{R}_x^n \times \mathbb{R}_x^n$, $DH(\,\cdot\,,\,\cdot\,)$ is equivalent to the $n \times n$ matrix

$$\left(m_{ij} = \sum_{\ell=1}^n \frac{\partial h_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^n h_k g_{\ell j} \Gamma_{ik}^\ell\right), \quad i,j = 1,\dots,n. \tag{2.24}$$

Hessian in the Riemannian metric g**.** Let $f \in C^2(\mathbb{R}^n)$. By definition, the Hessian of f with respect to the metric g is

$$D^2 f(X,X) \equiv \langle D_X(\nabla_g f), X\rangle_g \tag{2.25}$$

$$= \sum_{i,j=1}^n \xi_i \left(\sum_{\ell=1} \frac{\partial f_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^n f_k g_{\ell j} \Gamma_{ik}^\ell\right) \xi_j, \tag{2.26}$$

where, by (2.10), $f_\ell = (\nabla_g f)_\ell$ is the ℓ-th coordinate of $\nabla_g f$:

$$(\nabla_g f)_\ell = f_\ell = \sum_{p=1}^n a_{\ell p} \frac{\partial f}{\partial x_p}, \quad \ell = 1,2,\dots,n. \tag{2.27}$$

To prove (2.26), we recall (2.21) with $H = \nabla_g f$, hence with coordinates $h_\ell = (\nabla_g f)_\ell = f_\ell$ as in (2.27), and obtain by (2.19):

$$D_X(\nabla_g f) = \sum_{\ell=1}^n \left[\sum_{i=1}^n \xi_i \frac{\partial f_\ell}{\partial x_i} + \sum_{k,i=1}^n f_k \xi_i \Gamma_{ik}^\ell\right] \frac{\partial}{\partial x_\ell}. \tag{2.28}$$

Thus, (2.2), (2.16) for X and (2.28) yield

$$\langle D_X(\nabla_g f), X\rangle_g = \sum_{\ell,q=1}^n g_{\ell q} \left[\sum_{i=1}^n \xi_i \frac{\partial f_\ell}{\partial x_i} + \sum_{k,i=1}^n f_k \xi_i \Gamma_{ik}^\ell\right] \xi_q \tag{2.29}$$

$$= \sum_{\ell,q,i=1} g_{\ell q} \xi_i \frac{\partial f_\ell}{\partial x_i} \xi_q + \sum_{\ell,q,k,i=1}^n g_{\ell q} f_k \xi_i \Gamma_{ik}^\ell \xi_q \tag{2.30}$$

$$= \sum_{i,q=1}^n \xi_i \left(\sum_{\ell=1}^n g_{\ell q} \frac{\partial f_\ell}{\partial x_i}\right) \xi_q + \sum_{i,q=1}^n \xi_i \left(\sum_{\ell,k=1}^n g_{\ell q} f_k \Gamma_{ik}^\ell\right) \xi_q, \tag{2.31}$$

and (2.31) proves (2.26), as desired with $q = j$.

Thus, by (2.26), we have that

$$\begin{cases} D^2 f \text{ is positive on } \mathbb{R}^n_x \times \mathbb{R}^n_x \text{ if and only if the} \\ n \times n \text{ matrix } \left(m_{ij} = \sum_{\ell=1}^{n} \frac{\partial f_\ell}{\partial x_i} g_{\ell j} + \sum_{k,\ell=1}^{n} f_k g_{\ell j} \Gamma^\ell_{ik} \right), \\ i,j = 1,\dots,n, \text{ is positive, with } f_\ell \text{ given by (2.27).} \end{cases} \tag{2.32}$$

3 Main Results. Preliminaries

Let the domain Ω and the elliptic operator $\mathcal{A}$ in (1.1) be given satisfying the standing assumption (H.1)=(1.2). The additional hypothesis which we shall need to establish the continuous observability inequalities (1.5) and (1.8) is the following.

Main assumption (H.3). We assume that there exists a function $v_0 : \bar{\Omega} \to \mathbb{R}$ of class C^2 which is strictly convex on $\bar{\Omega}$, with respect to the Riemannian metric g defined in Section 2. For purposes of Eqn. (3.4) below, we translate $v(x)$ as to make it non-negative on $\bar{\Omega}$, and set

$$0 \le v(x) = v_0(x) - \min_{x\in\Omega} v_0(x). \tag{3.1}$$

This assumption means that the Hessian of v in the Riemannian metric g is positive on $\bar{\Omega}$, as defined by (2.25), (2.32):

$$D^2 v(X,X)(x) > 0, \quad \forall\, x \in \bar{\Omega},\ X \in \mathbb{R}^n_x. \tag{3.2a}$$

Since $\bar{\Omega}$ is compact, it follows from (3.2a) that there exists a positive constant $\rho > 0$ such that

$$D^2 v(X,X) \ge 2\rho |X|^2_g, \quad \forall\, x \in \bar{\Omega},\ X \in \mathbb{R}^n_x. \quad \square \tag{3.2b}$$

Under assumption (H.3), we then take the vector field

$$h(x) \equiv \nabla_g v(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_j} \right) \frac{\partial}{\partial x_i}, \tag{3.3}$$

defined as the gradient of $v(x)$ with respect to the Riemannian metric g, see (2.10).

Section 8 below will provide some non-trivial illustrations where the standing assumption (H.1) as well as the main assumption (H.3) are guaranteed to hold true.

Main results. Continuous observability inequalities. We are now in the position to state our main results concerning the validity of the continuous observability inequalities (1.5) and (1.8) for the Dirichlet and the Neumann case, respectively. First, define

$$T_0 = 2 \left(\frac{\max_{x\in\bar{\Omega}} v(x)}{\rho} \right)^{\frac{1}{2}}; \ \rho \text{ as in (3.2b).} \tag{3.4}$$

Remark 3.1. Both Theorems 3.1 and 3.2 below require a uniqueness continuation result for the hyperbolic ψ-problem (1.4), respectively (1.7), with over-determined B.C.:

$$\left.\frac{\partial \psi}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma_1} \equiv 0 \text{ for Theorem 3.1; } \psi|_{\Sigma} \equiv 0 \text{ for Theorem 3.2,} \tag{3.5}$$

which asserts that, then, $\psi \equiv 0$ in Q, for T as given. This uniqueness continuation result is needed to absorb the lower order term from estiamtes (5.2.13), respectively (7.1), though a (by now standard) compactness/uniqueness argument. Known uniqueness continuation results include the following cases:

(a) The case where the coefficients a_{ij} of the principal part are (time-independent) and of class $C^1(\bar{\Omega})$, as assumed; while the coefficients of the first-order operator F in (1.4a), or (1.7a), are analytic in time and in $L_\infty(Q)$, as assumed in (1.3). In this case, the required uniqueness continuation result, noted above, follows from [26, Section 5.1, p. 882], see also [7].

The subcase, where the coefficients of F are also *time-independent* and in $L_\infty(Q)$, appears also in [9, Corollary 3.4.3, p. 63]. Here, another approach is as follows. The aforementioned uniqueness continuation results for the hyperbolic over-determined problem can, in turn, be reduced [1] to a corresponding second-order over-determined elliptic problem to which we apply [6, Theorem 17.2.6, p. 14].

(b) The case of (real) analytic data covered by Homgren-John's theorem [9, p. 52], [6].

(c) The case where $-\mathcal{A} = \Delta/a_0$, with $a_0 \in C^1(\bar{\Omega})$, time-independent, $a_0 > 0$, but the coefficients of F possibly time-dependent in $L_\infty(Q)$, which is covered by [9, Theorem 3.4.1], with the boundary Γ_0 convex. This is particularly relevant in the Neumann case of Theorem 3.2. □

Theorem 3.1. (Dirichlet case) *Let $\Omega, \mathcal{A}$, and F satisfy the standing assumptions (H.1)=(1.2), (H.2)=(1.3). Let assumption (H.3)=(3.2) hold true and define $h(x)$ by (3.3). Let $T > T_0$, see (3.4). Assume that $h(x) \cdot \nu(x) \leq 0$ for $x \in \Gamma_0$, where we recall that $\nu(x) = [\nu_1(x), \dots, \nu_n(x)]$ is the unit outward normal vector to Γ, and where $h(x) \cdot \nu(x) = \sum_{i=1}^n h_i(x)\nu_i(x)$ is the dot product in $\mathbb{R}^n$. Assume the uniqueness continuation property of the over-determined problem (1.4) with $\left.\frac{\partial \psi}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma_1} \equiv 0$, as described in Remark 3.1 above. Then, the observability inequality (1.5) for the Dirichlet ψ-problem (1.4) holds true.*

Theorem 3.2. (Neumann case) *Let $\Omega, \mathcal{A}$, and F satisfy the standing assumption (H.1), (H.2). Let assumption (H.3) hold true and define $h(x)$ by (3.3). Let Γ_0 and Γ_1 be given, $\Gamma \equiv \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \neq \emptyset$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, and $h(x) \cdot \nu(x) \leq 0$ for $x \in \Gamma_0$. Let $T > T_0$, see (3.4). Assume the uniqueness continuation property of the over-determined problem (1.7) with $\psi|_\Sigma \equiv 0$, as described in Remark 3.1. Then, the observability inequality (1.8) for the Neumann ψ-problem (1.7) holds true.*

Remark 3.2. T_0, defined in (3.4), is sharp even in the case of constant coefficients with a radial vector field. If $v(x)$ is the square of the distance function, then $\rho = 1$ and T_0 is equal to the diameter of Ω.

Carleman estimates. The results of Theorems 3.1 and 3.2 can be shown as a consequence of suitable Carleman estimates for Eqn. (1.4a) with no boundary conditions imposed, which we now describe.

Let $v : \bar{\Omega} \to \mathbb{R}$ be the strictly convex function, with respect to the Riemannian metric g, provided by assumption (H.3). Define the function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(x,t) \equiv v(x) - c\left|t - \frac{T}{2}\right|^2, \tag{3.6a}$$

where $T > T_0$, see (3.4), and c is a constant chosen below as follows. Let $T > T_0$ be given. By (3.4), there is $\delta > 0$ such that

$$\rho T^2 > 4 \max_{x \in \Omega} v(x) + 4\delta.$$

For this δ, there is then a constant c, $0 < c < \rho$, such that

$$cT^2 > 4 \max_{x \in \Omega} v(x) + 4\delta, \quad 0 < c < \rho. \tag{3.6b}$$

Henceforth, let ϕ *be defined* by (3.6a) *with the above* c as in (3.6b) unless otherwise explicitly noted. Such function $\phi(x,t)$ has then the following properties:

(i)

$$\phi(x,0) < -\delta \text{ and } \phi(x,T) < -\delta \text{ uniformly in } x \in \Omega; \tag{3.6c}$$

(ii) there are t_0 and t_1 with $0 < t_0 < \frac{T}{2} < t_1 < T$ such that

$$\min_{x\in\Omega, t\in[t_0,t_1]} \phi(x,t) \geq -\frac{\delta}{2}, \tag{3.6d}$$

since $\phi\left(x, \frac{T}{2}\right) = v(x) \geq 0$ for all $x \in \Omega$;

(iii) recalling (3.3),

$$\nabla_g \phi = \nabla_g v = h;\ \phi_t(x,t) \equiv -2c\left(t - \frac{T}{2}\right),\ \phi_{tt} \equiv -2c;\ \phi_t(x,0) \equiv cT;\ \phi(x,T) \equiv -cT. \tag{3.6e}$$

The important property (3.6c) will be invoked in the proof of Eqn. (4.2.26) of Lemma 4.2.5 leading to Theorem 3.3, Eqn. (3.9a). The important property (3.6d) (in fact, only the weaker property: $\min \phi(x,t) \geq \sigma > -\delta$ is actually needed) will be invoked in going from Eqn. (3.9a) to Eqn. (3.9b) in the statement of Theorem 3.3 (Carleman estimates, first version), but not before (3.9a).

The following result is a counterpart of [17, Theorem 2.1.1].

Theorem 3.3. (Carleman estimates, first version) *Assume (H.1), (H.2), and (H.3). Let $f \in L_2(Q)$. Let w be a solution of the second-order hyperbolic equation*

$$w_{tt} + \mathcal{A}w = F_1(w) + f \text{ in } Q \tag{3.7}$$

[with no boundary conditions imposed], within the following class:

$$\begin{cases} w \in H^{1,1}(Q) \equiv L_2(0,T;H^1(\Omega)) \cap H^1(0,T;L_2(\Omega)) & (3.8a) \\ w_t,\ \dfrac{\partial w}{\partial \nu_{\mathcal{A}}} \in L_2(0,T;L_2(\Gamma)). & (3.8b) \end{cases}$$

Let $\phi(x,t)$ be the function defined by (3.6a), and C_T a generic constant.

Then, for $\tau > 0$, the following one-parameter family of estimates holds true:

$$\begin{aligned} &(BT_w)|_\Sigma + \frac{C_T}{\tau}\int_Q e^{\tau\phi} f^2 dQ + TC_T\ const_\tau \|w\|^2_{C([0,T];L_2(\Omega))} \\ &\geq \left(\rho - c - \frac{C_T}{\tau}\right)\int_Q e^{\tau\phi}[|\nabla_g w|_g^2 + w_t^2]dQ - C(1+\tau)e^{-\delta\tau}[E(T) + E(0)] && (3.9a) \\ &\geq \left(\rho - c - \frac{C_T}{\tau}\right) e^{-\frac{\tau\delta}{2}} \int_{t_0}^{t_1} E(t)dt - C(1+\tau)e^{-\delta\tau}[E(T)+E(0)], && (3.9b) \end{aligned}$$

where the boundary terms $(BT_w)|_\Sigma$ over $\Sigma = [0,T] \times \Gamma$ are given by

$$\begin{aligned} (BT_w)|_\Sigma = &\int_\Sigma e^{\tau\phi}\frac{\partial w}{\partial \nu_{\mathcal{A}}}[h(w) - \phi_t w_t]d\Sigma + \int_\Sigma \frac{\partial w}{\partial \nu_{\mathcal{A}}} w\left[\frac{1}{2}\mu - (1+c)e^{\tau\phi}\right]d\Sigma \\ &+ \frac{1}{2}\int_\Sigma e^{\tau\phi}[w_t^2 - |\nabla_g w|_g^2]h \cdot \nu\, d\Sigma, \end{aligned} \tag{3.10}$$

with $\mu(x,t)$ a suitable function depending on ϕ, defined in (4.2.14) below. Moreover, we have set for convenience

$$E(t) = E_w(t) = \int_\Omega [|\nabla_g w(t,x)|_g^2 + w_t^2(t,x)]d\Omega, \tag{3.11}$$

and we recall that $h(w) = \langle h, \nabla_g w\rangle_g = \langle \nabla_g v, \nabla_g w\rangle_g = \nabla_0 w \cdot h$ by (2.8), and (2.11), with h the vector field defined by (3.3). $\square$

Remark 3.3. By (2.13), (1.2a), we have

$$a|\nabla_0 w(t,x)|^2 \leq |\nabla_g w(t,x)|_g^2 = \nabla_0 w(t,x)\cdot A(x)\nabla_0 w(t,x)$$

$$\leq a_1|\nabla_0 w(t,x)|^2, \quad x\in\Omega, \tag{3.12}$$

where $a>0$ is the constant in (1.2a). Thus, by (3.11) and (3.12) we have that

$$E(t) \text{ is equivalent to } \|\{w(t),w_t(t)\}\|^2_{H^1_{\Gamma_0}(\Omega)\times L_2(\Omega)} \text{ if } w|_{\Sigma_0}\equiv 0, \quad \Gamma_0\neq\emptyset. \tag{3.13}$$

We shall henceforth use (3.13) freely, particularly for $t=0$ and $t=T$. $\square$

Remark 3.4. Property (3.6c) is used to obtain (3.9a). Property (3.6d) is used to obtain (3.9b). $\square$

The proof of Theorem 3.3 is given in Section 4. The counterpart of [17, Theorem 2.1.2] is

Theorem 3.4. (Carleman estimates, second version) *Assume the hypotheses of Theorem 3.3. Then, for all $\tau>0$ sufficiently large, there exists a constant $k_{\phi,\tau}>0$ such that the following one-parameter family of estimates holds true:*

$$(\overline{BT}_w)|_\Sigma + \frac{C_T}{\tau}\int_Q e^{\tau\phi}f^2dQ + C_T\ const_\tau\|w\|^2_{L_2(0,T;L_2(\Omega))}$$

$$\geq e^{-\frac{\tau\delta}{2}}\left\{\left(\rho - c - \frac{C_T}{\tau}\right)\frac{e^{-C_T T}}{2}(t_1-t_0) - C(1+\tau)e^{-\frac{\delta\tau}{2}}\right\}[E(T)+E(0)] \tag{3.14}$$

$$\geq k_{\phi,\tau}[E(T)+E(0)], \tag{3.15}$$

C_T a generic constant, where the boundary terms $(\overline{BT}_w)|_\Sigma$ over $\Sigma=(0,T]\times\Gamma$ are given by

$$(\overline{BT}_w)|_\Sigma = (BT_w)|_\Sigma + const_{\phi,\tau}\int_\Sigma\left|\frac{\partial w}{\partial\nu_A}w_t\right|d\Sigma, \tag{3.16}$$

with $(BT)_w|_\Sigma$ defined by (3.10).

(b) Assume, further, that the solution w of (3.7) satisfies

$$w|_{\Sigma_0}\equiv 0,\ \Sigma_0=(0,T]\times\Gamma_0,\ \text{and that } h(x)\cdot\nu(x)\leq 0,\ x\in\Gamma_0, \tag{3.17}$$

with $h=\nabla_g\phi=\nabla_g v$ by (3.6e), and $\nu(x)$ the unit outward normal vector at $x\in\Gamma$.

Then, estimate (3.15) holds true for $\tau>0$ sufficiently large, with the boundary terms $(\overline{BT})_w|_\Sigma$ replaced by $(\overline{BT})_w|_{\Sigma_1}$, i.e., evaluated only on $\Sigma_1=(0,T]\times\Gamma_1$, while the boundary terms $(\overline{BT})_w|_{\Sigma_0}$ evaluated on $\Sigma_0=(0,T]\times\Gamma_0$ are negative: $(\overline{BT})_w|_{\Sigma_0}\leq 0$ (see (6.2) below for the precise expression of $(\overline{BT})_w|_{\Sigma_0}$). $\square$

The proof of Theorem 3.4 is given in Section 5.1. Estimate (3.15) of Theorem 3.4 then readily yields Theorem 3.1 on the continuous observability inequality (1.5) in the Dirichlet case for $\psi=w$ with $f\equiv 0$, $\psi|_\Sigma\equiv 0$ and $h\cdot\nu\leq 0$ on Γ_0. This is done in Section 5.2. However, to prove Theorem 3.2 on the continuous observability inequality in the Neumann case for $\psi=w$ with $f\equiv 0$, $\psi|_{\Sigma_0}\equiv 0$, $\Gamma_0\neq\emptyset$, and $h\cdot\nu\leq 0$ on Γ_0, an additional non-trivial step is needed. This is provided by a key result of [15] which will be quoted in

Lemma 6.2 below. Combined with Theorem 3.4, this result will permit us to obtain the following theorem, which may be viewed as the *main estimate (at the energy level)* of the present paper, the counterpart of [17, Theorem 2.1.5].

Theorem 3.5. *Assume (H.1), (H.2), (H.3) and that* $f \in L_2(Q)$. *Let* w *be a solution of Eqn. (3.7) in the class (3.8).*

(a) Then, the following estimate holds true. There exists a constant $k_{\phi,\tau} > 0$ *for* τ *sufficiently large such that, for any* $\epsilon_0 > 0$:

$$\int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu_{\mathcal{A}}} \right)^2 + w_t^2 \right] d\Sigma + const_{\phi,\tau} \int_Q f^2 dQ + C_{\phi,\epsilon_0} \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}$$

$$\geq k_{\phi,\tau}[E(T) + E(0)]. \tag{3.18}$$

(b) Assume, further, that the solution w *of (3.7) satifies hypothesis (3.17).*
Then, estimate (3.18) holds true with $\int_\Gamma$ *replaced by* $\int_{\Gamma_1}$. □

Not only does estimate (3.18) imply the continuous observability inequality (1.8) for $\psi = w$, under the required assumption (3.17):

$$\psi|_{\Sigma_0} \equiv 0, \quad \Gamma_0 \neq \emptyset, \quad h \cdot \nu \leq 0 \text{ on } \Gamma_0; \text{ and } \left. \frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right|_{\Sigma_1} = 0,$$

by dropping $E(T)$ in (3.18) and by absorbing the lower-order interior term by compactness/uniqueness. Moreover, (3.18) implies also an inverse, or recovery, estimate for the following closed loop problem with explicit dissipative feedback in the Neumann B.C.

$$\begin{cases} w_{tt} + \mathcal{A}w = F_1(w) & (3.19a) \\ w(0, \cdot) = w_0, \ w_t(0, \cdot) = w_1 & (3.19b) \\ w|_{\Sigma_0} \equiv 0 & (3.19c) \\ \left. \dfrac{\partial w}{\partial \nu_{\mathcal{A}}} \right|_{\Sigma_1} = -w_t. & (3.19d) \end{cases}$$

Part (i) of the following result is standard (perturbation of the dissipative case $F_1 \equiv 0$, handled by Lumer-Phillips theorem); part (ii)—a recovery, or inverse, estimate—follows from Theorem 3.15 via compactness/uniqueness, see Remark 3.1.

Theorem 3.6. *With reference to the closed loop problem (3.19), we have:*

(i) when $\Gamma_0 \neq \phi$, *under assumptions (H.1) and (H.2) for* $\mathcal{A}$ *and* F_1, *problem (3.19) generates a s.c. semigroup* $\{w_0, w_1\} \in Y \to \{w(t), w_t(t)\} \in C([0,T]; Y)$, $Y = H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$;

(ii) when $\Gamma_0 = \phi$, *the same result, under (H.1), (H.2), holds true, with* $Y \equiv H^1(\Omega) \times L_2(\Omega)$ *replaced now by its proper subspace*

$$Y_0 \equiv \left\{ [u_1, u_2] \in Y : \int_{\Gamma_1} u_1 d\Gamma + \int_\Omega u_2 d\Omega = 0 \right\}, \tag{3.20}$$

topologized by (see (4.1.1) below)

$$\|\{u_1, u_2\}\|^2_{Y_0} = \int_\Omega [|\nabla_g u_1|^2_g + u_2^2] d\Omega, \tag{3.21}$$

which is a norm on Y_0 (but only a semi-norm on Y). The operator $\begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix}$, *with domain*

$$\mathcal{D} \equiv \left\{ [u_1, u_2] \in H^2(\Omega) \times H^1(\Omega) : \ \frac{\partial u_1}{\partial \nu_{\mathcal{A}}} = -u_2 \right\} \tag{3.22}$$

is dissipative on Y_0, since, by (4.1.1) below

$$\left(\begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right)_{Y_0} = -\int_\Gamma u_2^2 d\Gamma; \tag{3.23}$$

moreover, it is maximal dissipative on Y_0, since

$$\left[\lambda I - \begin{bmatrix} 0 & I \\ -\mathcal{A} & 0 \end{bmatrix} \right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad [u_1, u_2] \in \mathcal{D} \tag{3.24}$$

means, via (4.1.1) below,

$$\lambda \left[\int_\Gamma u_1 d\Gamma + \int_\Omega u_2 d\Omega \right] = \int_\Gamma f_1 d\Gamma + \int_\Omega f_2 d\Omega, \tag{3.25}$$

so that, given $[f_1, f_2] \in Y_0$, Eqn. (3.24) has a unique solution $[u_1, u_2] \in \mathcal{D}$ for $\lambda > 0$, which, moreover, satisfies the side condition of Y_0 in (3.20), by virtue of (3.25).

(iii) under the additional assumption (H.3), and the uniqueness continuation property of Remark 3.1, the following inequality holds: for all T sufficiently large, there exists a positive constant $k_{\phi,\tau} > 0$ such that

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial \nu_{\mathcal{A}}} \right)^2 d\Sigma_1 \geq \begin{cases} k_{\phi,\tau} E(T); \\ k_{\phi,\tau} E(0). \end{cases} \tag{3.26}$$

Remark 3.5. When $F_1 \equiv 0$ in (3.19a), estimate (3.26) implies (is equivalent to) uniform stabilization of problem (3.19): there exist constants $M \geq 1$, $a > 0$ such that $E(t) \leq M e^{-at} E(0)$, where $\sqrt{E(\,\cdot\,)}$ is the $Y \equiv H^1_{\Gamma_0}(\Omega) \times L_2(\Omega)$-norm (case (i)), or the Y_0-norm in (3.21) (case (ii), where Remark 3.3 is relevant) of the solution $\{w, w_t\}$ of (3.19). $\square$

4 Proof of Theorem 3.3: Carleman estimate (first version)

4.1 Preliminaries

We collect here below a few formulas to be invoked in the sequel.

A Green's formula. Below, in the proof of Proposition 4.2.1, Eqn. (4.2.7), as well as of Lemma 4.2.2, Eqn. (4.2.12), we shall make use of the following Green's formula. Let $z(x) \in C^1(\bar{\Omega})$. Then, the following identity holds true:

$$\int_\Omega (\mathcal{A} w) z \, d\Omega = \int_\Omega \langle \nabla_g w, \nabla_g z \rangle_g d\Omega - \int_\Gamma z \frac{\partial w}{\partial \nu_{\mathcal{A}}} d\Gamma. \tag{4.1.1}$$

In fact, to prove (4.1.1), we write by recalling (2.15) for $\mathcal{A}w$, and the usual divergence formula [14, (A.1)]

$$\int_\Omega (\mathcal{A} w) z \, d\Omega = -\int_\Omega z \, \text{div}_0(\nabla_g w) d\Omega \tag{4.1.2}$$

$$= \int_\Omega \nabla_g w \cdot \nabla_0 z \, d\Omega - \int_\Gamma z \, \nabla_g w \cdot \nu \, d\Gamma. \tag{4.1.3}$$

Then, recalling identity (2.11), and (2.12) for $\frac{\partial w}{\partial \nu_A}$, we see that (4.1.3) leads to (4.1.1), as desired.

An identity. Let ϕ be the function in (3.6a). Let $H = e^{\tau\phi}h$, with $h = \nabla_g \phi$ by (3.6e). Finally, let $X = \nabla_g w$. Then, the following identity to be invoked in the proof of Proposition 4.2.1, Eqn. (4.2.9) holds true:

$$DH(X,X) = \langle D_X H, X\rangle_g = \left\langle D_{\nabla_g w}(e^{\tau\phi}h), \nabla_g w \right\rangle_g \tag{4.1.4}$$

$$= \tau e^{\tau\phi}[h(w)]^2 + e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w), \tag{4.1.5}$$

where we have recalled (2.22), and where $h(w) = \langle \nabla_g w, h\rangle_g$ by (2.11).

Proof of (4.1.5). We preliminarily compute, by using the axioms of the connection D,

$$\begin{aligned} D_X H &= D_X(e^{\tau\phi}h) = X \cdot \nabla_0(e^{\tau\phi})\ h + e^{\tau\phi} D_X h \\ &= \tau e^{\tau\phi} X \cdot \nabla_0 \phi\ h + e^{\tau\phi} D_X h. \end{aligned} \tag{4.1.6}$$

Thus, (4.1.6) yields by (2.11),

$$\langle D_X H, X\rangle_g = \tau e^{\tau\phi} X(\phi)\langle h, X\rangle_g + e^{\tau\phi}\langle D_X h, X\rangle. \tag{4.1.7}$$

As to the second term in (4.1.7), with $h = \nabla_g \phi$ by (3.6e), we have, recalling definition (2.25) of Hessian of ϕ:

$$\langle D_X h, X\rangle_g = \langle D_X(\nabla_g \phi), X\rangle_g \equiv D^2\phi(X,X). \tag{4.1.8}$$

As to the first term in (4.1.17), we have with $X = \nabla_g w$, $h = \nabla_g \phi$, recaling (2.8) or (2.11):

$$X(\phi) = \langle \nabla_g \phi, X\rangle_g = \langle h, X\rangle_g = \langle h, \nabla_g w\rangle_g = h(w). \tag{4.1.9}$$

Thus, (4.1.8) and (4.1.9), used on the R.H.S. of (4.1.7) yields for $X = \nabla_g w$, $h = \nabla_g \phi$:

$$\langle D_X H, X\rangle_g = \tau e^{\tau\phi}[h(w)]^2 + e^{\tau\phi} D^2\phi(X,X), \tag{4.1.10}$$

which, in turn, proves (4.1.5).

A second identity. Let ϕ be the function in (3.6a). The following identity, to be invoked in the proof of Proposition 4.2.1, Eqn. (4.2.11), holds true

$$\langle \nabla_g w, \nabla_g[e^{\tau\phi}\phi_t w_t]\rangle_g = \tau e^{\tau\phi}\phi_t w_t h(w) + \frac{1}{2}\, e^{\tau\phi}\phi_t\, \frac{d}{dt}|\nabla_g w|_g^2. \tag{4.1.11}$$

Proof of (4.1.11). Since ϕ_t does not depend on x, by (3.6e), invoking identity (2.13), we obtain

$$\begin{aligned} \langle \nabla_g w, \nabla_g[e^{\tau\phi}\phi_t w_t]\rangle_g &= \nabla_0[e^{\tau\phi}\phi_t w_t] \cdot \nabla_g w \\ &= \tau e^{\tau\phi}\phi_t w_t \nabla_0 \phi \cdot \nabla_g w + e^{\tau\phi}\phi_t \nabla_0 w_t \cdot \nabla_g w, \end{aligned} \tag{4.1.12}$$

where, with $h = \nabla_g \phi$ by (3.6e), and recalling (2.11), we have

$$\nabla_0 \phi \cdot \nabla_g w = \langle \nabla_g \phi, \nabla_g w\rangle_g = \langle h, \nabla_g w\rangle_g = h(w); \tag{4.1.13}$$

$$\nabla_0 w_t \cdot \nabla_g w = \langle \nabla_g w_t, \nabla_g w\rangle_g = \frac{1}{2}\,\frac{d}{dt}|\nabla_g w|_g^2. \tag{4.1.14}$$

Inserting (4.1.13) and (4.1.14) in (4.1.12) yields (4.1.11), as desired.

4.2 Energy Methods in the Riemann Metric

We will complete the proof of Theorem 3.3 through several propositions. The strategy follows closely the proof of [17, Section 2] for constant coefficient principal part ($\mathcal{A} = -\Delta$), except that it is carried out in the Riemann metric g defined by (2.2), rather than in the Euclidean metric as in [17]. The close parallelism between the present treatment and that of [17] will be emphasized in the intermediate results as well. The counterpart of [17, Proposition 2.2.1] is

Step 1. Proposition 4.2.1. *Let w be a solution of Eqn. (3.7) within the class (3.8). Then the following identity holds true, where $\Sigma = [0,T] \times \Gamma$; $Q = [0,T] \times \Omega$:*

$$\begin{aligned}
&\int_\Sigma e^{\tau\phi} \frac{\partial w}{\partial \nu_{\mathcal{A}}}[h(w) - \phi_t w_t] d\Sigma + \frac{1}{2}\int_\Sigma e^{\tau\phi}[w_t^2 - |\nabla_g w|_g^2] h \cdot \nu \, d\Sigma \\
&= \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ + \frac{1}{2}\int_Q [w_t^2 - |\nabla_g w|_g^2] \, div_0(e^{\tau\phi} h) dQ \\
&+ \frac{1}{2}\int_Q [w_t^2 + |\nabla_g w|_g^2] \frac{d}{dt}(e^{\tau\phi}\phi_t) dQ + \tau \int_Q e^{\tau\phi}[h(w)]^2 dQ \\
&- 2\tau \int_Q e^{\tau\phi} h(w)\phi_t w_t dQ + \left[\int_\Omega e^{\tau\phi}\left[h(w) - \frac{1}{2}\phi_t w_t\right] w_t d\Omega\right]_0^T \\
&- \frac{1}{2}\left[\int_\Omega e^{\tau\phi}\phi_t |\nabla_g w|_g^2 d\Omega\right]_0^T - \int_Q [F_1(w) + f] e^{\tau\phi}[h(w) - w_t\phi_t] dQ. \qquad (4.2.1)
\end{aligned}$$

In (4.2.1), we have $h(x) = \nabla_g \phi = \nabla_g v(x)$, see (3.3), (3.6e), while $D^2\phi(\ ,\ )$ is the Hessian (as defined in (2.25)) of the function ϕ in (3.6a); finally, $h(w) = \langle h, \nabla_g w\rangle_g = \langle \nabla_g v, \nabla_g w\rangle_g = \nabla_0 w \cdot h$ by (2.8), and (2.11), with the vector field h defined by (3.3).

Proof. We multiply both sides of Eqn. (3.7) by the following main multiplier

$$e^{\tau\phi(x,t)}[h(w) - \phi_t w_t], \qquad (4.2.2)$$

counterpart of the one in [17, Eqn. (2.2.1a)] and integrate over $Q = [0,T] \times \Omega$ by parts.

Left-hand side. We shall show below that on the left-hand side (L.H.S.) of (3.7) we obtain, recalling Lemma 2.1(b)–(e),

$$\begin{aligned}
\text{L.H.S.} &= \int_0^T \int_\Omega w_{tt} e^{\tau\phi}[\nabla_g \phi(w) - \phi_t w_t] d\Omega \, dt \\
&= \frac{1}{2}\int_Q w_t^2 \left[\text{div}_0(e^{\tau\phi} h) + \frac{d}{dt}(e^{\tau\phi}\phi_t)\right] dQ - \tau \int_Q e^{\tau\phi} h(w)\phi_t w_t dQ \\
&- \frac{1}{2}\int_\Sigma e^{\tau\phi} w_t^2 h \cdot \nu \, d\Sigma + \left[\int_\Omega e^{\tau\phi}\left[h(w) - \frac{1}{2}\phi_t w_t\right] w_t d\Omega\right]_0^T, \qquad (4.2.3)
\end{aligned}$$

where $\nabla_g \phi = h$ by (3.6e), so that $\nabla_g \phi(w) = h(w) = \langle \nabla_g w, h\rangle_g = \nabla_0 w \cdot h$ by (2.8), (2.11). Indeed, integrating by parts in t, and recalling that $\nabla_g \phi = h(x)$ is time-independent, we compute

$$\int_\Omega \int_0^T w_{tt} e^{\tau\phi} h(w) dt \, d\Omega = \left[\int_\Omega e^{\tau\phi} w_t h(w) d\Omega\right]_0^T$$

$$- \tau \int_Q e^{\tau\phi} \phi_t w_t h(w) dQ - \int_Q e^{\tau\phi} w_t h(w_t) dQ. \tag{4.2.4}$$

Now the last term in (4.2.4), where $h(w_t) = h \cdot \nabla_0 w_t$ by (2.11), is rewritten, by the usual formula for divergence [14, (A.1)], as

$$\begin{aligned} \int_Q e^{\tau\phi} w_t h(w_t) dQ &= \frac{1}{2} \int_Q e^{\tau\phi} h(w_t^2) dQ = \frac{1}{2} \int_Q e^{\tau\phi} h \cdot \nabla_0 (w_t^2) dQ \\ &= \frac{1}{2} \int_\Sigma e^{\tau\phi} w_t^2 h \cdot \nu \, d\Sigma - \frac{1}{2} \int_Q w_t^2 \text{div}_0 (e^{\tau\phi} h) dQ. \end{aligned} \tag{4.2.5}$$

Similarly, integrating by parts in t, we compute

$$\begin{aligned} \int_\Omega \int_0^T w_{tt} e^{\tau\phi} \phi_t w_t dt \, d\Omega &= \frac{1}{2} \int_\Omega \int_0^T e^{\tau\phi} \phi_t \frac{d}{dt} (w_t^2) dt \, d\Omega \\ &= \left[\frac{1}{2} \int_\Omega e^{\tau\phi} \phi_t w_t^2 d\Omega \right]_0^T - \frac{1}{2} \int_Q w_t^2 \frac{d}{dt} (e^{\tau\phi} \phi_t) dQ. \end{aligned} \tag{4.2.6}$$

Using (4.2.5) in (4.2.4) and subtracting off (4.2.6) yields (4.2.3), as desired.

Right-hand side. Multiplying the right-hand side (R.H.S.) $[-\mathcal{A}w + F_1(w) + f]$ of Eqn. (3.7) by the multiplier in (4.2.2), $e^{\tau\phi}[h(w) - \phi_t w_t]$, and integrating over $Q = [0, T] \times \Omega$, we obtain

$$\begin{aligned} \text{R.H.S.} &= - \int_0^T \int_\Omega (\mathcal{A}w) e^{\tau\phi} [h(w) - \phi_t w_t] d\Omega \, dt + \int_Q [F_1(w) + f] e^{\tau\phi} [h(w) - \phi_t w_t] dQ \\ &= \int_\Sigma e^{\tau\phi} [h(w) - \phi_t w_t] \frac{\partial w}{\partial \nu_{\mathcal{A}}} d\Sigma - \frac{1}{2} \int_\Sigma |\nabla_g w|_g^2 e^{\tau\phi} h \cdot \nu \, d\Sigma \\ &+ \frac{1}{2} \int_Q |\nabla_g w|_g^2 \left\{ \text{div}_0 (e^{\tau\phi} h) - \frac{d}{dt} (e^{\tau\phi} \phi_t) \right\} dQ - \int_Q e^{\tau\phi} D^2 \phi (\nabla_g w, \nabla_g w) dQ \\ &- \tau \int_Q e^{\tau\phi} [h(w)]^2 dQ + \tau \int_Q e^{\tau\phi} \phi_t w_t h(w) dQ + \frac{1}{2} \left[\int_\Omega e^{\tau\phi} \phi_t |\nabla_g w|_g^2 d\Omega \right]_0^T \\ &+ \int_Q [F_1(w) + f] e^{\tau\phi} [h(w) - \phi_t w_t] dQ, \end{aligned} \tag{4.2.7}$$

where $D^2(\quad , \quad)$ is the Hessian (see (2.25)) of the function ϕ defined by (3.6a).

Proof of (4.2.7). Indeed, using Green's formula (4.1.1) with $z = e^{\tau\phi}[h(w) - \phi_t w_t]$, we compute

$$\begin{aligned} - \int_\Omega (\mathcal{A}w) e^{\tau\phi} [h(w) - \phi_t w_t] d\Gamma &= \int_\Gamma e^{\tau\phi} [h(w) - \phi_t w_t] \frac{\partial w}{\partial \nu_{\mathcal{A}}} d\Gamma \\ &- \int_\Omega \langle \nabla_g w, \nabla_g [e^{\tau\phi} h(w)] \rangle_g d\Omega + \int_\Omega \langle \nabla_g w, \nabla_g [e^{\tau\phi} \phi_t w_t] \rangle_g d\Omega. \end{aligned} \tag{4.2.8}$$

As to the first integral over Ω on the right-hand side of (4.2.8), we obtain from Lemma 2.1(f), Eqn. (2.14), with $H = e^{\tau\phi}h$, as well as by identity (4.1.5),

$$\begin{aligned}
\langle \nabla_g w, \nabla_g [e^{\tau\phi} h(w)]\rangle_g &= D(e^{\tau\phi}h)(\nabla_g w, \nabla_g w) + \frac{1}{2}\,\mathrm{div}_0(|\nabla_g w|_g^2 e^{\tau\phi} h) \\
&\quad - \frac{1}{2}|\nabla_g w|_g^2\,\mathrm{div}_0(e^{\tau\phi}h) \\
\text{(by (4.1.5))} \qquad &= \tau e^{\tau\phi}[h(w)]^2 + e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) \\
&\quad + \frac{1}{2}\,\mathrm{div}_0(|\nabla_g w|_g^2 e^{\tau\phi} h) - \frac{1}{2}\,|\nabla_g w|_g^2\,\mathrm{div}_0(e^{\tau\phi}h). \qquad (4.2.9)
\end{aligned}$$

We next integrate (4.2.9) over Q, apply the divergence theorem $\int_\Omega \mathrm{div}_0(zh)d\Omega = \int_\Gamma z\,h\cdot\nu\,d\Gamma$ with $z = |\nabla_g w|_g^2 e^{\tau\phi}$ to the third term of (4.2.9) and obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \langle \nabla_g w, \nabla_g [e^{\tau\phi}h(w)]\rangle_g d\Omega\, dt \\
&= \frac{1}{2}\int_\Sigma e^{\tau\phi}|\nabla_g w|_g^2 h\cdot\nu\, d\Sigma + \tau\int_Q e^{\tau\phi}[h(w)]^2 dQ \\
&\quad + \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w)dQ - \frac{1}{2}\int_Q |\nabla_g w|_g^2\,\mathrm{div}_0(e^{\tau\phi}h)dQ. \qquad (4.2.10)
\end{aligned}$$

As to the second integral term over Ω on the right-hand side of (4.2.8), we invoke identity (4.1.11) and integrate by parts,

$$\begin{aligned}
&\int_0^T \int_\Omega \langle \nabla_g w, \nabla_g [e^{\tau\phi}\phi_t w_t]\rangle_g dQ \\
\text{(by (4.1.11))} \qquad &= \tau\int_Q e^{\tau\phi}\phi_t w_t h(w)dQ + \frac{1}{2}\int_\Omega\int_0^T e^{\tau\phi}\phi_t\,\frac{d}{dt}\,|\nabla_g w|_g^2 dt\, d\Omega \\
&= \tau\int_Q e^{\tau\phi}\phi_t w_t h(w)dQ + \frac{1}{2}\left[\int_\Omega e^{\tau\phi}\phi_t|\nabla_g w|_g^2 d\Omega\right]_0^T \\
&\quad - \frac{1}{2}\int_Q |\nabla_g w|_g^2\,\frac{d}{dt}\,(e^{\tau\phi}\phi_t)dQ. \qquad (4.2.11)
\end{aligned}$$

Next, after (4.2.8) has been integrated over $[0,T]$, we insert (4.2.10) and (4.2.11) into it and obtain (4.2.7), as desired.

Finally, we combine the L.H.S. = (4.2.3) with the R.H.S. = (4.2.7), and we thus obtain (4.2.1). The proof of Proposition 4.2.1 is complete. □

Step 2. The following lemma will be invoked repeatedly for various suitable choices of the function $m(x,t)$.It is the counterpart of [17, Lemma 2.2.2].

Lemma 4.2.2. *Let w be a solution of Eqn. (3.7) in the class (3.8). Let $m(x,t)$ be C^1-function defined over Q. Then the following identity holds true*

$$\int_Q (w_t^2 - |\nabla_g w|_g^2) m\, dQ = \int_Q w\langle \nabla_g w, \nabla_g m\rangle_g dQ - \int_Q w w_t m_t dQ$$

$$- \int_Q [F_1(w) + f] wm \, dQ + \left[\int_\Omega w w_t m \, dQ \right]_0^T - \int_\Sigma wm \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Sigma. \tag{4.2.12}$$

Proof. We multiply both sides of Eqn. (3.7) by wm and integrate by parts, invoking the Green formula (4.1.1). This way, (4.2.12) is obtained. $\square$

Proposition 4.2.3. *Let w be a solution of Eqn. (3.7) in the class (3.8). Then the following identity holds true:*

$$\int_\Sigma e^{\tau\phi} [h(w) - \phi_t w_t] \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Sigma + \frac{1}{2} \int_\Sigma w \mu \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Sigma + \frac{1}{2} \int_\Sigma e^{\tau\phi} [w_t^2 - |\nabla_g w|_g^2] h \cdot \nu \, d\Sigma$$

$$= \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ - 2c \int_Q e^{\tau\phi} w_t^2 dQ + \tau \int_Q e^{\tau\phi} [h(w) - w_t \phi_t]^2 dQ$$

$$+ \frac{1}{2} \int_Q w \langle \nabla_g w, \nabla_g \mu \rangle_g dQ - \frac{1}{2} \int_Q w w_t \mu_t dQ - \frac{1}{2} \int_Q [F_1(w) + f] w \mu \, dQ$$

$$- \int_Q [F_1(w) + f] e^{\tau\phi} [h(w) - \phi_t w_t] dQ + [\alpha(t)]_0^T, \tag{4.2.13}$$

where $h(w) = \langle \nabla_g w, h \rangle_g$ by (2.11), $D^2\phi(\ \ ,\ \ )$ is the Hessian of ϕ (see (2.25)) and where we have set

$$\mu(x,t) = div_0(e^{\tau\phi} h) - \frac{d}{dt} (e^{\tau\phi} \phi_t) = e^{\tau\phi} [\tau |\nabla_g \phi|_g^2 - \tau \phi_t^2 - \mathcal{A}\phi + 2c]; \tag{4.2.14}$$

$$\alpha(t) = \int_\Omega e^{\tau\phi} w_t \left[h(w) - \frac{1}{2} \phi_t w_t \right] d\Omega - \frac{1}{2} \int_\Omega e^{\tau\phi} \phi_t |\nabla_g w|_g^2 d\Omega + \frac{1}{2} \int_\Omega w_t w \mu \, d\Omega. \tag{4.2.15}$$

Proof. We apply Lemma 4.2.2 with the choice $m = \mu$ in (4.2.14), and obtain from (4.2.12),

$$\frac{1}{2} \int_Q [w_t^2 - |\nabla_g w|_g^2] \text{div}_0(e^{\tau\phi} h) dQ$$

$$= \frac{1}{2} \int_Q [w_t^2 - |\nabla_g w|_g^2] \frac{d}{dt} (e^{\tau\phi} \phi_t) dQ + \frac{1}{2} \int_Q w \langle \nabla_g w, \nabla_g \mu \rangle_g dQ - \frac{1}{2} \int_Q w w_t \mu_t dQ$$

$$- \frac{1}{2} \int_Q [F_1(w) + f] w \mu \, dQ + \frac{1}{2} \left[\int_\Omega w w_t \mu \, d\Omega \right]_0^T - \frac{1}{2} \int_\Sigma w \mu \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Sigma. \tag{4.2.16}$$

Inserting (4.2.16) into the right-hand side of (4.2.1), to replace the second integral term over Q, yields after a cancellation,

$$\int_\Sigma e^{\tau\phi} [h(w) - \phi_t w_t] \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Sigma + \frac{1}{2} \int_\Sigma e^{\tau\phi} [w_t^2 - |\nabla_g w|_g^2] h \cdot \nu \, d\Sigma + \frac{1}{2} \int_\Sigma w \mu \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, d\Sigma$$

$$= \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ + \int_Q w_t^2 \frac{d}{dt} (e^{\tau\phi} \phi_t) dQ$$

$$+\frac{1}{2}\int_Q w\langle\nabla_g w,\nabla_g\mu\rangle_g dQ-\frac{1}{2}\int_Q ww_t\mu_t dQ$$

$$+\tau\int_Q e^{\tau\phi}[h(w)]^2 dQ-2\tau\int_Q e^{\tau\phi}h(w)\phi_t w_t dQ$$

$$+\frac{1}{2}\left[\int_\Omega ww_t\mu\, d\Omega\right]_0^T+\left[\int_\Omega e^{\tau\phi}\left[h(w)-\frac{1}{2}\phi_t w_t\right]w_t d\Omega\right]_0^T$$

$$-\frac{1}{2}\left[\int_\Omega e^{\tau\phi}\phi_t|\nabla_g w|_g^2 d\Omega\right]_0^T-\frac{1}{2}\int_Q[F_1(w)+f]w\mu\, dQ$$

$$-\int_Q[F_1(w)+f]e^{\tau\phi}[h(w)-w_t\phi_t]dQ. \tag{4.2.17}$$

We next combine the second, the fifth, and the sixth term on the right-hand side of (4.2.17) in a perfect square, as follows:

$$\int_Q w_t^2\frac{d}{dt}(e^{\tau\phi}\phi_t)dQ-2\tau\int_Q e^{\tau\phi}\phi_t w_t h(w)dQ+\tau\int_Q e^{\tau\phi}[h(w)]^2 dQ$$

$$=\tau\int_Q e^{\tau\phi}[h(w)-\phi_t w_t]^2 dQ-2c\int_Q e^{\tau\phi}w_t^2 dQ, \tag{4.2.18}$$

expanding $\frac{d}{dt}(e^{\tau\phi}\phi_t)=e^{\tau\phi}[\tau\phi_t^2+\phi_{tt}]=e^{\tau\phi}[\tau\phi_t^2-2c]$, see (3.6e). Using (4.2.18) into the right-hand side of (4.2.17) yields (4.2.13) via (4.2.15), as desired. □

The following result is the counterpart of [17, Theorem 2.2.4].

Step 3. Theorem 4.2.4. (Final identity) *Let w be a solution of Eqn. (3.7) in the class (3.8). Then the following identity holds true:*

$$(BT_w)|_\Sigma = \int_Q e^{\tau\phi}D^2\phi(\nabla_g w,\nabla_g w)dQ-2\rho\int_Q e^{\tau\phi}|\nabla_g w|_g^2 dQ$$

$$+(\rho-c)\int_Q e^{\tau\phi}[|\nabla_g w|_g^2+w_t^2]dQ+\tau\int_Q e^{\tau\phi}[h(w)-\phi_t w_t]^2 dQ$$

$$+\int_Q ww_t\left[(\rho+c)\frac{d}{dt}(e^{\tau\phi})-\frac{1}{2}\mu_t\right]dQ$$

$$+\int_Q w\left\langle\nabla_g w,\frac{1}{2}\nabla_g\mu-(\rho+c)\nabla_g(e^{\tau\phi})\right\rangle_g dQ$$

$$+\int_Q[F_1(w)+f]w\left[(\rho+c)e^{\tau\phi}-\frac{1}{2}\mu\right]dQ$$

$$-\int_Q[F_1(w)+f]e^{\tau\phi}[h(w)-\phi_t w_t]dQ+\beta(t)|_0^T, \tag{4.2.19}$$

where μ is defined by (4.2.14). Moreover, the boundary term $(BT_w)|_\Sigma$ is given by

$$\begin{aligned}(BT_w)|_\Sigma &= \int_\Sigma e^{\tau\phi}[h(w) - \phi_t w_t]\frac{\partial w}{\partial \nu_\mathcal{A}}\, d\Sigma + \int_\Sigma w \frac{\partial w}{\partial \nu_\mathcal{A}}\left[\frac{1}{2}\mu - (\rho + c)e^{\tau\phi}\right] d\Sigma \\ &+ \frac{1}{2}\int_\Sigma e^{\tau\phi}[w_t^2 - |\nabla_g w|_g^2] h \cdot \nu \, d\Sigma, \end{aligned} \tag{4.2.20}$$

and $\beta(t)$ is defined by

$$\beta(t) = \alpha(t) - (\rho + c)\int_\Omega e^{\tau\phi} w_t w \, d\Omega, \tag{4.2.21}$$

where $\alpha(t)$ is defined in (4.2.15).

Proof. We return to the first two integral terms in Q on the right-hand side of identity (4.2.13) and rewrite them, after adding and subtracting, as

$$\begin{aligned}&\int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ - 2c\int_Q e^{\tau\phi} w_t^2 dQ \\ &= \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ - 2\rho\int_Q e^{\tau\phi}|\nabla_g w|_g^2 dQ \\ &+ 2\rho\int_Q e^{\tau\phi}|\nabla_g w|_g^2 dQ - 2c\int_Q e^{\tau\phi} w_t^2 dQ \\ &= \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ - 2\rho\int_Q e^{\tau\phi}|\nabla_g w|_g^2 dQ \\ &+ (\rho - c)\int_Q e^{\tau\phi}|\nabla_g w|_g^2 dQ + (\rho - c)\int_Q e^{\tau\phi} w_t^2 dQ \\ &+(\rho + c)\int_Q e^{\tau\phi}[|\nabla_g w|_g^2 - w_t^2] dQ. \end{aligned} \tag{4.2.22}$$

Next, we apply Lemma 4.2.2, Eqn. (4.2.12) with the choice $m = e^{\tau\phi}$, and obtain

$$\begin{aligned}&\int_Q e^{\tau\phi}[|\nabla_g w|_g^2 - w_t^2] dQ \\ &= \int_Q w w_t \frac{d}{dt}(e^{\tau\phi}) dQ - \int_Q w\langle \nabla_g w, \nabla_g(e^{\tau\phi})\rangle_g dQ + \int_\Sigma \frac{\partial w}{\partial \nu_\mathcal{A}} w e^{\tau\phi} d\Sigma \\ &- \left[\int_Q w_t w e^{\tau\phi} d\Omega\right]_0^T + \int_Q [F_1(w) + f] w e^{\tau\phi} dQ. \end{aligned} \tag{4.2.23}$$

We then use (4.2.23) into the last term of (4.2.22) after inserting (4.2.22) into (4.2.13) and obtain (4.2.19). $\square$

Step 4. Henceforth, we concentrate our analysis on the right-hand side (R.H.S.) of the fundamental identity (4.2.19) of Theorem 4.2.4. So far, the parameter $\tau > 0$ has been arbitrary. The next lemma and its proof show the key virtue of the free parameter τ entering the present multiplier (4.2.2) in dealing with the general first-order differential operator $F_1(w)$ as in (1.3): choosing τ sufficiently large permits the absorption of a bad energy level term, which arises precisely because $F_1(w)$ is of order one.

Lemma 4.2.5. *Let w be a solution of Eqn. (3.7) in the class (3.8). With reference to some selected terms on the right-hand side of identity (4.2.19), we have:*

(i) For any $\epsilon > 0$, we have recalling (1.3) and $h(w) = \langle h, \nabla_g w\rangle_g$ with $h = \nabla_g \phi$:

$$\tau \int_Q e^{\tau\phi}[h(w) - \phi_t w_t]^2 dQ - \int_Q [F_1(w) + f]e^{\tau\phi}[h(w) - \phi_t w_t] dQ$$

$$\geq \left(\tau - \frac{C_T}{2\epsilon}\right)\int_Q e^{\tau\phi}[h(w) - \phi_t w_t]^2 dQ - \frac{\epsilon}{2}\int_Q [w_t^2 + |\nabla_g w|_g^2 + w^2 + f^2]e^{\tau\phi} dQ, \tag{4.2.24}$$

where C_T is the constant in (1.3).

(ii) Next, for any $\epsilon > 0$, we have

$$\int_Q w w_t \left[(\rho + c)\frac{d(e^{\tau\phi})}{dt} - \frac{1}{2}\mu_t\right] dQ + \int_Q w\left\langle \nabla_g w, \nabla_g\left[\frac{1}{2}\mu - (\rho + c)e^{\tau\phi}\right]\right\rangle_g dQ$$

$$+ \int_Q [F_1(w) + f]w\left[(\rho + c)e^{\tau\phi} - \frac{1}{2}\mu\right] dQ$$

$$\geq -\frac{\epsilon}{2}\int_Q e^{\tau\phi}[w_t^2 + |\nabla_g w|_g^2 + f^2] dQ - \frac{T\,\mathrm{const}_\tau}{2\epsilon}\|w\|^2_{C([0,T];L_2(\Omega))}, \tag{4.2.25}$$

where const_τ is a constant depending on τ.

(iii) Furthermore, recalling (4.2.21) and (4.2.15), we have

$$|\beta(t)|_0^T| \leq C(1 + \tau)e^{-\delta\tau}\left[E(0) + E(T) + \|w\|^2_{C([0,T];L_2(\Omega))}\right], \tag{4.2.26}$$

where the constant C is independent of T or τ, and where $E(\,\cdot\,)$ is defined in (3.11).

Proof. For both (i) and (ii) we use the inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$, where a denotes "energy terms": w_t, $|\nabla_g w|_g$, $F_1(w)$; while b denotes "lower-order terms" (i.e., w). Here, we recall (1.3) for $F_1(w)$ as well as (4.2.14) for μ and (3.6e).

(iii) Here we use (3.6c) in estimating $\alpha(t)$ and $\beta(t)$ in (4.2.15) and (4.2.21); see also (4.2.14) for μ.

Remark 4.2.1. In the second integral over Q on the left-hand side of (4.2.24), both factors $F_1(w)$ and $[h(w) - \phi_t w_t]^2$, $h(w) = \langle h, \nabla_g w\rangle_g$ by (2.11), are energy level, when F_1 is a general first-order operator. The virtue of the free parameter τ is seen in the first term on the right-hand side of (4.2.24), in making the coefficient $\tau - \frac{C_T}{2\epsilon} > 0$ after $\epsilon > 0$ has been fixed, and dropping that term, see next result. □

Step 5. We complete the proof of Theorem 3.3. As explained in the above Remark 4.2.1, with $\epsilon > 0$ given in Lemma 4.2.5, we select the parameter τ as $\tau = \frac{C_T}{\epsilon}$ so that $\tau - \frac{C_T}{2\epsilon} = \frac{C_T}{2\epsilon} > 0$ drop the first term on the R.H.S. of (4.2.24), then use the remaining version of inequality (4.2.24) along with (4.2.25) in the right-hand side of identity (4.2.19), We obtain

$$(BT_w)\Big|_\Sigma + T\frac{\mathrm{const}_\tau}{2\epsilon}\|w\|^2_{C([0,T];L_2(\Omega))} + \epsilon\int_Q e^{\tau\phi} f^2 dQ$$

$$\geq \int_Q e^{\tau\phi} D^2\phi(\nabla_g w, \nabla_g w) dQ - 2\rho \int_Q e^{\tau\phi} |\nabla_g w|_g^2 dQ$$

$$+ (\rho - c - \epsilon) \int_Q e^{\tau\phi} [|\nabla_g w|_g^2 + w_t^2] dQ - |[\beta(t)]_0^T|. \tag{4.2.27}$$

Next, we invoke assumption (H.3), in the form of Eqn. (3.2b), so that the first two terms on the right side of (4.2.27) vanish; moreover, we recall (4.2.26) for $[\beta(t)]_0^T$, and thus obtain the desired inequality (3.9a) from (4.2.27), where $\epsilon = \frac{C_T}{\tau}$. Then, inequality (3.9a) yields (3.9b), by recalling property (3.6d) of ϕ. Theorem 3.3 is proved. □

5 Proof of Theorem 3.4: Carleman estimates (second version); and of Theorem 3.1 (Dirichlet case)

5.1 Proof of Theorem 3.4

Having already established Eqn. (3.9) of Theorem 3.3, as proved in Section 4, we obtain then Eqn. (3.14) of Theorem 3.4, by simply using in the integral $\int_{t_0}^{t_1} E(t)dt$ on the R.H.S. of (3.9b) the inequality

$$E(t) \geq \frac{E(0) + E(T)}{2} \; e^{-C_T T} - \Lambda(T), \; 0 \leq t \leq T; \tag{5.1.1}$$

$$\Lambda(T) = \int_0^T \int_\Omega f^2 dQ + 2 \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, w_t \right| d\Sigma + C_T \int_0^T \|w\|_{L_2(\Omega)}^2 dt. \tag{5.1.2}$$

But all this follows directly from [17, Lemma 2.3.1] which yields for $T \geq t \geq s \geq 0$:

$$E(t) \leq [E(s) + \Lambda(T)] e^{C_T(t-s)}; \quad E(s) \leq [E(t) + \Lambda(T)] e^{C_T(t-s)} \tag{5.1.3}$$

in its proof [17, Eqn. (2.3.6)]. Next, the inequality on the right of (5.1.3) with $s = 0$, and that on the left with $t = T$ and $s = t$ yield then

$$E(0) \leq [E(t) + \Lambda(T)] e^{C_T T}; \quad E(T) \leq [E(t) + \Lambda(T)] e^{C_T T}. \tag{5.1.4}$$

Summing up these two inequalities in (5.1.4), we arrive at (5.1.1), as desired.

Thus, using (5.1.1) into the integral over $[t_0, t_1]$ on the right side of (3.9b), we obtain

$$(BT_w)\Big|_\Sigma + \left[\frac{C_T}{\tau} + \left(\rho - c - \frac{C_T}{\tau} \right) e^{-\frac{\tau\delta}{2}} (t_1 - t_0) \right] \int_Q f^2 dQ$$

$$+ 2 \int_0^T \int_\Gamma \left| \frac{\partial w}{\partial \nu_{\mathcal{A}}} \, w_t \right| d\Sigma + T C_T \; \text{const}_\tau \|w\|_{C([0,T]; L_2(\Omega))}^2$$

$$\geq e^{-\frac{\tau\delta}{2}} \left\{ \left(\rho - c - \frac{C_T}{\tau} \right) \frac{(t_1 - t_0)}{2} \; e^{-C_T T} - C(1 + \tau) e^{-\frac{\delta\tau}{2}} \right\} [E(T) + E(0)], \tag{5.1.5}$$

from which (3.14) of Theorem 3.4 is obtained, via (3.16). The proof of Theorem 3.4 is complete. □

5.2 Proof of Theorem 3.1 (Dirichlet case)

Let ψ be a solution of the ψ-problem in (1.4) (including the B.C. $\psi|_\Sigma \equiv 0$). We want to apply Theorem 3.4 to it.

Step 1. First, we deal with the values of $|\nabla_g\psi|_g^2$ and $h(\psi)$ on the boundary Γ, respectively, as required by $(BT_w)\Big|_\Sigma$ in (3.10).

Lemma 5.2.1. *Let ψ be the solution of problem (1.4) [including the B.C. $\psi|_\Sigma \equiv 0$]. then, in this case, the boundary term $(\overline{BT}_\psi)|_\Sigma$ defined by Eqns. (3.16) and (3.10) reduce to*

$$(\overline{BT}_\psi)|_\Sigma = (BT_\psi)|_\Sigma = \frac{1}{2}\int_\Sigma e^{\tau\phi}\left(\frac{\partial\psi}{\partial\nu_\mathcal{A}}\right)^2 \frac{h\cdot\nu}{|\nu_\mathcal{A}|_g^2}\,d\Sigma, \tag{5.2.1}$$

where, via (2.5), we define $\nu_\mathcal{A}(x)$,

$$\nu_\mathcal{A}(x) = \sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}(x)\nu_j(x)\right)\frac{\partial}{\partial x_i} = A(x)\nu(x). \tag{5.2.2}$$

to be the normal of the submanifold Γ in the Riemannian metric g.

Proof. Given $x\in\mathbb{R}^n$, the vector $\nabla_g\psi(x)$ has the decomposition into direct product in $(\mathbb{R}_x^n, g(x))$ as

$$\nabla_g\psi(x) = \left\langle\nabla_g\psi(x), \frac{\nu_\mathcal{A}}{|\nu_\mathcal{A}|_g}\right\rangle_g \frac{\nu_\mathcal{A}}{|\nu_\mathcal{A}|_g} + Y(x) = \left(\frac{1}{|\nu_\mathcal{A}|_g^2}\frac{\partial\psi}{\partial\nu_\mathcal{A}}\right)\nu_\mathcal{A} + \frac{\partial\psi}{\partial s}s. \tag{5.2.3}$$

Here, by (5.2.2), (2.9), (2.12),

$$\langle\nabla_g\psi(x), \nu_\mathcal{A}(x)\rangle_g = \langle\nabla_g\psi(x), A(x)\nu(x)\rangle_g = \nabla_g\psi(x)\cdot\nu(x) = \frac{\partial\psi}{\partial\nu_\mathcal{A}}. \tag{5.2.4a}$$

Moreover, $Y(x)\in\mathbb{R}_x^n$ satisfies $\langle Y(x),\nu_\mathcal{A}\rangle_g = 0$; consequently, by (2.9) and (5.2.2), $Y(x)\cdot\nu(x) = \langle Y(x),\nu_\mathcal{A}(x)\rangle_g = 0$, that is, $Y(x)\in\Gamma_x$, the tangent space of Γ at x. Therefore, if s denotes a unit tangent vector, then, by (2.11):

$$Y(x) = \langle\nabla_g\psi(x), s\rangle_g = \nabla_0\psi(x)\cdot s = \frac{\partial\psi(x)}{\partial s} \tag{5.2.4b}$$

is the tangential gradient. Thus, (5.2.4a–b) show the right side of (5.2.3). By (5.2.3), (2.11), we have

$$|\nabla_g\psi|_g^2 = \langle\nabla_g\psi,\nabla_g\psi\rangle_g = \nabla_g\psi(\psi) = \frac{1}{|\nu_\mathcal{A}(x)|_g^2}\langle\nabla_g\psi(x),\nu_\mathcal{A}(x)\rangle_g^2 + Y(\psi) \tag{5.2.5a}$$

$$= \frac{1}{|\nu_\mathcal{A}(x)|_g^2}\left(\frac{\partial\psi}{\partial\nu_\mathcal{A}}\right)^2, \tag{5.2.5b}$$

since $\psi|_\Sigma = 0$, hence $\nabla_0\psi \perp \Gamma$ and hence $Y(\psi) = \nabla_0\psi\cdot Y = 0$ by (2.11). Similarly, $h(x)$ has the decomposition into direct product

$$h(x) = \left\langle h(x), \frac{\nu_\mathcal{A}(x)}{|\nu_\mathcal{A}(x)|_g}\right\rangle_g \frac{\nu_\mathcal{A}(x)}{|\nu_\mathcal{A}(x)|_g} + Z(x), \tag{5.2.6}$$

where $Z(x)\in\Gamma_x$. Moreover, by (5.2.2), (2.12), (2.11), we have

$$\frac{\partial\psi}{\partial\nu_\mathcal{A}} = (A(x)\nabla_0\psi)\cdot\nu(x) = \nabla_0\psi\cdot A(x)\nu(x) = \nabla_0\psi\cdot\nu_\mathcal{A}(x) = \langle\nabla_g\psi,\nu_\mathcal{A}\rangle_g, \tag{5.2.7}$$

since the matrix $A(x)$ is symmetric. Hence, by (2.11), (5.2.2), (5.2.6), (5.2.7), (2.9),

$$h(\psi)(x) = \langle \nabla_g \psi, h \rangle_g = \left\langle h(x), \frac{\nu_{\mathcal{A}}(x)}{|\nu_{\mathcal{A}}(x)|_g^2} \right\rangle_g \langle \nabla_g \psi, \nu_{\mathcal{A}}(x) \rangle_g + \langle \nabla_g \psi, Z(x) \rangle_g \tag{5.2.8}$$

$$h(\psi)(x) = \frac{\langle h(x), \nu_{\mathcal{A}}(x) \rangle_g}{|\nu_{\mathcal{A}}(x)|_g^2} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right) = \frac{\langle h(x), \mathcal{A}(x)\nu(x) \rangle_g}{|\nu_{\mathcal{A}}(x)|_g^2} \frac{\partial \psi}{\partial \nu_{\mathcal{A}}}$$

$$(\text{by } (2.9)) \quad = \frac{h(x) \cdot \nu(x)}{|\nu_{\mathcal{A}}(x)|_g^2} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right), \tag{5.2.9}$$

since, as before, $\psi|_\Sigma \equiv 0$, hence, $\nabla_0 \psi \perp \Gamma$, and $\langle \nabla_g \psi, Z \rangle_g = \nabla_0 \psi \cdot Z = 0$, via (2.11).

Finally, we return to definition (3.10) for $BT\Big|_\Sigma$ (written for ψ), use here $\psi|_\Sigma \equiv 0$, hence $\psi_t \equiv 0$, as well as (5.2.5) and (5.2.9), to obtain

$$(BT_\psi)|_\Sigma = \int_\Sigma e^{\tau\phi} \frac{\partial \psi}{\partial \nu_{\mathcal{A}}} h(\psi) d\Sigma - \frac{1}{2} \int_\Sigma e^{\tau\phi} |\nabla_g \psi|_g^2 h \cdot \nu \, d\Sigma \tag{5.2.10}$$

$$= \int_\Sigma e^{\tau\phi} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right)^2 \frac{h \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Sigma - \frac{1}{2} \int_\Sigma e^{\tau\phi} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right)^2 \frac{h \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Sigma. \tag{5.2.11}$$

Then, (5.2.11) yields (5.2.1), as desired. □

Step 2. Completion of the proof of Theorem 3.1. In the Dirichlet case, to obtain the continuous observability inequality (1.5) from inequality (3.15) of Theorem 3.4 already proved, it suffices to return to (5.2.1); since $h(x) \cdot \nu(x) \leq 0$ for $x \in \Gamma_0$ by assumption, we readily have from (5.2.1),

$$\frac{1}{2} \max_{x \in \Gamma_1, 0 < t < T} \left(e^{\tau\phi} \frac{h(x) \cdot \nu(x)}{|\nu_{\mathcal{A}}(x)|_g^2} \right) \int_0^T \int_{\Gamma_1} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right)^2 d\Sigma \geq (\overline{BT}_\psi)|_\Sigma. \tag{5.2.12}$$

Then (5.2.12) used on the left-hand side of inequality (3.15) yields (when the parameter $\tau > 0$ is large enough) and $f \equiv 0$:

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial \psi}{\partial \nu_{\mathcal{A}}} \right)^2 d\Sigma + k_2 \|\psi\|_{C([0,T];L_2(\Omega))}^2 \geq k_1 E(0), \tag{5.2.13}$$

where $k_1, k_2 > 0$ are constants.

To get the sought-after inequality (1.5) from (5.2.13), we only need to drop the low order term $k_2 \|\psi\|_{C([0,T];L_2(\Omega))}^2$ in (5.2.13). This may be done, as usual, by a compactness/uniqueness argument [19], [14], see Remark 3.1. □

6 Proof of Theorem 3.5: Main inverse inequality

We prove the specialized version of Theorem 3.5 for w solution of Eqn. (3.7) within the class (3.8), which moreover satisfies hypothesis (3.17).

Step 1. Lemma 6.1. *Let w solve (3.7) and satisfy (3.17): $w|_{\Sigma_0} \equiv 0$ and $h \cdot \nu \leq 0$ on Γ_0.*
(a) Then, in this case, the boundary terms $(\overline{BT})_w|_\Sigma$ defined by (3.16), (3.10) reduce to

$$(\overline{BT}_w)|_\Sigma = (\overline{BT}_w)|_{\Sigma_0} + (\overline{BT}_w)|_{\Sigma_1}; \tag{6.1}$$

$$(\overline{BT}_w)|_{\Sigma_0} = (BT_w)|_{\Sigma_0} = \frac{1}{2}\int_0^T \int_{\Gamma_0} e^{\tau\phi} \frac{h(x)\cdot\nu(x)}{\nu_{\mathcal{A}}(x)} \left(\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right)^2 d\Sigma \le 0; \tag{6.2}$$

$$\left|(\overline{BT}_w)|_{\Sigma_1}\right| \le C\left\{\int_0^T \int_{\Gamma_1} \left[\left(\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right)^2 + \left(\frac{\partial w}{\partial s}\right)^2 + w_t^2\right] d\Sigma + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}\right\} \tag{6.3}$$

for any $\epsilon_0 > 0$, *where* $\frac{\partial w}{\partial s}$ *denotes, as before, the tangential gradient (derivative) of* w *on* Γ, *so that* $\left(\frac{\partial w}{\partial s}\right)^2 = |\nabla_{\text{tangential}} w|_0^2$.

(b) Moreover, if in addition, w *satisfies also* $\frac{\partial w}{\partial \nu_{\mathcal{A}}}|_{\Sigma_1} \equiv 0$, *then*

$$(\overline{BT}_w)|_{\Sigma_1} = (BT_w)|_{\Sigma_1} = \frac{1}{2}\int_0^T \int_{\Gamma_1} e^{\tau\phi} \left[w_t^2 - \left(\frac{\partial w}{\partial s}\right)^2\right] h\cdot\nu\, d\Sigma \quad \Box \tag{6.4}$$

Proof. We return to (3.16), (3.10): we then see that BT_w and $\overline{BT}_w$ coincide on $\Sigma_0 = (0,T]\times\Gamma_0$, since $w_t|_{\Sigma_0} \equiv 0$ by assumption. We may divide $BT_w|_\Sigma$ as in identity (6.1), where $BT_w|_{\Sigma_0}$ is given by (6.2) by virtue of the same argument of Lemma 5.2.1 (culminating in Eqns. (5.2.10) and (5.2.11) carried out this time on Σ_0). Similarly, from (3.16), (3.10), where $h(w) = \langle \nabla_g v, \nabla_g w\rangle_g$, we readily obtain

$$\begin{cases} (\overline{BT}_w)|_{\Sigma_1} = (BT_w)|_{\Sigma_1} = \frac{1}{2}\int_0^T \int_{\Gamma_1} e^{\tau\phi}[w_t^2 - |\nabla_g w|_g^2] h\cdot\nu\, d\Sigma \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } \left.\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma_1} \equiv 0; & (6.5a) \\ \left|(\overline{BT}_w)|_{\Sigma_1}\right| \le C\left\{\int_0^T \int_{\Gamma_1} \left[\left(\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right)^2 + |\nabla_g w|_g^2 + w_t^2\right] d\Sigma \right. \\ \qquad\qquad\qquad \left. + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))}\right\}, \text{ in general,} & (6.5b) \end{cases}$$

by use of trace theory applied to $w \in \Gamma_1$. Next, the decomposition (5.2.3) of $\nabla_g w$ in normal and tangential components yields by virtue of (5.2.5a)

$$\text{on } \Gamma_1 : |\nabla_g w|_g^2 = \begin{cases} \left(\frac{\partial^2 w}{\partial s}\right)^2, \quad \text{when } \left.\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right|_{\Sigma_1} = 0; & (6.6a) \\ \frac{1}{|\nu_{\mathcal{A}}(x)|_g^2} \left(\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right)^2 + \left(\frac{\partial^2 w}{\partial s}\right)^2, & (6.6b) \end{cases}$$

since, from (5.2.3), $Y(x) \in \Gamma_x$, the tangent space of Γ at x, we have $Y(w) = \nabla_0 w \cdot Y = \left(\frac{\partial w}{\partial s}\right)^2$ by (2.11), (5.2.4b). Then, (6.6a) and (6.6b), used in (6.5a) and (6.5b), yield (6.4) and (6.3), respectively. Lemma 6.1 is proved. $\Box$

Step 2. The following result is taken from [15, Section 7.2]. It is proved by micro-local analysis. It is critical in eliminating artificial geometrical conditions of the earlier literature on the controlled part Γ_1 of the boundary in the Neumann case.

Lemma 6.2. *Let* $f \in L_2(Q)$ *and let* w *be a solution of Eqn. (3.7) in the class (3.8).*
(a) Then, for any $\epsilon > 0$, $\epsilon_0 > 0$, *and* $T > 0$, *there exists a constant* $C_{\epsilon,\epsilon_0,T} > 0$ *such that*

$$\int_\epsilon^{T-\epsilon} \int_\Gamma \left(\frac{\partial w}{\partial s}\right)^2 d\Sigma \le C_{\epsilon,\epsilon_0,T}\left\{\int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu_{\mathcal{A}}}\right)^2 + w_t^2\right] d\Sigma + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|f\|^2_{L_2(Q)}\right\}. \tag{6.7}$$

(b) Moreover, if w satisfies in addition hypothesis (3.17): $w|_{\Sigma_0} \equiv 0$ and $h \cdot \nu \le 0$ on Γ_0, then inequality (6.7) holds true with Γ replaced by Γ_1. □

Step 4. We next use Lemma 6.2, Eqn. (6.7) to eliminate the tangential derivative from the estimate (6.3) [or identity (6.4)] of the boundary terms $(\overline{BT}_w)$ evaluated over $[\epsilon, T-\epsilon] \times \Gamma_1$.

Proposition 6.3. *Let $f \in L_2(Q)$ and let w be a solution of Eqn. (3.7) in the class (3.8). Moreover, let w satisfy hypothesis (3.17). Then, for all $\tau > 0$ sufficiently large, there exists a constant $k_{\phi,\tau} > 0$ such that*

$$\int_\epsilon^{T-\epsilon} \int_{\Gamma_1} \left[\left(\frac{\partial w}{\partial \nu_{\mathcal{A}}} \right)^2 + w_t^2 \right] d\Sigma + \frac{C_T}{\tau} \int_Q e^{\tau\phi} f^2 dQ + C_T \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \ge k_{\phi,\tau}[E(T) + E(0)]. \tag{6.8}$$

Proof. We apply Theorem 3.4, estimate (3.15), over $[\epsilon, T-\epsilon] \times \Gamma$ rather than $[0,T] \times \Gamma = \Sigma$. In so doing, we use hypothesis (3.17) to invoke (6.2) and conclude that $(\overline{BT}_w)|_{[\epsilon,T-\epsilon]\times\Gamma_0} \le 0$. Moreover, we invoke (6.3) for $(\overline{BT}_w)|_{[\epsilon,T-\epsilon]\times\Gamma_1}$ and use the key estimate (6.7). Finally, the right-hand side of (3.15) becomes $k_{\phi,\tau}[E(\epsilon) + E(T-\epsilon)]$. But

$$E(\epsilon) + E(T-\epsilon) \ge [E(0) + E(T)]e^{-C_T\epsilon} - 2\Lambda(T). \tag{6.9}$$

This can be proved as in the case of (5.1.1): by using the inequality on the right-hand side of (5.1.3) with $s = 0$ and $t = \epsilon$, and the inequality on the left-hand side of (5.1.3) with $t = T$ and $s = T - \epsilon$, and summing up the resulting inequalities. This yields (6.9). Then (6.8) is obtained. □

Step 5. Completion of the proof of Theorem 3.5. The sought-after inequality (3.18) of Theorem 3.5 now follows at once from (6.8) of Proposition 6.3, by further majorizing its left-hand side. Theorem 3.5 is proved. □

7 Proof of Theorem 3.2: Neumann case

We return to inequality (3.18) of Theorem 3.5(b), written for the solution $w = \psi$ of problem (1.7), with the boundary integral over Γ_1, since by assumption (3.17) holds true: $\psi|_{\Sigma_0} \equiv 0$ and $h \cdot \nu \le 0$ on Γ_0. Moreover, on Σ_1, it suffices to take $\beta \equiv 0$ in (1.7), i.e., $\frac{\partial \psi}{\partial \nu_{\mathcal{A}}}|_{\Sigma_1} \equiv 0$. Then, as $f \equiv 0$, (3.18) becomes the following inequality:

$$\int_{\Sigma_1} \psi_t^2 d\Sigma + k_1 \|\psi\|^2_{C([0,T];H^{\frac{1}{2}+\epsilon_0}(\Omega))} \ge k_2 E(0), \tag{7.1}$$

where $k_1, k_2 > 0$ are constants. Finally, by a compactness/uniqueness argument again, see Remark 3.1, we obtain the desired inequality in (1.8).

Remark 7.1. Given the ψ-problem (1.7), say with $\beta \equiv 0$, the proof of Theorem 3.5 uses (6.4) and (6.5a), (6.6a) rather than (6.3) and (6.5b), (6.6b), a streamlined procedure. □

8 Some illustrations where Assumptions (H.1) and (H.3) on $\mathcal{A}$ hold true

Example 8.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Assume that $\mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{A}u \;=\; & \frac{\partial}{\partial x}\left(\frac{1+y^6}{1+x^2+y^6} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial x}\left(\frac{xy^3}{1+x^2+y^6} \frac{\partial u}{\partial y} \right) \\ & + \frac{\partial}{\partial y}\left(\frac{xy^3}{1+x^2+y^6} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(\frac{1+x^2}{1+x^2+y^6} \frac{\partial u}{\partial y} \right). \end{aligned} \tag{8.1}$$

Set

$$A(x,y)=(a_{ij})=\begin{pmatrix}\dfrac{1+y^6}{1+x^2+y^6} & \dfrac{xy^3}{1+x^2+y^6}\\[2ex] \dfrac{xy^3}{1+x^2+y^6} & \dfrac{1+x^2}{1+x^2+y^6}\end{pmatrix}. \tag{8.2}$$

Then, $\det A(x,y)=1/(1+x^2+y^6)>0$, $\forall\,(x,y)\in\mathbb{R}^2$, and $A(x,y)$ is strictly positive definite on the bounded domain Ω. Thus, assumption (H.1) is verified.

The inverse of $A(x,y)$ is

$$G(x,y)=(g_{ij})=A^{-1}(x,y)=\begin{pmatrix}1+x^2 & -xy^3\\ -xy^3 & 1+y^6\end{pmatrix}. \tag{8.3}$$

Consider the Riemannian manifold $(\mathbb{R}^2,g)$, where the Riemannian metric g is defined in the natural coordinate system (x,y) via (8.3) by

$$g=(1+x^2)dx\,dx-xy^3dx\,dy-xy^3dy\,dx+(1+y^6)dy\,dy. \tag{8.4}$$

Consider the surface in $\mathbb{R}^3$ given by

$$M=\left\{(x,y,z)|z=f(x,y)=\frac{1}{2}x^2-\frac{1}{4}y^4\right\},$$

with the induced Riemannian metric g_M. Then the (projection) map $\Phi(x,y,z)=(x,y)$, for any $(x,y,z)\in M$, determines an isometry from M to $(\mathbb{R}^2,g)$. The Gaussian curvature of $(\mathbb{R}^2,g)$ at (x,y) is therefore

$$\begin{aligned} k(x,y) &= \text{the Gaussian curvature of } M \text{ at } (x,y,z)\\ &= \frac{\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2}{\left[1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]^2}=\frac{-3y^2}{(1+x^2+y^6)^2}\le 0,\quad \forall\ (x,y)\in\mathbb{R}^2. \end{aligned} \tag{8.5}$$

Since, by (8.5), the Gaussian curvature is non-positive, the function defined by

$$v(x)=d_g^2(x,x_0),\quad x_0 \text{ fixed } \in\mathbb{R}^2, \tag{8.6}$$

i.e., as the square of the distance $d_g(x,x_0)$, in the Riemann metric of (8.4), from x to a given fixed point $x_0\in\mathbb{R}^2$, is in fact strictly convex on $(\mathbb{R}^2,g)$ [29, p. 108]. Thus, assumption (H.3) also holds true in this case. □

Example 8.2. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and $a_i>0$ constants, $i=1,2,\dots,n$. Consider the operator on $\mathbb{R}^n$,

$$\mathcal{A}u=\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\frac{1+\sum\limits_{j\neq i}^n a_j^2x_j^2}{1+\sum\limits_{k=1}^n a_k^2x_k^2}\frac{\partial u}{\partial x_i}\right)-\sum_{i\neq j}\frac{\partial}{\partial x_i}\left(\frac{a_ia_jx_ix_j}{1+\sum\limits_{k=1}^n a_k^2x_k^2}\frac{\partial u}{\partial x_j}\right). \tag{8.7}$$

Set

$$A(x) = (a_{ij}) = \frac{1}{1+\sum_{k=1}^{n} a_k^2 x_k^2} \begin{pmatrix} 1+\sum_{i=2}^{n} a_i^2 x_i^2 & -a_1a_2x_1x_2 & \cdots & -a_1a_nx_1x_n \\ -a_2a_1x_2x_1 & 1+\sum_{i\neq 2} a_i^2x_i^2 & \cdots & -a_2a_nx_2x_n \\ \cdots & \cdots & \cdots & \cdots \\ -a_na_1x_nx_1 & -a_na_2x_nx_2 & \cdots & 1+\sum_{i=1}^{n-1} a_i^2x_i^2 \end{pmatrix}. \tag{8.8}$$

Then, the inverse of $A(x)$ is

$$G(x) = (g_{ij}) = A^{-1}(x) = \begin{pmatrix} 1+a_1^2x_1^2 & a_1a_2x_1x_2 & \cdots & a_1a_nx_1x_n \\ a_2a_1x_2x_1 & 1+a_2^2x_2^2 & \cdots & a_2a_nx_2x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_na_1x_nx_1 & a_na_2x_nx_2 & \cdots & 1+a_n^2x_n^2 \end{pmatrix}. \tag{8.9}$$

Consider the Riemannian manifold $(\mathbb{R}^n, g)$, where the Riemannian metric g is determined in the natural coordinate system $x = (x_1, x_2, \ldots, x_n)$ via (8.8) by

$$g = \sum_{i,j=1} g_{ij}dx_idx_j = \sum_{i,j=1}^{n} (\delta_{ij} + a_ia_jx_ix_j)dx_idx_j, \tag{8.10}$$

where δ_{ij} is 1 if $i = j$, and 0 if $i \neq j$. It follows that

$$\sum_{i,j=1} g_{ij}\xi_i\xi_j = \sum_{i,j=1}^{n} (\delta_{ij} + a_ia_jx_ix_j)\xi_i\xi_j \geq |\xi|_0^2, \quad \forall\ x,\ \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n. \tag{8.11}$$

It is easily checked from the above inequality that $(\mathbb{R}^n, g)$ is a complete non-compact Riemannian manifold.

Let M be the hypersurface in $\mathbb{R}^{n+1}$ given by

$$M = \left\{ [x_1, x_2, \ldots, x_n, x_{n+1}] | x_{n+1} = \frac{1}{2}\sum_{i=1}^{n} a_ix_i^2 \right\}, \tag{8.12}$$

with the induced Riemannian metric in $\mathbb{R}^n$. Then, by [28, Lemma 3.1], M is of everywhere positive sectional curvature. It is easily verified from (8.10) that the map $\Phi : M \to (\mathbb{R}^n, g)$, defined by

$$\Phi(p) = x = [x_1, \ldots, x_n], \quad \forall\ p = [x_1, \ldots, x_n, x_{n+1}] \in M,$$

is an isometry between M and $(\mathbb{R}^n, g)$. Thus, $(\mathbb{R}^n, g)$ itself is of everywhere positive sectional curvature. Since $(\mathbb{R}^n, g)$ is a non-compact, complete Riemannian manifold of everywehre positive sectional curvature, then there exists a C^∞ strictly convex function $v(x)$ on $(\mathbb{R}^n, g)$ by [4]. Assumptions (H.1) and (H.3) are verified.

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