Nonconservative Wave Equations with Unobserved Neumann B.C.: Global Uniqueness and Observability in One Shot

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Abstract

We consider a second-order hyperbolic equation on an open bounded domain Ω in $\mathbb{R}^n$, with C^1-boundary $\Gamma = \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, subject to Neumann boundary conditions on the *entire* boundary Γ. Here, Γ_0 (unobserved/uncontrolled part) and Γ_1 (observed/controlled part) are relatively open subsets of Γ. The principal part is of constant coefficients, while the energy level $(H^1(\Omega)-)$ terms may be variable in both space and time, and of low regularity $L_\infty(Q)$. Verifiable geometric conditions are imposed on the unobserved portion Γ_0. Then: we first establish a Carleman-type inequality for $H^{1,1}(Q)$-solutions of the hyperbolic equation *with no interior lower-order* terms. From here, we deduce *global uniqueness* results for $H^{1,1}(Q)$ solutions

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of the hyperbolic equation satisfying Neumann B.C. on all of Γ, and Dirichlet B.C. on Γ_0, over a time T greater than an explicit time T_0. T_0 is optimal if, e.g., Γ_0 is flat. Finally we obtain continuous observability (or stabilization) inequalities with an *explicit* constant. A three-part appendix, of purely geometric nature, provides several independent approaches leading to various general classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ which satisfy the geometric conditions of Section 1, and, more relevantly, the geometric conditions of the far more general Section 10: see Theorem C.1 in Appendix C. In particular: Γ_0 may be flat; Γ_0 may be either convex or concave; Γ_0 may be logarithm convex or concave; etc. In the case of a disk, we can take the unobserved part Γ_0 to be as close to a half-circumference as we please: an indication that our results in Section 10 are sharp. Finally, in line with the AMS Conference at the University of Colorado, we point out throughout some open geometric questions, as well as some potential extensions which would require geometric methods. Extension of the fundamental Lemma 3.1 to the case of variable (in space) coefficients of the principal part has already been accomplished [L-T-Y-Z.1] by means of Bochner's techniques in Riemann geometry, in the style of [L-T-Y.1–3], [Y.1].

1 Introduction. Problem statement

1.1 Problem statement. Assumptions. Consequences

Dynamical problem. Let Ω be an open bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ of class C^1, consisting of the closure of two disjoint parts: Γ_0 (uncontrolled or unobserved part) and Γ_1 (controlled or observed part), both relatively open in Γ : $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $\nu = [\nu_1, \ldots, \nu_n]$ be the unit outward normal vector on Γ, and let $\frac{\partial}{\partial\nu} = \nabla \cdot \nu$ denote the corresponding normal derivative. In this note we consider the following purely Neumann problem for a second-order hyperbolic equation in the unknown $w(t,x)$:

$$\begin{cases} w_{tt} - \Delta w = F(w) + f & \text{in } Q = (0,T] \times \Omega; \quad (1.1.1\text{a}) \\ w(0, \ \cdot \ ) = w_0, \ w_t(0, \ \cdot \ ) = w_1 & \text{in } \Omega; \quad (1.1.1\text{b}) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma} \equiv 0 & \text{in } \sum = (0,T] \times \Gamma. \quad (1.1.1\text{c}) \end{cases}$$

In (1.1.1a) we have set

$$F(w) = q_1(t,x)w + q_2(t,x)w_t + q_3(t,x) \cdot \nabla w, \tag{1.1.1d}$$

subject to the following standing *assumption* on the coefficients: $q_1, \ q_2, \ |q_3| \in L_\infty(Q)$, so that the following pointwise estimate holds true:

$$|F(w)| \le C_T[w_t^2 + |\nabla w|^2 + w^2], \quad (t,x) \in Q. \tag{1.1.2a}$$

Remark 1.1.1. In effect we could relax the standing assumption on the lower-order coefficient q_1, and just require $q_1 \in L_p(Q)$ for $p = n+1$, $n \geq 2$, by using a Sobolev embedding theorem to estimate

$$\left\{\int_Q e^{2\tau\phi}|q_1 w|^2 dQ\right\}^{\frac{1}{2}} \leq \|q_1\|_{L^p(Q)}\|e^{\tau\phi}w\|_{L^{\frac{2n+2}{n-1}}(Q)} \leq C\|e^{\tau\phi}w\|_{H^1(Q)}. \tag{1.1.2b}$$

We shall set

$$r \equiv \|q_1\|_{L^{n+1}(Q)} + \|q_2\|_{L^\infty(Q)} + \|q_3\|_{L^\infty(Q)}. \tag{1.1.2c}$$

Theorem 10.1.1a will give the observability/stabilization inequality with an *explicit* constant of the order of Ce^{Cr^2}, where C is a generic constant, and r is defined by (1.1.2c). □

Moreover, we assume throughout that the non-homogeneous term f satisfies

$$f \in L_2(Q). \tag{1.1.3}$$

Main assumptions. In addition to the standing assumptions (1.1.2), (1.1.3), on the first-order operator F, and on the forcing term f, the following assumption is postulated throughout Section 9 of this paper:

(A.1) Given the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}$, there exists a strictly convex function $d : \bar{\Omega} \to \mathbb{R}$, of class $C^3(\bar{\Omega})$, such that if we introduce the (conservative) vector field $h(x) = [h_1(x), \ldots, h_n(x)] \equiv \nabla d(x)$, $x \in \Omega$, then the following two properties hold true:

(i)

$$\left.\frac{\partial d}{\partial \nu}\right|_{\Gamma_0} = \nabla d \cdot \nu = h \cdot \nu = 0 \text{ on } \Gamma_0; \qquad h = \nabla d; \tag{1.1.4}$$

(ii) the (symmetric) Hessian matrix $\mathcal{H}_d$ of $d(x)$ [i.e., the Jacobian matrix J_h of $h(x)$] is strictly positive definite on $\bar{\Omega}$: there exists a constant $\rho_0 > 0$ such that for all $x \in \bar{\Omega}$

$$\mathcal{H}_d(x) = J_h(x) = \begin{bmatrix} d_{x_1x_1}, & \cdots, & d_{x_1x_n} \\ \vdots & & \vdots \\ d_{x_nx_1}, & \cdots, & d_{x_nx_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1}, & \cdots, & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial x_1}, & \cdots, & \frac{\partial h_n}{\partial x_n} \end{bmatrix} \geq \rho_0 I. \tag{1.1.5}$$

A working assumption throughout Section 9, to be later relaxed in our final results of Section 10, is that $d(x)$ has no critical point on $\bar{\Omega}$:

(A.2)

$$\inf_{x\in\Omega} |h(x)| = \inf_{x\in\Omega} |\nabla d(x)| = p > 0. \tag{1.1.6}$$

Without loss of generality as far as assumptions (A.1) and (A.2) are concerned, and for purposes of Eqn. (1.1.8b) below, we may always translate $d(x)$ as to make it positive on $\bar{\Omega}$: $\min d(x) = m > 0$ on $\bar{\Omega}$.

Remark 1.1.2. Assumption (A.1) is due to the Neumann B.C. Assumption (A.2) is needed for the validity of the key estimate (1.1.15b) which, in turn, is responsible for the elimination of the interior lower-order term in the final Carleman estimate of Theorem 5.1(ii), Eqn. (5.2). Assumption (A.1) was introduced in [Tr.1, Section 5] in dealing with a corresponding second-order hyperbolic problem with Neumann homogeneous B.C. on Γ_0 (uncontrolled/unobserved part of the boundary) and Dirichlet homogeneous B.C. on Γ_1 (controlled/observed part of the boundary); but it was not investigated in detail. Assumption (A.1) is much less restrictive than one would expect at first. Various classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ satisfying assumption (A.1) and, moreover, also (A.2), are given in Appendix A through Appendix C, which constitute an intrinsic part of this paper. See, in particular, Theorem A.2.1, Lemma A.2.2, Appendix B.2, via a constructively geometric approach; Theorem A.3.2 resting on a conformal mapping approach in the 2-dimensional case; Theorem A.4.1, based on a perturbation approach valid in any dimension; finally, the conclusive Theorem C.1 regarding the validity of the far more relaxed geometrical setting of Section 10. $\Box$

Remark 1.1.3. In effect, assumption (1.1.6) is needed to hold true only for $x \in \Gamma_0$ (uncontrolled or unobserved part of the boundary Γ):

$$\inf_{x \in \Gamma_0} |\nabla d(x)| = p > 0, \tag{1.1.6'}$$

for then a critical point of $d(x)$ at a point (necessarily interior) of Ω, or at a point $x \in \Gamma_1$ (controlled or observed part of Γ) can always be eliminated, by smoothly redefining locally $d(x)$ while preserving the positivity condition (1.1.5). $\Box$

Preliminary scaling. Assumptions (A.1) and (A.2) above yield, in effect, a full family of strictly convex functions $\{ad(x) + b\}$ for all constants $a > 0$ and b (scaling and translation of $d(x)$), each member of which satisfies (1.1.4), (1.1.5), (1.1.6). By translating and rescaling $d(x)$, with some $a > 1$, if necessary, we shall then operate a preliminary choice of 'normalization', in order to achieve the following outcomes:

$$\min_{\bar{\Omega}} d(x) = m > 0, \qquad \rho_0 \geq 2. \tag{1.1.7}$$

The first condition is only for convenience: we think, however, of m as arbitrarily small, in order not to deteriorate the threshold time T_0 in (1.1.8b) below. The second condition $\rho_0 \geq 2$ will allow us to automatically verify properties (p_1) and (p_2) below in (1.1.12) and (1.1.14), which are a consequence of (A.1). For instance, in the case where Γ_0 is *flat*, we can take $d(x) = |x - z|^2$, for some $z \in \mathbb{R}^n$ just *outside* Ω on the hyperplane containing Γ_0

(Fig. A.1, Appendix A.1), and then (A.1), (A.2) are satisfied, along with (1.1.7) where, in fact, $\rho_0 = 2$. See more details on this case at the end of this Section 1.1 below.

Pseudo-convex function. Having chosen, on the strength of assumption (A.1), a strictly convex potential function $d(x)$ satisfying the preliminary scaling condition (1.1.7), we next introduce the pseudo-convex function $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ of class C^3 by setting:

$$\phi(x,t) = d(x) - c\left(t - \frac{T}{2}\right)^2; \qquad 0 \le t \le T,\ x \in \Omega, \tag{1.1.8a}$$

where $T > 0$ and $0 < c < 1$ are selected as follows. We define first T_0 by setting

$$T_0^2 \equiv 4 \max_{x \in \bar{\Omega}} d(x). \tag{1.1.8b}$$

Let $T > T_0$ be given. By (1.1.8b), there exists $\delta > 0$ such that

$$T^2 > 4 \max_{x \in \bar{\Omega}} d(x) + 4\delta. \tag{1.1.8c}$$

For this $\delta > 0$, there exists a constant c, $0 < c < 1$ such that

$$cT^2 > 4 \max_{x \in \bar{\Omega}} d(x) + 4\delta. \tag{1.1.8d}$$

Henceforth, let $\phi(x,t)$ be defined by (1.1.8a) with T and c chosen as described above, unless otherwise explicitly noted. Such function $\phi(x,t)$ has the following properties:

(a) for the constant $\delta > 0$, fixed in (1.1.8c), we have via (1.1.8d),

$$\phi(x,0) \equiv \phi(x,T) = d(x) - c\frac{T^2}{4} \le \max_{\bar{\Omega}} d(x) - c\frac{T^2}{4} \le -\delta, \text{ uniformly in } x \in \Omega; \tag{1.1.9}$$

(b) there are t_0 and t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, say chosen symmetrically about $\frac{T}{2}$, such that

$$\min_{x \in \bar{\Omega}, t \in [t_0, t_1]} \phi(x,t) \ge \sigma, \quad 0 < \sigma < m, \tag{1.1.10}$$

since $\phi\left(x, \frac{T}{2}\right) = d(x) \ge m > 0$, under present choice: indeed, we take

$$m - c(t_1 - \frac{T}{2})^2 \equiv \sigma > 0, \text{ or } t_1 - \frac{T}{2} = \sqrt{\frac{m - \sigma}{c}}.$$

We remark that the property $\sigma > -\delta$, $\delta > 0$, here achieved from (1.1.10), having imposed $\sigma > 0$, will be critically invoked in going from Eqn. (6.5) to Eqn. (6.6) below.

Consequences of assumption (A.1), (A.2) and the scaling condition (1.1.7). Let $d(x)$ be a strictly convex (potential) function provided by assumptions (A.1), (A.2) and satisfying the scaling choice (1.1.7): $\rho_0 \geq 2$. It then follows readily that any such $d(x)$ fulfills automatically the following two properties (p_1) and (p_2): there exists a function $\alpha(x) \in C^1(\bar{\Omega})$, in fact, take

$$\alpha(x) \equiv \Delta d(x) - 2c - 1 + k \in C^1(\bar{\Omega}), \tag{1.1.11}$$

for a constant $0 < k < 1$ such that

(p_1)

$$\Delta d(x) - 2c - \alpha(x) \equiv 1 - k > 0, \quad x \in \bar{\Omega}; \tag{1.1.12}$$

(p_2) if we define γ by

$$\gamma \equiv \alpha(x) - 2c - \Delta d(x) \equiv -4c - 1 + k < 0, \quad x \in \bar{\Omega}, \tag{1.1.13}$$

the Hessian matrix $\mathcal{H}_d$ of any such $d(x)$ satisfies the following inequality for all $x \in \bar{\Omega}$:

$$\begin{aligned} &2\mathcal{H}_d(x) + [\alpha(x) - 2c - \Delta d(x)]I = 2\mathcal{H}_d(x) + \gamma I \\ &= \begin{bmatrix} 2d_{x_1x_1} + \gamma & 2d_{x_1x_2} & \cdots & 2d_{x_1x_n} \\ 2d_{x_2x_1} & 2d_{x_2x_2} + \gamma & \cdots & 2d_{x_2x_n} \\ \vdots & & \ddots & \vdots \\ 2d_{x_nx_1} & 2d_{x_nx_2} & \cdots & 2d_{x_nx_n} + \gamma \end{bmatrix} \geq \rho I, \ \forall\, x \in \bar{\Omega}, \end{aligned} \tag{1.1.14a}$$

for some constant $\rho > 0$, by virtue of assumption (1.1.5) and (1.1.13):

$$\rho \equiv 2\rho_0 + \gamma = 2\rho_0 + (-4c - 1 + k) > 0 \ \text{ for } 1 + 2(2c - \rho_0) < k < 1. \tag{1.1.14b}$$

We note that positivity of ρ in (1.1.14b) is obtained, since the constant c was selected below (1.1.8a) as $0 < c < 1$, and, moreover, by the scaling choice (1.1.7) we have $\rho_0 \geq 2$, so that $2c - \rho_0 < 0$ and then (1.1.14b) is achieved.

In addition, by additional rescaling, if necessary, we shall show below that any strictly convex function $d(x)$ provided by assumptions (A.1), (A.2), and satisfying the preliminary scaling condition (1.1.7), and, possibly, additional scaling, fulfills also the following property (p_3):

(p_3) noting via (1.1.12) that

$$6c + \Delta d(x) - \alpha(x) \equiv 8c + 1 - k, \quad x \in \bar{\Omega}, \tag{1.1.15a}$$

and, moreover, recalling (1.1.10) and $\nabla d = h$ from (1.1.4), we have that the following inequality holds true, by virtue of assumption (1.1.6), and of (1.1.13), (1.1.15a):

$$\begin{cases} (2c+\Delta d-\alpha)|\nabla d|^2+2\mathcal{H}_d\nabla d\cdot\nabla d-(6c+\Delta d-\alpha)4c^2\left(t-\dfrac{T}{2}\right)^2 \\ \equiv (4c+1-k)|\nabla d|^2+2\mathcal{H}_d\nabla d\cdot\nabla d-(8c+1-k)4c^2\left(t-\dfrac{T}{2}\right)^2 \geq \beta_1>0, \\ \hfill \forall (x,t)\in \text{ set } Q^*(\sigma^*), \end{cases} \tag{1.1.15b}$$

for a constant $\beta_1>0$, where the set $Q^*(\sigma^*)$ is defined by

$$Q^*(\sigma^*)\equiv\{(x,t):\, x\in\Omega,\ 0\leq t\leq T,\ \phi^*(x,t)\geq\sigma^*>0\}, \tag{1.1.16}$$

for a constant σ^* chosen to satisfy $0<\sigma^*<\sigma$, see (1.1.10), where in turn the function $\phi^*(x,t)$ is defined by

$$\phi^*(x,t)\equiv d(x)-c^2\left(t-\frac{T}{2}\right)^2, \quad x\in\Omega,\ 0\leq t\leq T. \tag{1.1.17}$$

Since $0<c<1$, we note, via (1.1.8a), (1.1.17), that

$$\phi^*(x,t)\geq\phi(x,t), \qquad x\in\Omega,\ 0\leq t\leq T. \tag{1.1.18}$$

Thus, if we define, in agreement with (1.1.17), the set $Q(\sigma)$ by

$$Q(\sigma)\equiv\{(x,t):\ x\in\Omega,\ 0\leq t\leq T,\ \phi(x,t)\geq\sigma>0\}, \tag{1.1.19}$$

we see, since $0<\sigma^*<\sigma$, and by virtue of (1.1.10), that

$$[t_0,t_1]\times\Omega\subset Q(\sigma)\subset Q^*(\sigma^*)\subset[0,T]\times\Omega, \tag{1.1.20}$$

see Figure 1 and Figure 2. The point of the set $Q^*(\sigma^*)$ is twofold: (1) it is a convenient subset of $[0,T]\times\Omega$ where to require the validity of inequality (1.1.15b); moreover, *it is comparable*, in the sense described by (1.1.20), *with the set $Q(\sigma)$ in (1.1.19), which instead is defined in terms of the level surface $\phi(x,t)=\sigma$, related to the original pseudo-convex function.* All this will be seen in the proof of Theorem 5.1.

A class where properties (p_1) = (1.1.12), (p_2) = (1.1.14), and (p_3) = (1.1.15b) are fulfilled without rescaling: the radial field case. Consider the special but important case where the unobserved boundary Γ_0 is *flat*. Then, take a point $z\in\mathbb{R}^n$ just *outside* Ω *on the hyperplane containing* Γ_0 (Fig. A.1, Appendix A.1). Define $d(x)=|x-z|^2$, so $d(x)\geq m>0$ on $\bar\Omega$, as desired. Assumptions (A.1) and (A.2) hold true, and moreover the constant ρ_0 in (1.1.5) is: $\rho_0=2$; moreover, $T>2$ (diameter of Ω):

$$\begin{cases} h(x)\equiv\nabla d(x)=2(x-z)=\text{a radial field centered at } z; & (1.1.21a) \\ |h(x)|^2=|\nabla d(x)|^2=4|x-z|^2=4d(x),\ T_0^2=4\max|x-z|^2; & (1.1.21b) \\ d_{x_ix_i}\equiv 2,\ i=1,\dots,n;\ \Delta d(x)=2n=2\dim\Omega;\ d_{x_ix_j}\equiv 0,\ i\neq j;\ \rho_0=2. & (1.1.21c) \end{cases}$$

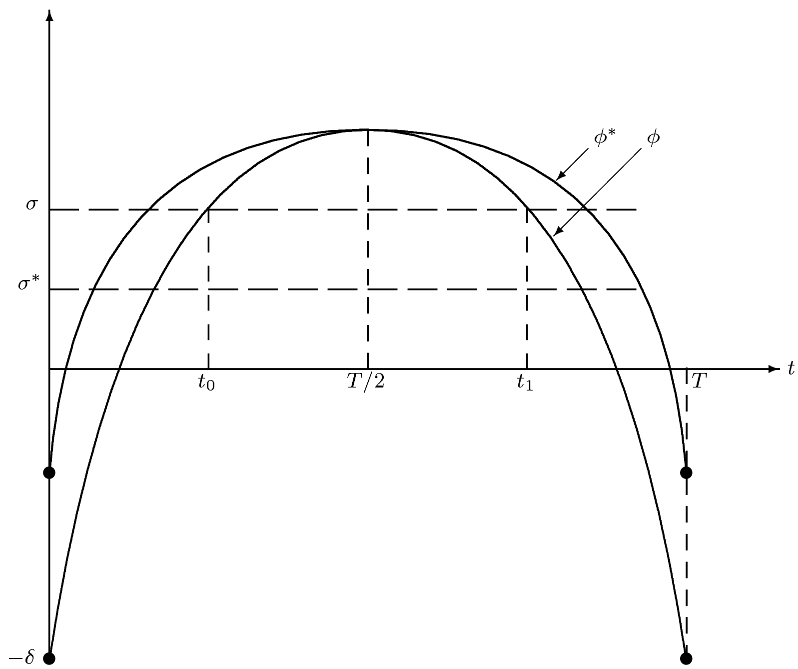

FIGURE 1: ϕ AND ϕ^* AT FIXED x, SAY WHERE $|x - z| = \min$ OVER Ω

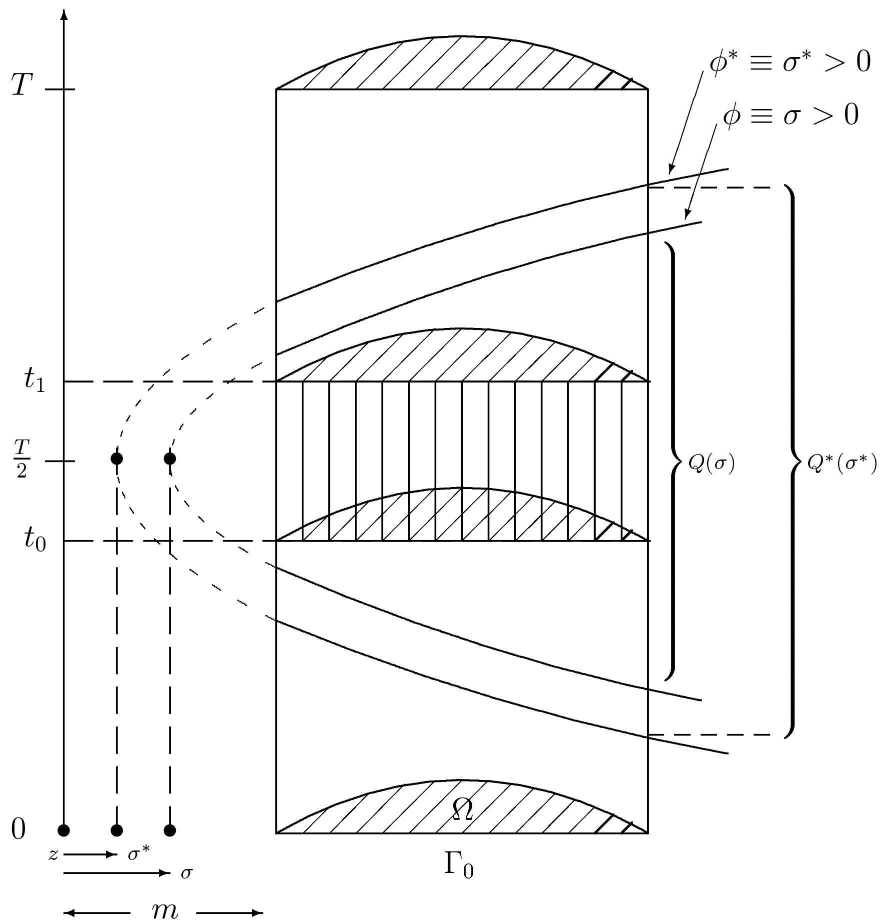

FIGURE 2: USE $d(x) = |x - z|^2$. AT $t = 0$ AND $t = T$, NO POINT OF $x \in \Omega$ BELONGS TO $Q^*(\sigma^*)$, BY VIRTUE OF (1.1.8D): $\phi(x, 0) = \phi\left(x, \frac{T}{2}\right) \leq$ MAX $d(x) - c\,\frac{T^2}{4} \leq -\delta$, WHERE $0 < \sigma^* < \sigma < m$, AND SO (1.1.16) IS VIOLATED.

We now verify the validity of (p_1), (p_2), (p_3) with no rescaling of $d(x)$:

$(p_1) = (1.1.12)$; $(p_2) = (1.1.14)$. Since now $\rho_0 = 2$, then (1.1.7) is satisfied and, as seen below (1.1.14), (p_1) and (p_2) are fulfilled.

$(p_3) = (1.1.15b)$. We premise verification of (p_3) with the following

Claim: *When $\rho_0 > 2c$, in particular, in the present radial vector field, we always have the following estimate*, which refers to the key expression in (1.1.15b); by (1.1.5) with $\rho_0 > 2c$:

$$(4c + 1 - k)|\nabla d|^2 + 2\mathcal{H}_d \nabla d \cdot \nabla d - (8c + 1 - k)4c^2 \left(t - \frac{T}{2}\right)^2$$

$$\geq \quad (4c + 1 - k + 4c)|\nabla d|^2 - (8c + 1 - k)4c^2 \left(t - \frac{T}{2}\right)^2 \qquad (1.1.22)$$

$$= \quad (8c+1-k)\left[|\nabla d|^2 - 4c^2\left(t-\frac{T}{2}\right)^2\right]. \tag{1.1.23}$$

We now verify (p_3). In addition, in the present case of a radial vector field, we make use of the (lucky) identity $|\nabla d(x)|^2 = 4d(x)$, see (1.1.21b), and obtain

$$|\nabla d|^2 - 4c^2\left(t-\frac{T}{2}\right)^2 = 4\left[d(x) - c^2\left(t-\frac{T}{2}\right)^2\right] = 4\phi^*(x,t), \tag{1.1.24}$$

recalling the function $\phi^*(x,t)$ defined in (1.1.17). Thus, for the present radial vector field, we obtain by (1.1.23) and (1.1.24):

$$(4c+1-k)|\nabla d|^2 + 2\mathcal{H}_d\nabla d\cdot\nabla d - (8c+1-k)4c^2\left(t-\frac{T}{2}\right)^2 \geq 4(8c+1-k)\phi^*(x,t). \tag{1.1.25}$$

We conclude: *the requirement* (p_3) = *(1.1.15b) always holds true with no rescaling of* $d(x) = |x-z|^2$ *in the present radial vector field case*, with controllability time $T > 2$ (diameter of Ω), which is optimal.

1.2 Rescaling of $d(x)$. Fulfillment of condition (p_3) =(1.1.15b) in general, under assumptions (A.1), (A.2).

Let $d(x)$ be a function satisfying (A.1), (A.2). Then property (p_1) = (1.1.12) is always satisfied with the choice of $\alpha(x)$ as in (1.1.11).

Rescaling $d(x)$**.** Here the point is the following: On the positive side, rescaling $d(x)$ will always allow us to achieve also properties (p_2) = (1.1.14) (by enforcing the choice $\rho_0 \geq 2$ in (1.1.7)) and, moreover, (p_3) = (1.1.15a-b), in fact, even (1.2.1) below. In fact, by rescaling $d(x)$ more, one can always achieve, by virtue of assumption (1.1.6), the validity of inequality (1.1.15b) even on *the entire cylinder* $[0,T]\times\Omega \equiv Q$:

$$(4c+1-k)|\nabla d|^2 + 2\mathcal{H}_d\nabla d\cdot\nabla d - (8c+1-k)4c^2\left(t-\frac{T}{2}\right)^2 \geq \beta_1 > 0.\ \forall\,(x,t)\in\Omega\times[0,T], \tag{1.2.1}$$

a stronger, and less desirable, requirement than (1.1.15b), as noted below. This is established below. Yet, on the negative side, rescaling $d(x)$ deteriorates the minimal observability time T_0 in (1.1.8b) while keeping $0 < c < 1$ fixed and close to 1. Thus, rescaling $d(x)$ calls for care. We have seen above that: *there is no need to rescale* $d(x)$ *in the special but important case where* $d(x) = |x-z|^2$, z a fixed point just outside Ω, which is relevant when the unobserved boundary Γ_0 is flat. See Appendix A.1.

Generally, if we rescale the original function $d(x)$, i.e., if we replace $d(x)$ by $d_{\text{new}}(x) = ad(x)$ for a constant $a > 1$, then:

$$\begin{cases} \nabla d_{\text{new}} = a\nabla d, \ |\nabla d_{\text{new}}|^2 = a^2|\nabla d|^2; \ \mathcal{H}_{d_{\text{new}}}(x) = a\mathcal{H}_d(x), \ \rho_{0,\text{new}} = a\rho_0; \\ T^2_{0,\text{new}} = aT_0^2; \ T^2_{\text{new}} = aT^2; \ \mathcal{H}_{d_{\text{new}}}\nabla_{d_{\text{new}}} \cdot \nabla_{d_{\text{new}}} = a^3\mathcal{H}_d\nabla d \cdot \nabla d, \end{cases} \tag{1.2.2}$$

and we *can then keep the same constant* $0 < c < 1$, and close to 1, of the original $d(x)$. We note that the time T was arbitrary $> T_0$. Thus, $\sqrt{a}\,T$ is arbitrary $> T_{0,\text{new}}$. Thus, in the critical expression on the left of, say (1.2.1), we have that the first two space terms grow, after rescaling, by a^2 and a^3 respectively, while the worst case time term (on $[0,T]$), i.e., $T^2 > T_0^2$, deteriorates only by a factor a. Thus, (1.2.1) can always be achieved with $a > 1$ sufficiently large, at the price of deteriorating the original T_0 in (1.1.8b) into $\sqrt{a}\,T_0$. More precisely,

Achievement, in general, of estimate $(p_3) = (1.1.15b)$ on $Q^*(\sigma^*)$ under rescaling of $d(x)$; indeed, of the stronger estimate (1.2.1). We now establish the following

Claim. Given a function $d(x) \geq 0$ satisfying assumptions (A.1), (A.2). Rescale it, if necessary, with a rescaling factor $a > 1$ sufficiently large as to achieve the condition

$$|\nabla d(x)|^2 - 4d(x) \geq 0, \quad \forall\, x \in \bar{\Omega}, \tag{1.2.3}$$

in addition to $\rho_0 \geq 2$ in (1.1.7). [This can always be done by (1.2.2)]. Then, in fact, inequality (1.2.1) on the entire cylinder $[0,T] \times \Omega \equiv Q$ holds true. Thus, *a-fortiori*, property $(p_3) = (1.1.15b)$ on the set $Q^*(\sigma^*)$ is fulfilled.

Proof of Claim. We can first achieve $\rho_0 \geq 2$ in (1.1.7). Thus, by the Claim at the end of Section 1.1, we obtain that inequality (1.1.23) holds true for $x \in \Omega, \ 0 \leq t \leq T$:

$$\begin{aligned} &(4c+1-k)|\nabla d|^2 + 2\mathcal{H}_d\nabla d \cdot \nabla d - (8c+1-k)4c^2\left(t - \frac{T}{2}\right)^2 \\ &\qquad \geq (8c+1-k)\left[|\nabla d|^2 - 4c^2\left(t - \frac{T}{2}\right)^2\right]. \end{aligned} \tag{1.2.4}$$

Next, recalling (1.1.17) for ϕ^* and for $x \in \Omega$, $0 \leq t \leq T$, we obtain:

$$\begin{aligned} |\nabla d(x)|^2 - 4c^2\left(t - \frac{T}{2}\right)^2 &= |\nabla d(x)|^2 - 4d(x) + 4\left[d(x) - c^2\left(t - \frac{T}{2}\right)^2\right] \\ &= |\nabla d(x)|^2 - 4d(x) + 4\phi^*(x,t). \end{aligned} \tag{1.2.5}$$

Then, assumption (1.2.3) used in (1.2.5) yields, by use of (1.2.4),

$$(4c+1-k)|\nabla d|^2 + 2\mathcal{H}_d \nabla d \cdot \nabla d - (8c+1-k)4c^2 \left(t - \frac{T}{2}\right)^2$$

$$\geq \ 4(8c+1-k)\phi^*(x,t) \quad \text{on } Q = (0,T] \times \Omega, \tag{1.2.6}$$

and (1.2.6) establishes (1.2.1), as claimed. □

Remark 1.2.1. The proof in Sections 5 and 6 will markedly simplify if we assumed the validity of (1.2.1) on the whole cylinder $[0,T]\times\Omega$, rather than the validity of (1.1.15b) on its smaller subset $Q^*(\sigma^*)$. See Remark 5.1. □

2 Main results under assumptions (A.1) and (A.2)

In this section we state our main results under assumptions (A.1) [(1.1.4), (1.1.5)] and (A.2) = (1.1.6). In Section 10, assumption (A.2) will be suitably relaxed, in fact in many cases it will be dispensed with altogether. While it is possible to construct interesting examples of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ which satisfy assumptions (A.1) and (A.2)—see Appendix A through Appendix C—nevertheless, assumption (A.2) introduces undesirable limitations in some key examples, such as $\Omega =$ disk. These can be eliminated by increasing "by ϵ" the observed boundary Γ_0, making it non-connected, see Example B.1 in Appendix B. Thus, omitting assumption (A.2) is a worthwhile endeavor. This will be done in Section 10.

Essentially, the setting of Section 10 will consist of splitting the original domain Ω into two subdomains Ω_1 and Ω_2: $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$, where the present framework of Section 1, based on assumptions (A.1) and (A.2), applies to each subdomain Ω_i separately, $i = 1, 2$. This avoids imposing assumption (A.2) on the entire domain Ω. The corresponding proofs in Section 10 become much more complicated, even though to each subproblem on Ω_i, we apply the key results of the preceding sections such as Corollary 4.3, obtained under assumptions (A.1) and (A.2). Thus, a separate treatment of the more general setting (without assumption (A.2)), is to be postponed until Section 10. This is justified in name of clarity. In conclusion, the most general results of this paper may essentially be obtained from those given below in this section, by replacing assumption (A.2) with a similar assumption on Ω_1 and Ω_2, however, often a more relaxed condition. See Section 10. The validity of the setting of Section 10 for large classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ is provided by Theorem C.1 in Appendix C.

2.1 Continuous observability. Global uniqueness

We first list our main continuous observability inequality, and related global uniqueness result of our treatment in Sections 1 through 9. In the next subsection, we shall present a corresponding uniform stabilization result. All these results are obtained under assumptions (A.1) and (A.2). In effect, they are derived from a corresponding Carleman-type estimate to be established in Section 6 (Theorem 6.1), which actually holds true in a more general setting (that of hypotheses (H.1) = (4.17), (H.2) = (4.18), and (H.3) = (4.19), which is *a-fortiori* satisfied by the checkable assumptions (A.1) and (A.2)). Throughout this paper we introduce

$$E(t) = \int_{\Omega} [w_t^2(t) + |\nabla w(t)|^2 + w^2(t)] d\Omega = \|\{w(t), w_t(t)\}\|^2_{H^1(\Omega)\times L_2(\Omega)}. \tag{2.1.1}$$

Theorem 2.1.1. With reference to problem (1.1.1), let F satisfy (1.1.2). Let assumptions (A.1) and (A.2) hold true, so that there is a (coercive) conservative vector field $h(x) = \nabla d(x) \in [C^2(\bar{\Omega})]^n$ such that $h \cdot \nu = 0$ on Γ_0. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$ and let $T_0 > 0$ be the constant defined in (1.1.8b). Then

(a) for all $T > T_0$, the following continuous observability inequality holds true for $H^{1,1}(Q)$-solutions: there exists a constant $C_T > 0$ such that, with $\Sigma_1 \equiv (0,T] \times \Gamma_1$, we have:

$$C_T E(0) \leq \int_0^T \int_{\Gamma_1} [w_t^2 + w^2] d\Sigma_1 + \int_0^T \int_{\Omega} f^2 dQ. \tag{2.1.2}$$

(b) *A-fortiori*, the following global uniqueness result holds true: Let $T > T_0$ and let w be an $H^{1,1}(Q)$-solution of problem (1.1a) with $f \equiv 0$ along with the B.C.

$$\left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} = 0 \quad \text{and} \quad w|_{\Sigma_1} = 0 \quad \text{where } h \cdot \nu = 0 \text{ on } \Gamma_0,\ \Sigma_1 \equiv (0,T] \times \Gamma_1. \tag{2.1.3}$$

Then, in fact, $w \equiv 0$ in Q (in fact, $w \equiv 0$ in $\mathbb{R}_t \times \Omega$). □

Indeed, we shall first prove the uniqueness statement of part (b) in Theorem 7.1 [as a direct consequence of the Carleman estimates of Theorem 6.1 for $H^{2,2}(Q)$-solutions], as supplemented by Section 8, which provides the extension to $H^{1,1}(Q)$-solutions. Next, part (b) will be used to establish part (a), in Theorem 9.2, by virtue also of the trace Lemma 9.1.

Duality between continuous observability and exact controllability. We now explain the control-theoretic terminology so far used. Consider the mixed hyperbolic problem:

$$\begin{cases} v_{tt} - \Delta v = F_1(v) & \text{in } Q; & (2.1.4\text{a}) \\ v(0, \,\cdot\,) = 0,\ v_t(0, \,\cdot\,) = 0 & \text{in } \Omega; & (2.1.4\text{b}) \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Sigma_0} \equiv 0;\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\Sigma_1} \equiv g & \text{in } \Sigma, & (2.1.4\text{c}) \end{cases}$$

where F_1 is the first-order differential operator such as F in (1.1.1d), satisfying therefore hypothesis (1.1.2) [and Remark 1.1.1]. Problem (2.1.4) is called exactly controllable in the space $H^1(\Omega) \times L_2(\Omega)$ within the class of $L_2(0,T;L_2(\Gamma_1))$-controls, for $0 < T < \infty$, in case: given such $T > 0$ and given any pair $\{v_{0,T}, v_{1,T}\} \in H^1(\Omega) \times L_2(\Omega)$, there exists a control function $g \in L_2(0,T;L_2(\Gamma_1))$ on the universal time interval $[0,T]$, such that the corresponding solution to problem (2.1.4) satisfies the terminal condition: $v(T,\,\cdot\,) = v_{0,T};\ v_t(T,\,\cdot\,) = v_{1,T}$.

In other words, the input-solution map $g \to \mathcal{L}_T g \equiv \{v(T,\,\cdot\,), v_t(T,\,\cdot\,)\}$ of problem (2.1.4) is surjective:

$$\mathcal{L}_T : L_2(0,T;L_2(\Gamma_1)) \supset \mathcal{D}(\mathcal{L}_T) \text{ onto } H^1(\Omega) \times L_2(\Omega). \tag{2.1.5}$$

By a standard Functional Analysis result [T-L.1, p. 235], the surjectivity condition (2.1.5) is *equivalent* to the condition that the adjoint $\mathcal{L}_T^*$ be bounded below: there exists $C_T > 0$ such that the following continuous observability inequality holds true:

$$\|\mathcal{L}_T^* z\|_{L_2(0,T;L_2(\Gamma_1))} \geq C_T \|z\|_{H^1(\Omega)\times L_2(\Omega)}, \tag{2.1.6}$$

for all $z \in \mathcal{D}(\mathcal{L}_T^*) \subset L_2(0,T;L_2(\Gamma_1))$, so that the left-hand side is finite. Conditions (2.1.6) turns out to be the counterpart, modulo $\ell.o.t.$, to inequality (2.1.2) for $f \equiv 0$ (e.g., [L-T.2–4], [L-T-Y.1], [Tr.1]). Thus, inequality (2.1.2) of Theorem 2.1.1 is the crux in establishing the exact controllability property for problem (2.1.4), as defined before, on a universal time $T > T_0$.

2.2 Uniform stabilization

In this section we consider the damped problem

$$\begin{cases} w_{tt} = \Delta w + Fw & \text{in } (0,T] \times \Omega = Q; & (2.2.1\text{a}) \\ w(0,\,\cdot\,) = w_0,\ w_t(0,\,\cdot\,) = w_1 & \text{in } \Omega; & (2.2.1\text{b}) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_0} \equiv 0 & \text{in } (0,T] \times \Gamma_0 = \Sigma_0; & (2.2.1\text{c}) \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} = -w_t & \text{in } (0,T] \times \Gamma_1 = \Sigma_1. & (2.2.1\text{d}) \end{cases}$$

Theorem 2.2.1. With reference to problem (2.2.1), let F satisfy hypothesis (1.1.2).

(i) Then, problem (2.2.1) is well posed on $Y = H^1(\Omega) \times L_2(\Omega)$ in the semigroup sense: the map

$$\{w_0, w_1\} \to e^{A_N t}\{w_0, w_1\} \equiv \{w(t), w_t(t)\} : Y \to C([0,T];Y), \tag{2.2.2}$$

defines a strongly continuous semigroup $e^{A_N t}$ on Y.

(ii) Let assumptions (A.1) and (A.2) hold true, so that there exists a (coercive) conservative vector field $h(x) = \nabla d(x) \in [C^2(\Omega)]^n$ such that $h \cdot \nu = 0$ on Γ_0. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$.

Then: the following estimates holds: for $T > T_0$, T_0 defined in (1.1.8b), there is a constant $c_T > 0$ such that

$$c_T E(T) \leq \int_0^T \int_{\Gamma_1} \left[\left(\frac{\partial w}{\partial \nu} \right)^2 + w^2 \right] d\Sigma_1. \tag{2.2.3}$$

We now explain the terminology of the present subsection's title. Let $F \equiv 0$ in (2.2.1a). It then follows readily from inequality (2.2.3) via the dissipation identity (obtained by multiplying problem (2.2.1) by w_t and integrating by parts) that $E(T) \leq c_T E(0)$ for some $c_T < 1$. Or, in semigroup terms, $\|e^{A_N T}\| \leq C_T < 1$ in the uniform norm of $\mathcal{L}(Y)$. A standard semigroup result then yields that: there exist constants $a > 0$ and $M \geq 1$, such that $\|e^{A_N t}\| \leq M e^{-at}$, $t \geq 0$. Thus, the feedback $-w_t$ in (2.2.1d) uniformly stabilizes, when $F \equiv 0$, the corresponding problem where $-w_t$ is replaced by zero, which is a conservative, energy preserving problem.

2.3 Literature and overview

Since the inception of this topic, continuous observability/stabilization inequalities for second-order hyperbolic equations have been established, almost exclusively, in the case where the unobserved/uncontrolled part Γ_0 of the boundary Γ is subject to homogeneous *Dirichlet*, rather than Neumann, B.C. To begin with, this is the case for the original uniform stabilization (hence [R.1], exact controllability) results of the wave equation: in $H^1(\Omega) \times L_2(\Omega)$ with Neumann $L_2(\Sigma)$-boundary feedback [C.1], [Lag.1], [Tr.2], [L-T.3] and in $L_2(\Omega) \times H^{-1}(\Omega)$ with Dirichlet $L_2(\Sigma)$-boundary feedback [L-T.1], [L-T.3], [L-T-Y.3]. Moreover, this is the case also for most of the subsequent works aimed at a *direct* establishment of continuous observability inequalities (which then, by duality, yield exact controllability results, without passing through the generally more, or even much more, demanding stabilization problem): [H.1], [L-T.2], [Li.1–2]. See also an account in [K.1]. An excellent summary of the earlier literature is given in [R.2]. A first exception to the above statement is paper [Tr.1], which deals also with the observability/controllability issue of the *wave* equation subject to homogeneous *Neumann* B.C. on Γ_0, though under Dirichlet control on Γ_1. It was this reference [Tr.1] that introduced the vector field condition, $h \cdot \nu \equiv 0$ on Γ_0 in (1.1.4), of parallelism along Γ_0, for a coercive vector field h on Ω, see (1.1.5), to obtain continuous observability estimates: in the last step, appeal to compactness/uniqueness (Holmgren) is made, to absorb an interior lower-order term. Though several classes of examples were given in [Tr.1], this geometrical condition (A.1) (i.e., (1.1.4) and (1.1.5)) due to the homogeneous Neumann B.C. on Γ_0 was not analyzed there.

In the present paper, the inclusion of an energy level term F with coefficients both space and time dependent produces serious additional difficulties when coupled with the geometric condition (A.1) in particular on Γ_0.

First, the original Holmgren unique continuation result across the non-characteristic surface is not available with only $L_\infty(Q)$-regularity of the coefficients of the energy level terms. However, a local unique continuation result of $H_m^{(loc)}$ solutions across a strongly pseudo-convex surface is available for (linear) differential operators of order m with, say, real C^1-coefficients of the principal part and L_∞-coefficient of lower-order terms [Ho.1, Theorem 8.9.1, p. 224]. (A non-characteristic surface is strongly pseudo-convex, but not conversely.) Furthermore, we refer to [Ho.1–2], [Ta.2–3] for some very recent references where at least partial analyticity is needed, and [I.1] for a recent research monograph on this subject. (If these coefficients were *only space dependent*, appeal to the unique continuation results for the corresponding *elliptic* problem [Ho.1, Theorem 17.2.6, p. 14] would do the job.) A contemporaneous work [Lit.3] provides a much welcome *global* uniqueness theorem, say for hyperbolic second-order equations, starting, however, from a *local* uniqueness result. Thus, this is a result, proved in [Lit.3] by purely geometric means, that is a counterpart of the 'triangle lemma' in [I.1, Lemma 3.4.6, p. 67], which by contrast is established by analytic means. On the other hand, a main goal of the present paper is precisely to *eliminate* from the continuous observability/stabilization estimates *interior lower-order terms*, thus the need of appealing to external (until now apparently non-existent) *global* unique continuation results, to absorb them. This is achieved by virtue of the additional requirement (A.2) = (1.1.6) on the vector field h, which is responsible for obtaining the key estimate (1.1.15b), in the treatment of Sections 1 through 9 (or the additional requirement (A.2i) = (10.1.4), which is responsible for the key estimate (10.1.14) in the treatment of Section 10). In fact, a sub-goal of the present paper is to obtain, *a-fortiori*, directly, new global unique continuation results for $T > T_0$, from the preliminary Carleman estimates established this time without lower-order terms. The consequent global unique continuation results in Theorem 7.1 are obtained in precisely the form that is then needed to eliminate the interior lower-order terms arising, however, from a different source: the goal, this time, of eliminating the traditional star-shaped restrictions of the literature on the observed/controlled part Γ_1 of the boundary Γ. This step requires Lemma 9.1, Eqn. (9.1), to control the tangential trace of the solutions to Eqn. (1.1.11) in terms of their normal and velocity traces, and hence the global uniqueness Theorem 7.1 (and Remark 8.1) to eliminate the corresponding interior lower-order term in (9.1): see Step 2, in the proof of Theorem 9.2. Alternatively, if one wishes to retain the star-shaped geometrical conditions $h \cdot \nu \geq 0$ on the observed/controlled portion Γ_1 of the boundary Γ, this method has then the virtue over the literature of yielding an *explicit* constant in the continuous observability/stabilization estimates of the order Ce^{Cr^2}, where C is a constant and r is the Hilbert norm of the involved coefficients, see (1.1.2c). This is the case, e.g., in the more general setting of Section 10, see Theorem 10.1.1.

Second, the further effort in Section 10 to weaken in many cases (including the case of flat Γ_0) the geometrical conditions (A.1), (A.2), relies on a domain decomposition $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$, with consequent cut-off functions χ_i on each Ω_i. It is because of the Neumann (rather than Dirichlet) B.C., that these cut-off functions have to be non-trivially selected as to be, among other features, only time-dependent on a small interior layer of the boundary Γ, see (10.2.10), (10.2.11) and Remark 10.2.1. A relevant reference for general second-order hyperbolic equations giving sharp (geometric optics) sufficient conditions for observability/stabilization at the energy $H^1 \times L_2$-level (as in the present paper) is the [B-L-R.1] and its precursor [Lit.1] for first-order hyperbolic systems. A comparison of techniques in the literature is attempted in the introduction of [L-T-Y.1], as well as of [I-Y.1], to which we refer. Generally, the geometric optics conditions are not readily checkable.

Conceptually, this paper is motivated by the desire to give a self-contained treatment of continuous observability/stabilization inequalities without passing through the preliminary traditional step of *first establishing these inequalities polluted by lower-order terms*, as done essentially in almost all of the literature, including [Lag.1], [B-L-R], [L-T.2–4], [Lio.1–2], [Ta.1], etc. To this end, we would then seek to introduce an additional degree of flexibility to the explicit computational treatment of [L-T.4] in obtaining Carleman estimates. The inspiration for this comes from the Russian literature, in particular [L-R-S.1, Lemma 1, p. 124], which is apparently not well known outside its original circle. This gives a complicated, yet very useful, *pointwise* Carleman-type inequality with, apparently, one further degree of freedom over [L-T.4], and other literature, where the benefits for the estimates are obtained not pointwise, but after integration on Q. In [L-R-S.1] Lemma 1, p. 124 of this reference is used precisely to obtain unique continuation results (pp. 133–142). This aforementioned result [L-R-S.1, Lemma 1, p. 124] forms also the basic starting point in other investigations on inverse problems and on stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities. See [K-K.1] for the latter, and references therein for the former.

Comparison with [K-K.1]. The main result of [K-K.1] is an *a-priori* stability estimate for $H^{2,2}(Q)$-solutions of a hyperbolic problem such as (1.1.1) and (1.1.2), which in particular yields uniqueness $w \equiv 0$ in Q of such solutions, if *zero Dirichlet* B.C. $w|_\Sigma \equiv 0$ is imposed on the *whole boundary* Γ, while zero Neumann B.C. $\frac{\partial w}{\partial \nu}|_{\Sigma_1} \equiv 0$ is imposed on a subportion Γ_1 where $h \cdot \nu \geq 0$, $h(x) = x - y$. Yet the stability estimate in [K-K.1, Theorem 2.2 or Theorem 3.1] is not quite the continuous observability estimate, as it needs to be extended to $H^{1,1}(Q)$-solutions. Apart from the higher *a-priori* regularity ($H^{2,2}(Q)$ rather than $H^{1,1}(Q)$) that [K-K.1] requires of the wave equation solution, its main estimate [K-K.1, Theorem 3.1, p. 101] would yield continuous observability inequalities in the following two 'classical' cases of the mid-eighties: (i) either for the case where $w|_\Sigma \equiv 0$ on the *entire* boundary Γ, while the *observed* portion of the boundary Γ_1 satisfies the (star-shaped) geometrical condition $h \cdot \nu \geq 0$, $h(x) = x - x_0$; (ii) or else for the case where $w|_{\Sigma_0} \equiv 0$ and $\frac{\partial w}{\partial \nu}|_{\Sigma_1} \equiv 0$, with $h \cdot \nu \geq 0$, $h(x) = x - x_0$, on the *observed* portion of the boundary Γ_1 (in our present notation) with, in addition, a *full* $H^{1,1}(\Sigma_1)$-norm on Σ_1. (In this case, the passage from $H^{2,2}(Q)$ to $H^{1,1}(Q)$ solutions is technical, see our Section 8.)

By contrast, our present paper assumes the definitively more challenging and non-classical B.C. $\frac{\partial w}{\partial \nu}|_{\Sigma_0} \equiv 0$ as well on the *unobserved* portion Γ_0 of the boundary Γ, and, moreover, it manages to dispense altogether—in the final estimates (2.1.2), or (10.1.20)—of the tangential H^1-norm on Γ_1 with no star-shaped geometrical condition imposed on the observed portion Γ_1; or else, if a star-shaped condition is imposed on the observed portion Γ_1, then the resulting observability constant is *explicit* (and of the order of Ce^{cr^2}, where C is a generic constant and r is the norm in (1.1.2c) on the involved coefficients).

In terms of control rather than observation, the preceding considerations can be expressed as follows:

Case (i) above in [K-K.1] refers to Dirichlet control on Γ_1, while $w|_{\Sigma_0} \equiv 0$ on Γ_0.

Case (ii) above in [K-K.1] refers to Neumann control on Γ_1, while $w|_{\Sigma_0} \equiv 0$ on Γ_0.

The present paper refers to Neumann control on Γ_1, while $\frac{\partial w}{\partial \nu}|_{\Sigma_0} \equiv 0$ on Γ_0.

The cut-off functions in [K-K.1] are not really needed (as shown in an unpublished report by the present authors, Fall 1998). By contrast, our present paper requires the choice of sophisticated cut-off functions, see Remark 10.2.1, to achieve its maximally claimed generality in Theorem 10.1.1, in addition to a new version of the fundamental pointwise Lemma 3.1, over [L-R-S.1, Lemma 1, p. 124].

The method of [K-K.1] suitably refined permits to obtain an *explicit* constant in the corresponding observability/stabilization inequalities [Z.1–3], which has beneficial consequences in semilinear problems, in the style of the present paper.

A key feature of the pointwise Carlemann-type estimate given by [L-R-S.1, Lemma 1, p. 124] is that the interior lower-order term comes with a controlled 'right' sign $c\tau^3$ (as in our Eqns. (4.21) and (4.26)) in a suitable set in time and space (akin to our $Q^*(\sigma^*)$).

For the aforementioned reasons stated at the outset of this subsection, the Neumann (rather than Dirichlet) problem is technically much more demanding. In our first effort to the Neumann problem of the present paper (Fall 1998), we also took an approach that relied, in its starting point, on the pointwise Carlemann-type inequality for C^2-solutions of problem (1.1.1), (1.1.2) given by [L-R-S.1, Lemma 1, p. 124]. This approach led to a sharp result, precisely the one of our present paper, in the case of a *flat unobserved/uncontrolled boundary* Γ_0. Here, the special feature $|\nabla d(x)|^2 = 4|x-z|^2$ noted in (1.1.21a) was useful. An apparently somewhat similar treatment was carried out in the almost contemporaneous, and surely independent, work of [I-Y.1], presented at the Colorado's Conference. However, in the general case of a *curved unobserved/uncontrolled boundary* Γ_0, the approach based on [L-R-S.1, Lemma 1, p. 124] of both our first Fall 1998 effort and [I-Y.1] inherits unfortunately additional geometrical conditions. For instance, in the case where Ω is the $\mathbb{R}^2$-unit disk centered at the origin, this method required the uncontrolled/unobserved portion Γ_0 of the boundary Γ to be arbitrarily close to $\frac{1}{4}$ of the circumference Γ (if connected, see Example B.1.1 in Appendix B).

By contrast, we expect by known control theory results [Lio.1–2], [Tr.1], that Γ_0 should be arbitrarily close to $\frac{1}{2}$ of the circumference Γ. The improvement from $\frac{1}{4}$ of Γ to $\frac{1}{2}$ of Γ is highly non-trivial and is achieved in our Section 10.

In order to relax the additional geometrical conditions, our present second effort (Spring 1999) obtains another more suitable pointwise Carleman-type estimate for C^2-solutions of problems (1.1.1), (1.1.2) in a form (our present Lemma 3.1), which is a sufficiently noteworthy variation of [L-R-S.1, Lemma 1, p. 124] to warrant an explicit, complete proof. This we provide in Section 3. Building upon this new pointwise estimate of our Lemma 3.1, we are thus able to relax the geometric conditions on Γ_0 when Γ_0 is curved, even in the framework of Sections 1 through 9, and more so in the much less restrictive setting of Section 10, at the price of serious additional technical difficulties (see cut-off functions χ_i of Section 10.2). It is through the treatment of Section 10 (of which there is no counterpart either in our first Fall 1998 effort, or in [I-Y.1]) that we are finally able to reach our goal to increase, in the case of an $\mathbb{R}^2$-disk, the unobserved/uncontrolled portion Γ_0 of the boundary from almost $\frac{1}{4}$ to almost $\frac{1}{2}$ of the whole circumference Γ. A similar result holds for the $\mathbb{R}^n$-sphere for any n. See Appendix C, Theorem C.1, Fig. C.2. In common with the original Russian approach, our present methodology shares the key observation and the benefit of penalizing the interior lower-order term with a controlled 'right' sign (our Eqns. (4.21) and (4.24) with $\tau^3\tilde{\beta}$ or $\tau^3\beta$, $\tilde{\beta}, \beta > 0$) on a suitable time and space—set $Q^*(\sigma^*)$, which may, at worst, be the entire cylinder $Q = (0, T] \times \Omega$. This then allows one *to drop* such interior lower-order term, see the one-paragraph argument below (5.18), in the proof of the final Carleman estimate, Theorem 5.1, part (ii), Eqn. (5.2), or in the proof of Theorem 10.4.1, Step 2. This step is the main virtue of all these [L-R-S]-based, or [L-R-S]-inspired approaches: the elimination of the interior lower-order term in the final Carleman estimate. As the method accomodates, with no extra difficulty, energy level terms which may be space as well as *time* dependent and of low regularity $L_\infty(Q)$, the resulting global uniqueness result Theorem 7.1 for $T > T_0$ and $H^{1,1}(Q)$-solutions is, apparently, new. Here, time-dependence prevents appealing to elliptic theory. In the case of *flat* Γ_0, the uniqueness time T_0 is optimal. In general, for Γ_0 curved, the time T_0 of observability/controllability/global uniqueness is subject to scaling, as explained in Section 1.2, and is not necessarily guaranteed (nor do we expect it) to be optimal. We finally remark that, in the Neumann case, an additional technical difficulty not present in the Dirichlet case is the passage from $H^{2,2}(Q)$-solutions to $H^{1,1}(Q)$-solutions for the final Carleman estimates. This step is carried out in Section 8, via an approximation argument. This step is non-trivial, since finite energy solutions subject to Neumann B.C. do not produce (in dimension dim $\Omega \geq 2$) H^1-traces on the boundary, see [L-T.5].

This paper leads naturally to some geometrical open questions. To keep in line with the spirit of the AMS-IMS-SIAM Summer Research Conference held at the University of Colorado, Boulder, June 27–July 2, 1998, these are duly noted. One is the fulfillment of assumptions (A.1), (A.2) of Section 1, or (A.1i), (A.2i) of Section 10: see the appendices. Another is the possibility of extending the present paper to the general case where $(-\Delta)$ is replaced by a strongly elliptic operator with C^1-space dependent coefficients. Such an extension will have to rest on the counterpart of the basic Lemma 3.1 of this paper, given, however, in terms of a corresponding Riemann metric, as in [L-T-Y.1–3], [Y.1].

The first key step of such an extension—the proof of the basic Lemma 3.1 to this variable coefficient situation—has been (December 1999) recently carried out successfully [L-T-Y-Z.1]. It uses Bochner's techniques in Riemann geometry, in the style of [L-T-Y.1–3], [Y.1], which closely patterns the present proof in the Euclidean environment, thus confirming the validity of the aim of the Colorado's Conference.

3 A fundamental lemma

The starting point of our proofs is the following pointwise estimate. This is a sufficiently noteworthy variation of a result in the literature [L-R-S.1, Lemma 1, p. 124] to warrant an explicit proof. Such estimate will then be applied to (smooth) solutions of the wave equation (1.1.1a). With $t \in \mathbb{R}_t$ and $x = [x_1, \dots, x_n] \in \mathbb{R}^n_x$, we shall indicate the partial derivatives as follows: $\frac{\partial}{\partial x_j}\frac{\partial}{\partial t} f = f_{tx_j}$, etc. [K-K.1], by contast, uses [L-R-S.1, Lemma 1, p. 124].

Lemma 3.1. Let

$$w(t,x) \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x);\ \ell(t,x) \in C^3((\mathbb{R}_t \times \mathbb{R}^n_x);\ \psi(t,x) \in C^2 \text{ in } t \text{ and } C^1 \text{ in } x \tag{3.1}$$

be three given functions and set $\theta(t,x) = e^{\ell(t,x)}$. Let $\epsilon > 0$ be arbitrary.

Then, the following pointwise inequality holds true

$$\theta^2(w_{tt} - \Delta w)^2 - \frac{\partial M}{\partial t} + \text{div } V \geq -8 v_t \nabla \ell_t \cdot \nabla v + 2(\Delta \ell + \ell_{tt} - \psi) v_t^2$$

$$+2\left(\psi - \frac{\epsilon}{2} - \Delta \ell + \ell_{tt}\right)|\nabla v|^2 + 4\left(\sum_{i,j=1}^{n} \ell_{x_i x_j} v_{x_i} v_{x_j}\right) + \theta^2 \tilde{B} w^2, \tag{3.2}$$

where we have set

$$\begin{aligned} M &= M(w) \equiv \theta^2\{-2\ell_t(w_t^2 + |\nabla w|^2) + 4\nabla \ell \cdot \nabla w\, w_t \\ &\quad + 2(-2\ell_t^2 + 2|\nabla \ell|^2 + \psi) w_t w + (-2A\ell_t - 2\ell_t^3 + 2\ell_t |\nabla \ell|^2 - \psi_t) w^2\}; \end{aligned} \tag{3.3}$$

$$V = [V_1, \dots, V_j, \dots, V_n]; \tag{3.4a}$$

$$\begin{aligned} V_j &= V_j(w) \equiv 2\theta^2\Big\{\ell_{x_j}(w_t^2 - |\nabla w|^2) - 2w_{x_j}(\ell_t w_t - \nabla \ell \cdot \nabla w) \\ &\quad + 2\Big(|\nabla \ell|^2 - \ell_t^2 + \frac{\psi}{2}\Big) w_{x_j} w + \ell_{x_j}(|\nabla \ell|^2 - \ell_t^2 - A) w^2\Big\}, \end{aligned} \tag{3.4b}$$

and, moreover,

$$A \equiv (\ell_t^2 - \ell_{tt}) - |\nabla \ell|^2 + \Delta \ell - \psi; \tag{3.5}$$

$$\tilde{B} = 2A\psi - 2\left[\sum_{j=1}^{n} \frac{\partial}{\partial x_j}((A+\psi)\ell_{x_j}) - \frac{\partial}{\partial t}((A+\psi)\ell_t) - \frac{1}{\epsilon}|\nabla \psi|^2 + \psi_{tt}\right]. \tag{3.6}$$

[(3.6) shows the need for ψ to be C^2 in t, and C^1 in x.]

Proof. Step 1. We let $v(t,x) = \theta(t,x)w(t,x) = e^{\ell(t,x)}w(t,x)$, $(t,x) \in Q = (0,T] \times \Omega$. By direct differentiation, we get

$$\begin{cases} \theta w_{tt} = v_{tt} - 2\ell_t v_t + (\ell_t^2 - \ell_{tt})v & (3.7) \\ \theta w_{x_jx_j} = v_{x_jx_j} - 2\ell_{x_j}v_{x_j} + (\ell_{x_j}^2 - \ell_{x_jx_j})v, \; j = 1, \dots, n & (3.8) \\ \theta \Delta w = \theta \sum_{j=1}^{n} w_{x_jx_j} = \Delta v - 2\nabla\ell \cdot \nabla v + (|\nabla \ell|^2 - \Delta \ell)v. & (3.9) \end{cases}$$

Thus, from (3.7) and (3.9) we obtain

$$\begin{aligned} \theta^2[w_{tt} - \Delta w]^2 &= \{[v_{tt} - 2\ell_t v_t + (\ell_t^2 - \ell_{tt})v] \\ &\quad -[\Delta v - 2\nabla \ell \cdot \nabla v + (|\nabla \ell|^2 - \Delta \ell)v]\}^2 && (3.9a) \\ &= |I_1 + I_2 + I_3|^2. && (3.9b) \end{aligned}$$

Here we have set (after adding and subtracting ψv)

$$I_1 = v_{tt} - \Delta v + Av, \; I_2 = -2\ell_t v_t + 2\nabla \ell \cdot \nabla v; \; I_3 = \psi v, \tag{3.10}$$

where A is defined by (3.5). From (3.9b), we obtain

$$\theta^2[w_{tt} - \Delta w]^2 \geq 2(I_1 I_2 + I_2 I_3 + I_1 I_3). \tag{3.11}$$

Step 2. With reference to (3.10), we shall prove in this Step 2 that

$$\begin{aligned} 2I_1 I_2 &= \frac{\partial}{\partial t}\left\{-2\ell_t[v_t^2 + |\nabla v|^2 + Av^2] + 4v_t \nabla \ell \cdot \nabla v\right\} \\ &\quad - 2\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left\{2v_{x_j}\nabla \ell \cdot \nabla v - \ell_{x_j}|\nabla v|^2 - 2\ell_t v_t v_{x_j} + \ell_{x_j}v_t^2 - A\ell_{x_j}v^2\right\} \\ &\quad - 8v_t \nabla \ell_t \cdot \nabla v + 2(\Delta \ell + \ell_{tt})v_t^2 + 4\sum_{i,j}^{n} \ell_{x_ix_j}v_{x_i}v_{x_j} \\ &\quad - 2(\Delta \ell - \ell_{tt})|\nabla v|^2 - 2\left[\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(A\ell_{x_j}) - \frac{\partial}{\partial t}(A\ell_t)\right] v^2. \end{aligned} \tag{3.12}$$

Proof of (3.12). From the definitions of I_1, I_2 in (3.10), we obtain after using $2v_tv_{tt} = \frac{\partial}{\partial t}(v_t^2)$, $2vv_t = \frac{\partial}{\partial t}(v^2)$, and $2v\nabla v = \nabla(v^2)$:

$$\begin{aligned} 2I_1I_2 &= 2(v_{tt} - \Delta v + Av)(-2\ell_t v_t + 2\nabla\ell \cdot \nabla v) \\ &= -2\ell_t \frac{\partial}{\partial t}(v_t^2) - 2A\ell_t \frac{\partial}{\partial t}(v^2) + 2A\nabla\ell \cdot \nabla(v^2) \\ &\quad + 4v_{tt}\nabla\ell \cdot \nabla v + 4\Delta v v_t \ell_t - 4\Delta v \nabla\ell \cdot \nabla v. \end{aligned} \tag{3.13}$$

But, with reference to the last three terms in (3.13), we have by direct computations:

$$4v_{tt}\nabla\ell \cdot \nabla v = 4\tfrac{\partial}{\partial t}(v_t\nabla\ell \cdot \nabla v) - 4v_t\nabla\ell_t \cdot \nabla v - 4v_t\nabla\ell \cdot \nabla v_t \tag{3.14a}$$

$$= 4\sum_{j=1}^{n} \frac{\partial}{\partial t}(\ell_{x_j} v_{x_j} v_t) - 4v_t\nabla\ell_t \cdot \nabla v - 2\nabla\ell \cdot \nabla(v_t^2); \tag{3.14b}$$

$$4\Delta v v_t \ell_t = 4\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(v_t\ell_t v_{x_j}) - 2\ell_t \frac{\partial}{\partial t}(|\nabla v|^2) - 4v_t\nabla\ell_t \cdot \nabla v; \tag{3.15}$$

$$\begin{aligned} -4\Delta v \nabla\ell \cdot \nabla v &= -4\sum_{i,j=1}^{n} v_{x_jx_j}\ell_{x_i}v_{x_i} = -4\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(v_{x_j}\ell_{x_i}v_{x_i}) \\ &\quad + 4\sum_{i,j=1}^{n} \ell_{x_ix_j}v_{x_i}v_{x_j} + 2\sum_{i,j=1}^{n} \ell_{x_i}\frac{\partial}{\partial x_i}(v_{x_j}^2). \end{aligned} \tag{3.16}$$

Next, we substitute (3.14), (3.15) and (3.16) into (3.13), thus obtaining, after a re-arrangement of terms

$$\begin{aligned} 2I_1I_2 &= -2\ell_t \frac{\partial}{\partial t}(v_t^2) - 2A\ell_t \frac{\partial}{\partial t}(v^2) + 4\frac{\partial}{\partial t}(v_t\nabla\ell \cdot \nabla v) - 2\ell_t \frac{\partial}{\partial t}(|\nabla v|^2) \\ &\quad + 2A\sum_{j=1}^{n} \ell_{x_j}\frac{\partial}{\partial x_j}(v^2) - 2\sum_{j=1}^{n} \ell_{x_j}\frac{\partial}{\partial x_j}(v_t^2) \\ &\quad + 4\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(v_t\ell_t v_{x_j}) - 4\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(v_{x_j}\ell_{x_i}v_{x_i}) + 2\sum_{i,j=1}^{n} \ell_{x_j}\frac{\partial}{\partial x_j}(v_{x_i}^2) \\ &\quad - 4v_t\nabla\ell_t \cdot \nabla v - 4v_t\nabla\ell_t \cdot \nabla v + 4\sum_{i,j=1}^{n} \ell_{x_ix_j}v_{x_i}v_{x_j}. \end{aligned} \tag{3.17}$$

Next, regarding the first four-term group in (3.17) (with $\frac{\partial}{\partial t}$), we can rewrite as follows:

$$
\begin{aligned}
&-2\ell_t\frac{\partial}{\partial t}(v_t^2) - 2A\ell_t\frac{\partial}{\partial t}(v^2) + 4\frac{\partial}{\partial t}(v_t\nabla\ell\cdot\nabla v) - 2\ell_t\frac{\partial}{\partial t}(|\nabla v|^2) \\
&\quad = \frac{\partial}{\partial t}\left\{-2\ell_t(v_t^2 + Av^2 + |\nabla v|^2) + 4v_t\nabla\ell\cdot\nabla v\right\} + 2\ell_{tt}v_t^2 + 2v^2\frac{\partial}{\partial t}(A\ell_t) + 2\ell_{tt}|\nabla v|^2.
\end{aligned}
\tag{3.18}
$$

Similarly, regarding the second five-term group in (3.17) (with $\frac{\partial}{\partial x_j}$), we can rewrite as follows:

$$
\begin{aligned}
&2A\sum_{j=1}^n \ell_{x_j}\frac{\partial}{\partial x_j}(v^2) - 2\sum_{j=1}^n \ell_{x_j}\frac{\partial}{\partial x_j}(v_t^2) \\
&\quad + 4\sum_{j=1}^n \frac{\partial}{\partial x_j}(v_t\ell_t v_{x_j}) - 4\sum_{i,j=1}^n \frac{\partial}{\partial x_j}(v_{x_j}\ell_{x_i}v_{x_i}) + 2\sum_{i,j=1}^n \ell_{x_j}\frac{\partial}{\partial x_j}(v_{x_i}^2) \\
&= \sum_{j=1}^n \frac{\partial}{\partial x_j}\left\{2A\ell_{x_j}v^2 - 2\ell_{x_j}v_t^2 + 4v_t\ell_t v_{x_j} - 4v_{x_j}\sum_{i=1}^n \ell_{x_i}v_{x_i} + 2\ell_{x_j}\sum_{i=1}^n v_{x_i}^2\right\} \\
&\quad - 2\sum_{j=1}^n\left[\frac{\partial}{\partial x_j}(A\ell_{x_j})\right]v^2 + 2\left(\sum_{j=1}^n \ell_{x_jx_j}\right)v_t^2 - 2\left(\sum_{j=1}^n \ell_{x_jx_j}\right)|\nabla v|^2.
\end{aligned}
\tag{3.19}
$$

Substituting (3.18) and (3.19) into (3.17) yields (3.12), as desired.

Step 3. With reference to (3.10) we shall prove in this Step 3 that

$$
\begin{aligned}
2I_1I_3 &= \frac{\partial}{\partial t}[2\psi vv_t - \psi_t v^2] + [\psi_{tt} + 2A\psi]v^2 - 2\psi v_t^2 \\
&\quad - 2\sum_{j=1}^n \frac{\partial}{\partial x_j}(\psi v_{x_j}v) + 2v\nabla\psi\cdot\nabla v + 2\psi|\nabla v|^2
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\geq \frac{\partial}{\partial t}[2\psi vv_t - \psi_t v^2] + \left[\psi_{tt} + 2A\psi - \frac{1}{\epsilon}|\nabla\psi|^2\right]v^2 \\
&\quad - 2\psi v_t^2 + [2\psi - \epsilon]|\nabla v|^2 - 2\sum_{j=1}^n \frac{\partial}{\partial x_j}(\psi v_{x_j}v),
\end{aligned}
\tag{3.21}
$$

where, in (3.21), $\epsilon > 0$ is arbitrary.

Proof of (3.20), (3.21). From the definition of I_1 and I_3 in (3.10), we obtain

$$2I_1I_3 = 2(v_{tt} - \Delta v + Av)\psi v = 2\psi v v_{tt} - 2\psi \Delta v v + 2A\psi v^2, \tag{3.22}$$

where

$$2\psi v v_{tt} = 2\frac{\partial}{\partial t}(\psi v v_t) - 2\psi_t v v_t - 2\psi v_t^2; \tag{3.23}$$

$$\begin{aligned} -2\psi v \Delta v &= -2\sum_{i=1}^{n} \psi v v_{x_i x_i} \\ &= -2\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\psi v v_{x_j}) + 2v\nabla\psi \cdot \nabla v + 2\psi|\nabla v|^2. \end{aligned} \tag{3.24}$$

Finally, to obtain (3.20), we insert (3.23) and (3.24) into (3.22) and use $-2\psi_t v v_t = -\frac{\partial}{\partial t}(\psi_t v^2) + \psi_{tt}v^2$. Then, the estimate $2v\nabla\psi \cdot \nabla v \geq -\epsilon|\nabla v|^2 - \frac{1}{\epsilon}|\nabla\psi|^2 v^2$, for $\epsilon > 0$, used in the penultimate term in (3.20) yields (3.21).

Step 4. With reference to (3.10), we shall prove in this Step 4 that

$$\begin{aligned} 2I_2I_3 &= \frac{\partial}{\partial t}[-2\ell_t\psi v^2] + \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(2\psi\ell_{x_j}v^2) \\ &\quad + 2\left[\frac{\partial}{\partial t}(\ell_t\psi) - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}(\psi\ell_{x_j})\right]v^2. \end{aligned} \tag{3.25}$$

Proof of (3.25). From the definition of I_2 and I_3 in (3.10), we obtain

$$2I_2I_3 = 2[-2\ell_t v_t + 2\nabla\ell\cdot\nabla v]\psi v = -2\ell_t\psi\frac{\partial}{\partial t}(v^2) + 2\psi\nabla\ell\cdot\nabla(v^2), \tag{3.26}$$

where

$$-2\ell_t\psi\frac{\partial}{\partial t}(v^2) = \frac{\partial}{\partial t}(-2\ell_t\psi v^2) + \left[\frac{\partial}{\partial t}(2\ell_t\psi)\right]v^2 \tag{3.27}$$

$$\begin{aligned} 2\psi\nabla\ell\cdot\nabla(v^2) &= 2\sum_{j=1}^{n}\psi\ell_{x_j}\frac{\partial(v^2)}{\partial x_j} \\ &= 2\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(\psi\ell_{x_j}v^2) - 2\sum_{j=1}^{n}\left[\frac{\partial}{\partial x_j}(\psi\ell_{x_j})\right]v^2. \end{aligned} \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.26) yields (3.25), as desired.

Step 5. With reference to (3.11), in this Step 5 we prove that

$$\theta^2[w_{tt}-\Delta w]^2 \geq \frac{\partial}{\partial t}\Big\{-2\ell_t(v_t^2+|\nabla v|^2)+4v_t\nabla\ell\cdot\nabla v+2\psi v_t v-2\ell_t(A+\psi)v^2-\psi_t v^2\Big\}$$

$$-2\sum_{j=1}^{n}\frac{\partial}{\partial x_j}[2v_{x_j}\nabla\ell\cdot\nabla v-\ell_{x_j}|\nabla v|^2-2\ell_t v_{x_j}v_t+\ell_{x_j}v_t^2+\psi v_{x_j}v-(A+\psi)\ell_{x_j}v^2]$$

$$-8v_t\nabla\ell_t\cdot\nabla v+2(\Delta\ell+\ell_{tt}-\psi)v_t^2$$

$$+2\left(\psi-\frac{\epsilon}{2}-\Delta\ell+\ell_{tt}\right)|\nabla v|^2+4\sum_{i,j=1}^{n}\ell_{x_ix_j}v_{x_i}v_{x_j}$$

$$+\left\{2A\psi-2\left[\sum_{j=1}^{n}\frac{\partial}{\partial x_j}((A+\psi)\ell_{x_j})-\frac{\partial}{\partial t}((A+\psi)\ell_t)\right]-\frac{1}{\epsilon}|\nabla\psi|^2+\psi_{tt}\right\}v^2. \quad (3.29)$$

Proof of (3.29). We return to (3.11), where we use (3.12) for $2I_1I_2$, (3.21) for $2I_1I_3$, and (3.25) for $2I_2I_3$. Combining all '$\frac{\partial}{\partial t}$-terms,' all '$\frac{\partial}{\partial x_j}$-terms,' all $|\nabla v|^2$-terms and all v^2-terms, we arrive at (3.29), as desired.

Step 6. Henceforth, we specialize (3.29) with

$$\begin{cases} v=\theta w,\ \theta=e^{\ell};\ \text{hence } v_t=\theta[w_t+\ell_t w]; \\ v_{x_j}=\theta[w_{x_j}+\ell_{x_j}w];\ |\nabla v|^2=\theta^2\sum_{j=1}^{n}[w_{x_j}+\ell_{x_j}w]^2. \end{cases} \quad (3.30)$$

Then, the terms under $\frac{\partial}{\partial t}$ in (3.29) become via (3.30):

$$\frac{\partial}{\partial t}\left\{-2\ell_t(v_t^2+|\nabla v|^2)+4v_t\sum_{j=1}^{n}\ell_{x_j}v_{x_j}+2\psi v_t v-2\ell_t(A+\psi)v^2-\psi_t v^2\right\}$$

$$=\frac{\partial}{\partial t}\Big\{\theta^2\Big[-2\ell_t(w_t^2+|\nabla w|^2)+4w_t\nabla\ell\cdot\nabla w+2(2|\nabla\ell|^2-2\ell_t^2+\psi)w_t w$$

$$+(2\ell_t|\nabla\ell|^2-2\ell_t^3-2A\ell_t-\psi_t)w^2\Big]\Big\}. \quad (3.31)$$

Proof of (3.31). Using (3.30) in the terms under $\frac{\partial}{\partial t}$ in (3.29) we obtain

$$\frac{\partial}{\partial t}\Big\{ -2\ell_t(v_t^2+|\nabla v|^2)+4\sum_{j=1}^n \ell_{x_j}v_{x_j}v_t+2\psi v_t v-2\ell_t(A+\psi)v^2+\psi_t v^2\Big\}$$

$$\begin{aligned} &= \frac{\partial}{\partial t}\Big\{\theta^2\Big[-2\ell_t\Big((w_t+\ell_t w)^2+\sum_{j=1}^n(w_{x_j}+\ell_{x_j}w)^2\Big) \\ &\quad + 4\sum_{j=1}^n \ell_{x_j}(w_{x_j}+\ell_{x_j}w)(w_t+\ell_t w) \\ &\quad + 2\psi(\ell_t w+w_t)w-2\ell_t(A+\psi)w^2-\psi_t w^2\Big]\Big\} \qquad (3.32) \\ &= \frac{\partial}{\partial t}\Big\{\theta^2\Big[-2\ell_t(w_t^2+|\nabla w|^2)+4\sum_{j=1}^n \ell_{x_j}w_{x_j}w_t \\ &\quad + 2(-2\ell_t^2+2|\nabla \ell|^2+\psi)w_t w+(-2A\ell_t-2\ell_t^3+2\ell_t|\nabla\ell|^2-\psi_t)w^2\Big]\Big\}, \qquad (3.33) \end{aligned}$$

after two cancellations: of $2\psi\ell_t w^2$ and of $4\ell_t\sum_j \ell_{x_j}w_{x_j}w$, and (3.31) is proved.

Step 7. Under the specialization $v=\theta w$ as in (3.30), we have that the block of 'divergence terms' in (3.29) (modulo the coefficient -2) becomes

$$\sum_{j=1}^n \frac{\partial}{\partial x_j}[2v_{x_j}\nabla\ell\cdot\nabla v-\ell_{x_j}|\nabla v|^2-2\ell_t v_{x_j}v_t+\ell_{x_j}v_t^2+\psi v_{x_j}v-(A+\psi)\ell_{x_j}v^2]$$

$$\begin{aligned} &= \sum_{j=1}^n \frac{\partial}{\partial x_j}\Big\{\theta^2\Big[2w_{x_j}\nabla\ell\cdot\nabla w-\ell_{x_j}|\nabla w|^2-2\ell_t w_{x_j}w_t+\ell_{x_j}w_t^2 \\ &\quad + 2\Big(|\nabla\ell|^2-\ell_t^2+\frac{\psi}{2}\Big)w_{x_j}w+\ell_{x_j}(|\nabla\ell|^2-\ell_t^2-A)w^2\Big]\Big\}. \qquad (3.34) \end{aligned}$$

Proof of (3.34). Using (3.30) for v_{x_j} and v_{x_i}, we preliminarily compute

$$\begin{aligned} 2v_{x_j}\nabla\ell\cdot\nabla v &= 2v_{x_j}\sum_{i=1}^n \ell_{x_i}v_{x_i}=2\theta[\ell_{x_j}w+w_{x_j}]\sum_{i=1}^n \ell_{x_i}\theta[\ell_{x_i}w+w_{x_i}] \\ &= 2\theta^2(\ell_{x_j}w+w_{x_j})(|\nabla\ell|^2 w+\nabla\ell\cdot\nabla w). \qquad (3.35) \end{aligned}$$

Next, we use (3.35), as well as the expressions in (3.30) for $|\nabla v|$, v_{x_j}, v_t, and v, into the left side of (3.34) to rewrite it as

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[2v_{x_j}\nabla \ell \cdot \nabla v - \ell_{x_j}|\nabla v|^2 - 2\ell_t v_{x_j} v_t + \ell_{x_j} v_t^2 + \psi v_{x_j} v - (A+\psi)\ell_{x_j} v^2]$$

$$\begin{aligned} = \quad & \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big[2\theta^2(\ell_{x_j} w + w_{x_j})(|\nabla \ell|^2 w + \nabla \ell \cdot \nabla w) \\ & - \ell_{x_j}\theta^2 \sum_{i=1}^{n}(\ell_{x_i} w + w_{x_i})^2 - 2\ell_t \theta(\ell_{x_j} w + w_{x_j})\theta(\ell_t w + w_t) \\ & + \ell_{x_j}\theta^2(\ell_t w + w_t)^2 + \psi\theta(\ell_{x_j} w + w_{x_j})\theta w - (A+\psi)\ell_{x_j}\theta^2 w^2\Big] \end{aligned} \tag{3.36}$$

$$\begin{aligned} = \quad & \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big\{\theta^2\Big[2|\nabla \ell|^2 \ell_{x_j} w^2 + 2\ell_{x_j}\nabla \ell \cdot \nabla w w \\ & + 2|\nabla \ell|^2 w_{x_j} w + 2\nabla \ell \cdot \nabla w w_{x_j} - \ell_{x_j}(|\nabla \ell|^2 w^2 + |\nabla w|^2 + 2w\nabla \ell \cdot \nabla w) \\ & - 2\ell_t^2 \ell_{x_j} w^2 - 2\ell_t \ell_{x_j} w w_t - 2\ell_t^2 w w_{x_j} - 2\ell_t w_t w_{x_j} \\ & + \ell_{x_j}\ell_t^2 w^2 + \ell_{x_j} w_t^2 + 2\ell_t \ell_{x_j} w w_t + \psi \ell_{x_j} w^2 + \psi w_{x_j} w - (A+\psi)\ell_{x_j} w^2\Big]\Big\}. \end{aligned} \tag{3.37}$$

After a cancellation of $2\ell_{x_j}\nabla \ell \cdot \nabla w w$, $2\ell_t \ell_{x_j} w w_t$, and $\psi \ell_{x_j} w^2$, then (3.37) becomes (3.34) as desired.

Step 8. We finally insert (3.31) and (3.34) in the right side of (3.29), recall the definition of M, V, and $\tilde{B}$ in (3.3), (3.4), and (3.6), respectively, and finally arrive at

$$\begin{aligned} & \theta^2(w_{tt} - \Delta w)^2 - \frac{\partial M}{\partial t} + \text{div } V \\ & \qquad \geq \quad -8v_t \nabla \ell_t \cdot \nabla v + 2(\Delta \ell + \ell_{tt} - \psi)v_t^2 \\ & \qquad \qquad +2\left(\psi - \frac{\epsilon}{2} - \Delta \ell + \ell_{tt}\right)|\nabla v|^2 + 4\sum_{i,j=1}^{n} \ell_{x_i x_j} v_{x_i} v_{x_j} + \tilde{B} v^2, \end{aligned} \tag{3.38}$$

which is precisely the sought-after Eqn. (3.2). The proof of Lemma 3.1 is complete. □

Remark 3.1. The above proof has been recently extended (December 1999) to the case where the coefficients of the principal part are variable in space [L-T-Y-Z.1], by using Bochner's techniques in Riemann geometry, in the style of [L-T-Y.1–3], [Y.1]. Extension of the present paper to that variable coefficient case is in progress. □

4 A basic pointwise inequality

We now make suitable choices in the functions $\ell(t,x)$ and $\psi(x)$ involved in Lemma 3.1.

Theorem 4.1. Let

$$w(t,x) \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x);\ d(x) \in C^3(\mathbb{R}^n_x),\ \alpha(x) \in C^1(\mathbb{R}^n_x) \tag{4.1}$$

be three given functions [at this stage, w and d need not be the solution of Eqn. (1.1.1a), and the function provided by assumptions (A.1) and (A.2), respectively]. If $\tau > 0$ is a parameter, we introduce the functions

$$\ell(t,x) \equiv \tau \left[d(x) - c\left(t - \frac{T}{2}\right)^2 \right] \equiv \tau\phi(t,x); \tag{4.2}$$

$$\psi(x) \equiv \tau\alpha(x); \qquad \theta(t,x) = e^{\ell(t,x)} = e^{\tau\phi(t,x)}, \tag{4.3}$$

where $\phi(t,x)$ is defined consistently with (1.1.8a), with a constant $0 < c < 1$ selected as in (1.1.8d). Then, with the above choices, Lemma 3.1 specializes as follows: setting $h = \nabla d$:

$$\ell_{x_i} = \tau d_{x_i}; \quad |\nabla \ell|^2 = \tau^2 |\nabla d|^2 = \tau^2 |h|^2; \quad \ell_{x_i x_j} = \tau d_{x_i x_j}; \quad \Delta\ell = \tau\Delta d; \tag{4.4}$$

$$\ell_t = -2c\tau\left(t - \frac{T}{2}\right);\ \ell_{tt} = -2c\tau;\ \ell_{tx_j} \equiv 0;\ \psi_t = 0;\ |\nabla\psi| = \tau|\nabla\alpha|, \tag{4.5}$$

so that the pointwise estimate (3.2) becomes

$$\begin{aligned} \theta^2(w_{tt} - \Delta w)^2 - \frac{\partial M}{\partial t} + \operatorname{div} V \ \geq\ & 2\tau[\Delta d - 2c - \alpha]v_t^2 \\ & + 2\tau\left[\alpha - \frac{\epsilon}{2\tau} - \Delta d - 2c\right]|\nabla v|^2 \\ & + 4\tau\left[\sum_{i,j=1}^n d_{x_i x_j} v_{x_i} v_{x_j}\right] + \theta^2 \tilde{B} w^2, \end{aligned} \tag{4.6}$$

where M and $V = [V_1, \dots, V_n]$ are given by (3.3) and (3.4b) respectively, as functions of w, as specialized via (4.4); while for A and $\tilde{B}$, we now obtain from (3.5), (3.6), via (4.3)–(4.5):

$$A = \tau^2\left[4c^2\left(t - \frac{T}{2}\right)^2 - |\nabla d|^2\right] + \tau[2c + \Delta d - \alpha] \tag{4.7}$$

$$\tilde{B} = 2\tau^3 \left\{ [2c + \Delta d - \alpha]|\nabla d|^2 + 2\mathcal{H}_d \nabla d \cdot \nabla d - (6c + \Delta d - \alpha) 4c^2 \left(t - \frac{T}{2} \right)^2 \right\} + \mathcal{O}(\tau^2). \tag{4.8}$$

Notice that it is the coefficient $\{\quad\}$ of the principal part of $\tilde{B}$ that justifies our interest in property $(p_3) = (1.1.15\text{b})$.

Proof. The proof is a direct computation starting from Lemma 3.1 and using the choice of functions made in (4.2), (4.3). First, A in (3.5) becomes at once the expression (4.7) via (4.3)–(4.5). We then verify (4.8) for $\tilde{B}$. By recalling $\psi(x) = \tau\alpha(x)$ in (4.3), we obtain from (4.7), via (4.4)–(4.5):

$$2A\psi = 2\tau^3 \alpha \left[4c^2 \left(t - \frac{T}{2} \right)^2 - |\nabla d|^2 \right] + \mathcal{O}(\tau^2); \tag{4.9}$$

$$(A + \psi)\ell_{x_j} = \tau^3 \left[4c^2 \left(t - \frac{T}{2} \right)^2 - |\nabla d|^2 \right] d_{x_j} + \tau^2 [\Delta d + 2c] d_{x_j}; \tag{4.10}$$

$$\frac{\partial}{\partial x_j}[(A + \psi)\ell_{x_j}] = \tau^3 \left\{ \left[4c^2 \left(t - \frac{T}{2} \right)^2 - |\nabla d|^2 \right] d_{x_j x_j} - \left(\frac{\partial}{\partial x_j} |\nabla d|^2 \right) d_{x_j} \right\} + \mathcal{O}(\tau^2); \tag{4.11}$$

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[(A + \psi)\ell_{x_j}] = \tau^3 \left\{ \left[4c^2 \left(t - \frac{T}{2} \right)^2 - |\nabla d|^2 \right] \Delta d - \nabla(|\nabla d|^2) \cdot \nabla d \right\} + \mathcal{O}(\tau^2); \tag{4.12}$$

$$(A + \psi)\ell_t = \tau^2 \left[4c^2 \left(t - \frac{T}{2} \right)^2 - |\nabla d|^2 \right] \ell_t + \tau(\Delta d + 2c)\ell_t; \tag{4.13}$$

$$\frac{\partial}{\partial t}[(A + \psi)\ell_t] = 2c\tau^3 \left[|\nabla d|^2 - 12c^2 \left(t - \frac{T}{2} \right)^2 \right] + \mathcal{O}(\tau^2). \tag{4.14}$$

Finally, one either verifies (or recalls from say [L-T.2]) the following identity, where $h = \nabla d$, and $J_h = \mathcal{H}_d$ are defined in (1.1.5):

$$\nabla(|\nabla d|^2) \cdot \nabla d = \nabla(h \cdot \nabla d) \cdot \nabla d = J_h \nabla d \cdot \nabla d + \frac{1}{2} \nabla d \cdot \nabla(|\nabla d|^2), \tag{4.15}$$

hence

$$\nabla(|\nabla d|^2) \cdot \nabla d = 2\mathcal{H}_d \nabla d \cdot \nabla d = 2J_h h \cdot h. \tag{4.16}$$

Finally, first inserting (4.16) into (4.12), and next inserting the resulting (4.12) along with (4.9), (4.14), and (4.5) for ψ in the definition (3.6) of $\tilde{B}$, we readily obtain (4.8) for $\tilde{B}$, as desired. $\Box$

The pointwise estimate of interest in Corollary 4.2 below is then obtained for functions $d(x) \in C^3(\mathbb{R}^n_x)$, $\alpha(x) \in C^1(\mathbb{R}^n_x)$, such that the following three estimates hold true, for suitable positive constants $\rho > 0$, $\tilde{\beta} > 0$:

(H.1)

$$\Delta d - 2c - \alpha \geq \rho > 0; \ \forall \ x \in \bar{\Omega}; \tag{4.17}$$

(H.2)

$$\begin{bmatrix} 2d_{x_1x_1} + \gamma & 2d_{x_1x_2} & \cdots & 2d_{x_1x_n} \\ 2d_{x_2x_1} & 2d_{x_2x_2} + \gamma & \cdots & 2d_{x_2x_n} \\ \vdots & & & \vdots \\ 2d_{x_nx_1} & 2d_{x_nx_2} & \cdots & 2d_{x_nx_n} + \gamma \end{bmatrix} \geq \rho I, \ \forall x \in \bar{\Omega}, \tag{4.18}$$

where we have set $\gamma(x) = \alpha(x) - \Delta d(x) - 2c$;

(H.3)

$$[2c + \Delta d - \alpha]|\nabla d|^2 + 2\mathcal{H}_d \nabla d \cdot \nabla d - (6c + \Delta d - \alpha)4c^2 \left(t - \frac{T}{2}\right)^2 \geq \tilde{\beta} > 0$$

$$\forall \ (t, x) \in Q^*(\sigma^*), \tag{4.19}$$

where $Q^*(\sigma^*)$ is the subset of $[0, T] \times \Omega$ defined in (1.1.16). But, as we have seen in Section 1, these three inequalities hold true, in particular, in the case of our interest where assumptions (A.1) and (A.2) hold true. Then (A.1) provides a strictly convex (positive potential) function $d(x)$, and we then choose $\alpha(x) = \Delta d(x) - 2c - 1 + k$ as in (1.1.11), to obtain (when $d(x)$ is, possibly, suitably rescaled, see Section 1.2) properties (p.1), (p.2), (p.3) listed in (1.1.12), (1.1.14), (1.1.15b), which then verify inequalities (4.17)–(4.19), respectively. We thus obtain from (4.6):

Corollary 4.2. With $0 < c < 1$ chosen in (1.1.8d), let $d(x) \in C^3(\mathbb{R}^n_x)$, and $\alpha(x) \in C^1(\mathbb{R}^n_x)$ be two functions such that inequalities (H.1) = (4.17), (H.2) = (4.18), (H.3) = (4.19) hold true. This is the case, in particular, if $d(x)$ is a (suitably rescaled, see Section 1.2) strictly convex function provided by assumptions (A.1) and (A.2), and then $\alpha(x) = \Delta d(x) - 2c - 1 + k$, as in (1.1.11) with k subject to (1.1.14b). Let $w \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x)$. Then, with such choices in (4.2), (4.3) for $\ell(t, x)$ and $\psi(x)$, respectively, Theorem 4.1, Eqn. (4.6), specializes to

$$\theta^2(w_{tt} - \Delta w)^2 - \frac{\partial M}{\partial t} + \text{div } V \geq 2\tau\rho[v_t^2 + |\nabla v|^2] + \tilde{B}v^2, \ 0 \leq t \leq T, \ x \in \bar{\Omega}, \tag{4.20}$$

where, with the constant $\tilde{\beta} > 0$, we have via (4.8), (4.19):

$$\tilde{B}v^2 \geq [2\tau^3\tilde{\beta} + \mathcal{O}(\tau^2)]v^2, \quad \forall (t,x) \in Q^*(\sigma^*). \tag{4.21}$$

Moreover, the scalar function M and the vector function V are given by (3.3) and (3.4), respectively, as functions of w, as specialized via (4.4). In particular, for future use below, we note that (3.4) yields on the boundary $\Gamma = \partial\Omega$, with outward unit normal $\nu = [\nu_1, \ldots, \nu_n]$, the following identity where $\nabla \ell = \tau \nabla d = \tau h$:

$$\begin{aligned} \text{on } \Gamma: \ V \cdot \nu &= \sum_{j=1}^{n} V_j \nu_j \\ &= 2\theta^2 \Big\{ (w_t^2 - |\nabla w|^2)\nabla \ell \cdot \nu - 2\ell_t w_t \nabla w \cdot \nu + 2(\nabla \ell \cdot \nabla w)\nabla w \cdot \nu \\ &\quad + 2\left(|\nabla \ell|^2 - \ell_t^2 + \tau\frac{\alpha}{2}\right) w \nabla w \cdot \nu + (|\nabla \ell|^2 - \ell_t^2 - A) w^2 \nabla \ell \cdot \nu \Big\} \end{aligned} \tag{4.22}$$

Moreover, via (4.5) for ℓ_t and (4.7) for A, we have:

$$(|\nabla \ell|^2 - \ell_t^2 - A) = 2\tau^2 \left[|h|^2 - 4c^2 \left(t - \frac{T}{2}\right)^2 \right] + \tau(\alpha - \Delta d - 2c). \qquad \square \tag{4.23}$$

Notice that, through M in (3.3) and V in (3.4), the left-hand side of (4.20) is expressed in terms of w, while instead the right-hand side of (4.20) is still expressed in terms of $v = \theta w$, see (3.30). We no remedy this, and obtain a further corollary involving only w: it is then this corollary which, of course, will be used in the sequel.

Corollary 4.3. With $0 < c < 1$ chosen in (1.1.8d), let $d(x) \in C^3(\mathbb{R}_x^n)$, and $\alpha(x) \in C^1(\mathbb{R}_x^n)$ be two functions such that inequalities (H.1) - (4.17), (H.2) - (4.18), (H.3) = (4.19) hold true. This is the case, in particular, if $d(x)$ is any strictly convex function (suitably rescaled, see Section 1.2) provided by assumptions (A.1) and (A.2), and then $\alpha(x) \equiv \Delta d(x) - 2c - 1 + k$ as in (1.1.11), with k subject to (1.1.14b). Let $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x^n)$.

Then, with such choices in (4.2), (4.3) for $\ell(t,x)$ and $\psi(x)$, respectively, Corollary 4.2 becomes: for any $1 > \epsilon > 0$, we obtain

$$\theta^2(w_{tt} - \Delta w)^2 = \frac{\partial M}{\partial t} + \text{div } V \geq \epsilon\tau\rho\theta^2[w_t^2 + |\nabla w|^2] + \theta^2 B w^2, \qquad 0 \leq t \leq T, \quad x \in \bar{\Omega}, \tag{4.24}$$

where, recalling $\tilde{B}$ from (4.8), we have

$$\begin{cases} B \equiv \tilde{B} - 2\epsilon\rho\tau^3(\phi_t^2 + |\nabla\phi|^2) \geq \tilde{B} - 2\epsilon\rho\tau^3 r; & (4.25\text{a}) \\ r = \max_{\bar{Q}}(\phi_t^2 + |\nabla\phi|^2). & (4.25\text{b}) \end{cases}$$

Thus, for $\epsilon > 0$ suitably small, the cosntant $\beta \equiv \beta_\epsilon \equiv (\tilde{\beta} - \epsilon\rho r)$ is positive, via (4.19), and recalling (4.8), (4.19), (4.21), we obtain from (4.25):

$$Bw^2 \geq [2\tau^3\beta + \mathcal{O}(\tau^2)]w^2, \quad \forall\ (t,x) \in Q^*(\sigma^*); \tag{4.26}$$

$$\beta = \beta_\epsilon = (\tilde{\beta} - \epsilon\rho r) > 0;\ B = \mathcal{O}(\tau^3)\ \text{ in } Q = (0,T] \times \Omega. \tag{4.27}$$

Proof. With $\ell_t = \tau\phi_t$ and $\ell_{x_j} = \tau\phi_{x_j} = \tau d_{x_j}$ from (4.2), we specialize v_t and v_x; in (3.30) and obtain from there that $\theta w_t = v_t - \theta\tau\phi_t w$; $\theta w_{x_j} = v_{x_j} - \theta\tau d_{x_j} w$, hence

$$2v_t^2 \geq \theta^2 w_t^2 - 2\tau^2\phi_t^2 v^2;\ 2|\nabla v|^2 \geq \theta^2|\nabla w|^2 - 2\tau^2|\nabla d|^2 v^2. \tag{4.28}$$

Returning to the right-hand side of (4.20), we then obtain via (4.28), for any $1 > \epsilon > 0$:

$$2\tau\rho[v_t^2 + |\nabla v|^2] + \tilde{B}v^2 \ \geq\ \epsilon 2\tau\rho[v_t^2 + |\nabla v|^2] + \tilde{B}v^2 \tag{4.29}$$

$$(\text{by (4.28)}) \quad \geq\ \epsilon\tau\rho\theta^2[w_t^2 + |\nabla w|^2] + \tilde{B}v^2 - 2\epsilon\tau^3\rho[\phi_t^2 + |\nabla d|^2]v^2, \tag{4.30}$$

and (4.30) yields (4.24) as desired, via (4.25), as well as (4.26) and (4.27), as described below (4.25). $\square$

5 Carleman estimates for smooth solutions of Eqn. (1.1a). First version

The next key result yields a Carleman-type estimate.

Theorem 5.1. With $0 < c < 1$ chosen in (1.1.8d), let $d(x) \in C^3(\bar{\Omega})$, $\alpha(x) \in C^1(\bar{\Omega})$ be two functions such that inequalities (H.1) = (4.17), (H.2) = (4.18), (H.3) = (4.19) hold true. This is the case, in particular, if $d(x)$ is a (suitably rescaled, see Section 1.2) strictly convex function provided by assumptions(A.1) and (A.2), and then $\alpha(x) = \Delta d(x) - 2c - 1 + k$, as in (1.1.11), with k subject to (1.1.14b). Let $\phi(x,t)$ be the pseudo-convex function defined by (1.1.8). Let $w \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x)$ be a solution of Eqn. (1.1.1a) [and no B.C.], under the standing assumptions (1.1.2) for $F(w)$ and (1.1.3) for f. Then, the following one parameter family of estimates hold true, with $\rho > 0$, $\beta > 0$, as in (4.17)–(4.19); or (1.1.14a), (1.1.15b):

(i) for all $\tau > 0$ sufficiently large, and any $0 < \epsilon$ small:

$$\begin{aligned} BT|_\Sigma + 2\int_0^T\int_\Omega e^{2\tau\phi}f^2 dQ + C_{1,T}e^{2\tau\sigma}\int_0^T\int_\Omega w^2 dQ \\ \geq\ (\tau\epsilon\rho - 2C_T)\int_0^T\int_\Omega e^{2\tau\phi}[w_t^2 + |\nabla w|^2]dQ \\ + [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T]\int_{Q(\sigma)} e^{2\tau\phi}w^2 dx\,dt - c_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)], \end{aligned} \tag{5.1}$$

where $Q(\sigma)$ is the subset of $[0,T]\times\Omega \equiv Q$ defined by (1.1.19), where we recall from (4.26) that β depends on ϵ:

(ii) for all $\tau > 0$ sufficiently large and any $\epsilon > 0$ small,

$$BT|_\Sigma + 2\int_0^T \int_\Omega e^{2\tau\phi} f^2 dQ \geq (\tau\epsilon\rho - 2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_\Omega [w_t^2 + |\nabla w|^2] d\Omega\, dt$$

$$-C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt - c_T\tau^3 e^{-2\tau\delta}[E(0)+E(T)]. \tag{5.2}$$

Here $\delta > 0$, $\sigma > 0$, and $\sigma > -\delta$, are the constants in (1.1.9), (1.1.10), while $C_T > 0$ is a positive constant depending on T, as well as d. Moreover, the boundary terms $BT|_\Sigma$, $\Sigma = [0,T]\times\Gamma$, are defined by

$$BT|_\Sigma \equiv \int_0^T \int_\Omega \text{div } V\, d\Omega\, dt = \int_0^T \int_\Gamma V\cdot\nu\, d\Gamma\, dt, \tag{5.3}$$

via the divergence theorem, and are explicitly given via (4.22), (4.23), $h \equiv \nabla d$ and $\nabla \ell = \tau\nabla d = \tau h$, ℓ_t in (4.5) and A in (4.7), and $\theta = \exp(\tau\phi)$ in (4.3), by:

$$\begin{aligned}
BT|_\Sigma &= 2\tau\int_0^T\int_{\Gamma_1} e^{2\tau\phi}(w_t^2 - |\nabla w|^2)h\cdot\nu\, d\Gamma\, dt \\
&+ 8c\tau\int_0^T\int_\Gamma e^{2\tau\phi}\left(t-\frac{T}{2}\right)w_t\,\frac{\partial w}{\partial\nu}\, d\Gamma\, dt \\
&+ 4\tau\int_0^T\int_\Gamma e^{2\tau\phi}(h\cdot\nabla w)\frac{\partial w}{\partial \nu}\, d\Gamma\, dt \\
&+ 4\tau^2\int_0^T\int_\Gamma e^{2\tau\phi}\left[|h|^2 - 4c^2\left(t-\frac{T}{2}\right)^2 + \frac{\alpha}{2\tau}\right] w\,\frac{\partial w}{\partial\nu}\, d\Gamma\, dt \\
&+ 2\tau\int_0^T\int_{\Gamma_1} e^{2\tau\phi}\left[2\tau^2\left(|h|^2 - 4c^2\left(t-\frac{T}{2}\right)^2\right) + \tau(\alpha - \Delta d - 2c)\right] w^2 h\cdot\nu\, d\Gamma\, dt,
\end{aligned} \tag{5.4}$$

since $h\cdot\nu = 0$ on Γ_0 by assumption (1.1.4). Moreover, as in (2.1.1), we have set

$$E(t) \equiv \int_\Omega [w_t^2(t,x) + |\nabla w(t,x)|^2 + w^2(t,x)]d\Omega. \tag{5.5}$$

(iii) The above inequality (5.2) may be then extended to all $w \in H^{2,2}(Q) = L_2(0,T;H^2(\Omega)) \cap H^2(0,T;L_2(\Omega))$. $\square$

Proof. (i) **Step 1.** With $w \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x)$, we return to inequality (4.24) of Corollary 4.3, supplemented by estimate (4.26) for Bw^2, and identity (4.22) for $V \cdot \nu$. Invoking the divergence theorem, we then obtain, with $\epsilon > 0$ fixed and small as in (4.27),

$$\int_0^T \int_\Omega \theta^2 (w_{tt} - \Delta w)^2 d\Omega\, dt - \left[\int_\Omega M\, d\Omega\right]_0^T + \int_0^T \int_\Gamma V \cdot \nu\, d\Gamma\, dt$$

$$\geq \epsilon\tau\rho \int_0^T \int_\Omega \theta^2 [w_t^2 + |\nabla w|^2] d\Omega\, dt + \int_0^T \int_\Omega \theta^2 B w^2 d\Omega\, dt. \tag{5.6}$$

Next, in view of estimate (4.26) for Bw^2 which holds true only on the subset $Q^*(\sigma^*)$ of Q defined by (1.1.16), split $Q \equiv [0,T] \times \Omega = Q^*(\sigma^*) \cup [Q^*(\sigma^*)]^c$ where $[\quad]^c$ denotes the complement in Q. See Figure . Thus, we then obtain by (4.26):

$$\int_0^T \int_\Omega \theta^2 B w^2 d\Omega\, dt = \int_{Q^*(\sigma^*)} \theta^2 B w^2 dx\, dt + \int_{[Q^*(\sigma^*)]^c} \theta^2 B w^2 dx\, dt$$

$$\geq [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q^*(\sigma^*)} \theta^2 w^2 dx\, dt + \int_{[Q^*(\sigma^*)]^c} \theta^2 B w^2 dx\, dt. \tag{5.7}$$

Since the right side of (1.1a) is subject to estimate (1.2), we then obtain

$$\int_0^T \int_\Omega \theta^2 (w_{tt} - \Delta w)^2 d\Omega\, dt \leq 2C_T \left[\int_0^T \int_\Omega \theta^2 [w_t^2 + |\nabla w|^2 + w^2] d\Omega\, dt\right] + 2\int_0^T \int_\Omega \theta^2 f^2 d\Omega\, dt. \tag{5.8}$$

As to the term $[\quad]_0^T$ at the time endpoints, if we recall M from (3.3), as well as (4.2)–(4.5), and A from (4.7), we then obtain (τ^3 comes from ℓ_t^3):

$$\left|\left[\int_\Omega M\, d\Omega\right]_0^T\right| \leq c_T \tau^3 \left[\int_\Omega e^{2\tau\phi} [w_t^2 + |\nabla w|^2 + w^2] d\Omega\right]_0^T$$

$$\text{(by (1.1.9))} \qquad \leq c_T \tau^3 e^{-2\tau\delta} \left[\int_\Omega [w_t^2 + |\nabla w|^2 + w^2] d\Omega\right]_0^T$$

$$\text{(by (5.5))} \qquad \leq c_T \tau^3 e^{-2\tau\delta} [E(0) + E(T)], \tag{5.9}$$

where in the last two steps we have recalled the critical property (1.1.9) for ϕ at $t = 0$ and $t = T$, as well as the definition of $E(t)$ in (5.5).

Next, via (4.22), (4.23), supplemented by (4.3)–(4.5), we obtain that the boundary terms $BT|_\Sigma$, as defined by (5.3), are explicitly given by (5.4).

Finally, we use (5.7), as well as (5.9), and (5.4) on the left side of inequality (5.6), and readily obtain for τ sufficiently large:

$$BT|_\Sigma + 2\int_0^T \int_\Omega e^{2\tau\phi} f^2 dQ - \int_{[Q^*(\sigma^*)]^c} e^{2\tau\phi} B w^2 dx\, dt$$

$$\geq \quad (\epsilon\tau\rho - 2C_T)\int_0^T\int_\Omega e^{2\tau\phi}[w_t^2 + |\nabla w|^2]dQ$$

$$+ [2\tau^3\beta + \mathcal{O}(\tau^2)]\int_{Q^*(\sigma^*)} e^{2\tau\phi}w^2 dx\,dt$$

$$- 2C_T\int_0^T\int_\Omega e^{2\tau\phi}w^2 dQ - c_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)]. \tag{5.10}$$

Step 2. By (4.27), we have $B = \mathcal{O}(\tau^3)$ on $[0,T]\times\Omega$. Moreover, we have that $\phi \leq \phi^* \leq \sigma^*$ on $[Q^*(\sigma^*)]^c$ by the very definition (1.1.16) and (1.1.18). Hence

$$-\int_{[Q^*(\sigma^*)]^c} e^{2\tau\phi}Bw^2 dx\,dt = \mathcal{O}(\tau^3)e^{2\tau\sigma^*}\int_{[Q^*(\sigma^*)]^c} w^2 dx\,dt. \tag{5.11}$$

Step 3. Recalling the subset $Q(\sigma)$ of $[0,T]\times\Omega \equiv Q$ in (1.1.19), we split $Q \equiv [0,T]\times\Omega = Q(\sigma)\cup[Q(\sigma)]^c$, where $[\quad]^c$ denotes complement in Q. Accordingly,

$$\int_0^T\int_\Omega e^{2\tau\phi}w^2 dQ = \int_{Q(\sigma)} e^{2\tau\phi}w^2 dx\,dt + \int_{[Q(\sigma)]^c} e^{2\tau\phi}w^2 dx\,dt. \tag{5.12}$$

Moreover, since $Q^*(\sigma^*) \supset Q(\sigma)$, see (1.1.20), we have via (5.12) for two right terms of (5.10):

$$[2\tau^3\beta + \mathcal{O}(\tau^2)]\int_{Q^*(\sigma^*)} e^{2\tau\phi}w^2 dx\,dt - 2C_T\int_0^T\int_\Omega e^{2\tau\phi}w^2 dQ$$

$$\geq \quad [2\tau^3\beta + \mathcal{O}(\tau^2)]\int_{Q(\sigma)} e^{2\tau\phi}w^2 dx\,dt - 2C_T\int_{Q(\sigma)} e^{2\tau\phi}w^2 dx\,dt - 2C_T\int_{[Q(\sigma)]^c} e^{2\tau\phi}w^2 dx\,dt. \tag{5.13}$$

Finally, in the last integral term in (5.13), we use that $\phi \leq \sigma$ in $[Q(\sigma)]^c$, by the very definition (1.1.19), so that

$$-2C_T\int_{[Q(\sigma)]^c} e^{2\tau\phi}w^2 dx\,dt \geq -2C_T e^{2\tau\sigma}\int_{[Q(\sigma)]^c} w^2 dx\,dt. \tag{5.14}$$

Using (5.14) in (5.13), we then conclude that

$$[2\tau^3\beta + \mathcal{O}(\tau^2)]\int_{Q^*(\sigma^*)} e^{2\tau\phi}w^2 dx\,dt - 2C_T\int_0^T\int_\Omega e^{2\tau\phi}w^2 dQ$$

$$\geq \quad [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T]\int_{Q(\sigma)} e^{2\tau\phi}w^2 dx\,dt - 2C_T e^{2\tau\sigma}\int_{[Q(\sigma)]^c} w^2 dx\,dt. \tag{5.15}$$

Step 4. Using (5.11) and (5.15) in (5.10) yields

$$
\begin{aligned}
&BT|_\Sigma + 2\int_0^T\int_\Omega e^{2\tau\phi} f^2 dQ + \mathcal{O}(\tau^3)e^{2\tau\sigma^*}\int_{[Q^*(\sigma^*)]^c} w^2 dx\,dt + 2C_T e^{2\tau\sigma}\int_{[Q(\sigma)]^c} w^2 dx\,dt \\
&\geq \ (\epsilon\tau\rho - 2C_T)\int_0^T\int_\Omega e^{2\tau\phi}[w_t^2 + |\nabla w|^2]dQ \\
&\quad + [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T]\int_{Q(\sigma)} e^{2\tau\phi} w^2 dx\,dt - c_T\tau^3 e^{-2\tau\delta}[E(0)+E(T)]. \quad (5.16)
\end{aligned}
$$

Finally, we use that, by construction, both $[Q^*(\sigma^*)]^c$ and $[Q(\sigma)]^c$ are subsets of $[0,T]\times\Omega$; and that, moreover, by the selection process in (1.1.16) through (1.1.20), we have chosen $0<\sigma^*<\sigma$. Hence, these two facts yield

$$
\begin{aligned}
&\mathcal{O}(\tau^3)e^{2\tau\sigma^*}\int_{[Q^*(\sigma^*)]^c} w^2 dx\,dt + 2C_T e^{2\tau\sigma}\int_{[Q(\sigma)]^c} w^2 dx\,dt \\
&\leq \ [\mathcal{O}(\tau^3)e^{2\tau\sigma^*} + 2C_T e^{2\tau\sigma}]\int_0^T\int_\Omega w^2 dQ \quad (5.17) \\
&\leq \ C_{1,T} e^{2\tau\sigma}\int_0^T\int_\Omega w^2 dQ, \quad (5.18)
\end{aligned}
$$

for all τ sufficiently large. Inserting (5.18) into (5.16) yields (5.1), as desired.

(ii) We take τ sufficiently large so that, since $\beta>0$ by assumption, see (4.27), we then have that the term $[2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T]$ is positive, and we then drop the corresponding lower order *interior* term involving w^2 in (5.1). Moreover, we invoke the critical property (1.1.10) for ϕ on the first integral term on the right side of (5.1). Finally we majorize $\int_\Omega w^2(t)d\Omega$ by $E(t)$, see (5.5). This way, (5.1) readily yields (5.2). □

Remark 5.1. The statement and the proof of Theorem 5.1 use—as a consequence of assumptions (A.1) and (A.2)—that the principal part of the coefficient B acting on w^2 is *positive only on the set* $Q^*(\sigma^*)$; see (4.21), (4.27), and, ultimately, estimate (1.1.15b). If, instead, through possibly further rescaling of $d(x)$ [and consequent deterioration of the minimal time T_0, see Section 1.2], we assume, as in (1.2.1), that the principal part of the coefficient of B_1, hence of B, is positive *on the entire cylinder* $[0,T]\times\Omega$, then the proof and statement of Theorem 5.1 simplify. In particular: the term $e^{2\tau\sigma}\int_0^T\int_\Omega w^2 dQ$ is omitted in (5.1); thus the term $-e^{2\tau\sigma}\int_0^T E(t)dt$ is omitted in (5.2). The subsequent proof in Section 6 simplifies accordingly. □

6 Carleman estimate for smooth solutions of Eqn. (1.1a). Second version

A preliminary equivalence. Let $u \in H^1(\Omega)$. Then the following inequality holds true: there exist positive constants $0 < k_1 < k_2 < \infty$, independent of u, such that

$$k_1 \int_\Omega [u^2 + |\nabla u|^2] d\Omega \leq \int_\Omega |\nabla u|^2 d\Omega + \int_{\tilde{\Gamma}_1} u^2 d\Gamma \leq k_2 \int_\Omega [u^2 + |\nabla u|^2] d\Omega, \tag{6.1}$$

where $\tilde{\Gamma}_1$ is any (fixed) portion of the boundary Γ with positive measure. Inequality (6.1) is obtained by combining the following two inequalities:

$$\int_\Omega u^2 d\Omega \leq c_1 \left[\int_\Omega |\nabla u|^2 d\Omega + \int_{\tilde{\Gamma}_1} u^2 d\Gamma \right]; \quad \int_{\tilde{\Gamma}_1} u^2 d\Gamma \leq c_2 \int_\Omega [u^2 + |\nabla u|^2] d\Omega. \tag{6.2}$$

The inequality on the left of (6.2) [ ] replaces Poincaré's inequality, while the inequality on the right of (6.2) stems from (a conservative version of) trace theory. Thus, for $w \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x)$, in fact, in $H^{2,2}(Q)$, if we introduce

$$\mathcal{E}(t) \equiv \int_\Omega [|\nabla w(t)|^2 + w_t^2(t)] d\Omega + \int_{\Gamma_1} w^2(t) d\Gamma_1, \tag{6.3}$$

with $\Gamma_1 = \Gamma \setminus \Gamma_0$, where Γ_0 is defined by (1.4), and recall $E(t)$ from (2.1.1) = (5.5), then (6.1) yields the equivalence

$$a\, E(t) \leq \mathcal{E}(t) \leq b\, E(t), \tag{6.4}$$

for some positive constant $a > 0$, $b > 0$.

We can now state the main result of the present section.

Theorem 6.1. With $0 < c < 1$ chosen in (1.1.8d), let $d(x) \in C^3(\bar{\Omega})$, $\alpha(x) \in C^1(\bar{\Omega})$ be two function such that inequalities (H.1) = (4.17), (H.2) = (4.18), (H.3) = (4.19) hold true. This is the case, in particular, if $d(x)$ is a (possibly, suitably rescaled, see Section 1.2) strictly convex function provided by assumptions (A.1) and (A.2), and then $\alpha(x) = \Delta d(x) - 2c - 1 + k$ as in (1.1.11), with k subject to (1.1.14b). Let $\phi(t,x)$ be the pseudo-convex function defined by (1.7) and define $\theta(t,x) = \exp(\tau\phi(t,x))$ as in (4.3). Finally, let $w \in H^{2,2}(Q)$ be a solution of Eqn. (1.1.1a) [and no B.C.], subject to the standing assumptions (1.1.2) on $F(w)$ and (1.1.3) on f. Then, the following one-parameter family of estimates hold true, for all τ sufficiently large, and any $\epsilon > 0$ small as in Corollary 4.3:

$$\overline{BT}|_\Sigma + 2\int_0^T \int_\Omega e^{2\tau\phi} f^2 dQ + \text{const}_\phi \int_0^T \int_\Omega f^2 dQ$$

$$\geq \left\{ \left[\frac{a}{2}(\epsilon\tau\rho - 2C_T)(t_1 - t_0) e^{-C_T T} - \frac{C_{1,T} b}{2a}\, T e^{C_T T} \right] e^{2\tau\sigma} - c_T \tau^3 e^{-2\tau\delta} \right\} [E(0) + E(T)] \tag{6.5}$$

$$\geq k_\phi [E(0) + E(T)], \text{ for a constant } k_\phi > 0, \tag{6.6}$$

since $\sigma > -\delta$, see (1.1.10).

(a) Here, with $h = \nabla d = \nabla\phi$, the boundary terms $\overline{BT}|_\Sigma$ are given in terms of the boundary terms $BT|_\Sigma$ in (5.4) by

$$\overline{BT}|_\Sigma = BT|_\Sigma + \text{const}_\phi\Big[\int_0^T\int_\Gamma \left|\frac{\partial w}{\partial \nu}\, w_t\right| d\Sigma + \int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt + \int_0^T\int_{\Gamma_1} |ww_t| d\Sigma_1\Big] \tag{6.7}$$

$$\begin{aligned}
(\text{by (5.4)}) \;=\;& 2\tau\int_0^T\int_\Gamma e^{2\tau\phi}(w_t^2 - |\nabla w|^2)h\cdot\nu\, d\Gamma\, dt \\
&+ 8c\tau\int_0^T\int_\Gamma e^{2\tau\phi}\left(t-\frac{T}{2}\right) w_t\,\frac{\partial w}{\partial\nu}\, d\Gamma\, dt \\
&+ 4\tau\int_0^T\int_\Gamma e^{2\tau\phi}\,(h\cdot\nabla w)\,\frac{\partial w}{\partial\nu}\, d\Gamma\, dt \\
&+ 4\tau^2\int_0^T\int_\Gamma e^{2\tau\phi}\left[|h|^2 - 4c^2\left(t-\frac{T}{2}\right)^2 + \frac{\alpha}{2\tau}\right] w\,\frac{\partial w}{\partial\nu}\, d\Gamma\, dt \\
&+ 2\tau\int_0^T\int_\Gamma e^{2\tau\phi}\Bigg[2\tau^2\left(|h|^2 - 4c^2\left(t-\frac{T}{2}\right)^2\right) \\
&\qquad\qquad +\tau(\alpha-\Delta d-2c)\Bigg] w^2 h\cdot\nu\, d\Sigma \\
&+ \text{const}_\phi\left\{\int_0^T\int_\Gamma\left|\frac{\partial w}{\partial\nu}\, w_t\right| d\Sigma + \int_0^T\int_{\Gamma_1}|ww_t|d\Sigma_1 + \int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt\right\}.
\end{aligned} \tag{6.8a}$$

(b) Moreover, if in addition, w satisfies the pure Neumann B.C. $\frac{\partial w}{\partial\nu}|_\Sigma \equiv 0$ in (1.1.1c), so that $\nabla w = \nabla_{\tan} w$ (tangential gradient), then (6.8a) specializes (with $\alpha - \Delta d - 2c \equiv -4c - 1 + k$ by (1.1.13)) to:

$$\begin{aligned}
\overline{BT}|_\Sigma \;=\;& 2\tau\int_0^T\int_\Gamma e^{2\tau\phi}(w_t^2 - |\nabla_{\tan} w|^2)h\cdot\nu\, d\Sigma \\
&+ 2\tau\int_0^T\int_\Gamma e^{2\tau\phi}\left[2\tau^2\left(|h|^2 - 4c^2\left(t-\frac{T}{2}\right)^2\right) - (4c+1-k)\tau\right] w^2 h\cdot\nu\, d\Sigma \\
&+ \text{const}_\phi\left[\int_0^T\int_{\Gamma_1}|ww_t|d\Sigma_1 + \int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt\right].
\end{aligned} \tag{6.8b}$$

Proof. Step 1. We return to estimate (5.2) of Theorem 5.1(ii), add the term $(\tau\epsilon\rho -$

$2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt$ to both sides, recall (6.3) for $\mathcal{E}(t)$ and obtain

$$\begin{aligned} BT|_\Sigma + (\tau\epsilon\rho - 2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt + 2\int_0^T\int_\Omega e^{2\tau\phi} f^2 dQ \\ \geq \ (\tau\epsilon\rho - 2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\mathcal{E}(t)dt - C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt \\ - c_T\tau^3 e^{-2\tau\delta}[E(0)+E(T)]. \end{aligned} \tag{6.9}$$

Step 2. In a standard way, multiplying Eqn. (1.1a) by w_t and integrating over Ω yields, after an application of the first Green's identity

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega [w_t^2 + |\nabla w|^2] d\Omega + \int_{\Gamma_1} w^2 d\Gamma_1\right) \\ = \int_\Gamma \frac{\partial w}{\partial \nu} w_t d\Gamma + \int_{\Gamma_1} w w_t d\Gamma_1 + \int_\Omega [F(w)+f] w_t d\Omega. \end{aligned} \tag{6.10}$$

Notice that on both sides of (6.10) we have added the term $\frac{1}{2}\frac{\partial}{\partial t}\int_{\Gamma_1} w^2 d\Gamma_1 = \int_{\Gamma_1} w w_t d\Gamma_1$. Recalling $\mathcal{E}(t)$ in (6.3), we integrate (6.10) over (s,t) and obtain

$$\mathcal{E}(t) = \mathcal{E}(s) + 2\int_s^t \left[\int_\Gamma \frac{\partial w}{\partial \nu} w_t d\Gamma + \int_{\Gamma_1} w w_t d\Gamma_1\right] dr + 2\int_s^t [F(w)+f] w_t d\Omega\, dr. \tag{6.11}$$

We apply Schwarz inequality on $[F(w)+f]w_t$, recall estimate (1.2) for $F(w)$, invoke the left side $E(t) \leq \frac{1}{a}\mathcal{E}(t)$ of equivalence (6.4), and obtain

$$\mathcal{E}(t) \ \leq \ [\mathcal{E}(s) + N(T)] + C_T\int_s^t \mathcal{E}(r)dr; \tag{6.12}$$

$$\mathcal{E}(s) \ \leq \ [\mathcal{E}(t) + N(T)] + C_T\int_s^t \mathcal{E}(r)dr, \tag{6.13}$$

(C_T includes the constant $\frac{1}{a}$ of equivalence), where we have set

$$N(T) = \int_0^T\int_\Omega f^2 dQ + 2\int_0^T\int_\Gamma \left|\frac{\partial w}{\partial \nu} w_t\right| d\Sigma + 2\int_0^T\int_{\Gamma_1} |w w_t| d\Sigma_1. \tag{6.14}$$

Gronwall's inequality applied on (6.12), (6.13) then yields for $0 \leq s \leq t \leq T$,

$$\mathcal{E}(t) \leq [\mathcal{E}(s) + N(T)]e^{C_T(t-s)}; \ \ \mathcal{E}(s) \leq [\mathcal{E}(t) + N(T)]e^{C_T(t-s)}. \tag{6.15}$$

Set $t = T$ and $s = t$ in the first (left) inequality of (6.15); and set $s = 0$ in the second (right) inequality of (6.15), to obtain

$$\mathcal{E}(T) \leq [\mathcal{E}(t) + N(T)]e^{C_T T}; \quad \mathcal{E}(0) \leq [\mathcal{E}(t) + N(T)]e^{C_T T}. \tag{6.16}$$

Summing up these two inequalites in (6.16) yields for $0 \leq t \leq T$,

$$\mathcal{E}(t) \geq \frac{\mathcal{E}(T) + \mathcal{E}(0)}{2} e^{-C_T T} - N(T) \tag{6.17}$$

$$\geq \frac{a}{2}[E(T) + E(0)]e^{-C_T T} - N(T), \tag{6.18}$$

after recalling the left side of the equivalence in (6.4). Similarly, summing up the left inequality of (6.15) for $s = 0$ and the right inequality of (6.15) for $s = t$ and $t = T$, and using the equivalence (6.4) yields for $0 \leq t \leq T$,

$$E(t) \leq \frac{1}{a}\left[\frac{b(E(0) + E(T))}{2} + N(T)\right] e^{C_T T}, \tag{6.19}$$

and hence, by (6.19),

$$-C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt \geq -\frac{C_{1,T}b}{2a} Te^{C_T T}e^{2\tau\sigma}[E(0) + E(T)]$$

$$-\frac{C_{1,T}T}{a} e^{C_T T}e^{2\tau\sigma}N(T). \tag{6.20}$$

Step 3. We insert (6.18) into the first integral on the right side of (6.9) and use (6.20) and readily obtain (6.5), (6.7), by invoking (6.14) for $N(T)$.

Finally, we recall the critical relation $\sigma > 0$, $\delta > 0$, $\sigma > -\delta$ from (1.1.9), (1.1.10), so that $[\epsilon\tau e^{2\tau\sigma} - \tau^3 e^{-2\tau\delta}]$ is positive for all τ large enough. Thus, (6.5) yields (6.6). □

7 A global uniqueness theorem with pure homogeneous Neumann B.C. on Σ

We consider the following over-determined problem with Γ_0 in (1.1.4) and $\Gamma_1 = \Gamma \setminus \Gamma_0$:

$$\begin{cases} w_{tt} - \Delta w = F(w) & \text{in } (0,T] \times \Omega = Q; & \text{(7.1a)} \\ \left.\frac{\partial w}{\partial \nu}\right|_{\Sigma} \equiv 0 & \text{in } (0,T] \times \Gamma = \Sigma; & \text{(7.1b)} \\ w|_{\Sigma_1} \equiv 0 & \text{in } (0,T] \times \Gamma_1 = \Sigma_1. & \text{(7.1c)} \end{cases}$$

As a corollary of Theorem 6.1, we then obtain the following global uniqueness theorem.

Theorem 7.1. Assume hypotheses (A.1) and (A.2): thus there exists a strictly convex (possibly suitably rescaled, see Section 1.2) function $d(x)$, which along with the choice $\alpha(x) = \Delta d(x) - 2c - 1 + k$ in (1.1.11) [where $0 < c < 1$ chosen in (1.1.8d) and k subject to (1.1.14b)], satisfies properties $(p_1) = (1.1.12)$, $(p_2) = (1.1.14)$, $(p_3) = (1.1.15\text{b})$, so that inequalities (H.1) = (4.17), (H.2) = (4.18), (H.3) = (4.19) hold true. Moreover, with $h = \nabla d$ as usual, we have that $h \cdot \nu = 0$ on Γ_0. Let $\Gamma_1 = \Gamma \setminus \Gamma_0$ as usual, and let $T > T_0$ in problem (7.1), with T_0 the constant in the definition (1.1.8b) of the pseudo-convex function $\phi(x,t)$ in (1.1.8a). Let $w \in H^{2,2}(Q)$ be a solution of problem (7.1a-b-c). Then, in fact, $w \equiv 0$ in Q; indeed, in $\mathbb{R}_t \times \Omega$.

Proof. Theorem 6.1 applies under the present assumptions (A.1) and (A.2). Thus estimate (6.6) holds true, where, because of the B.C. (7.1b), $\overline{BT}|_\Sigma$ is given by (6.8b). Moreover, we presently have two additional pieces of information: (i) $h \cdot \nu = 0$ on Γ_0 by assumption (A.1), and (ii) $w|_{\Sigma_1} \equiv 0$ by (7.1c); hence $w_t|_{\Sigma_1} \equiv |\nabla_{\tan} w|_{\Sigma_1} \equiv 0$, $\Sigma_1 = (0,T] \times \Gamma_1$, $\Gamma_0 \cup \Gamma_1 = \Gamma$. Thus, returning to (6.8b), we see that we now obtain

$$\overline{BT}|_\Sigma = 0, \text{ hence, by (6.6), } \Rightarrow E(0) = 0, \text{ or } w_0 = w_1 = 0,$$

since $f = 0$ in (7.1). Then, as problem (7.1) is well-posed forward (and backward) in time as a s.c. group (F being a bounded operator: $\{w_1, w_2\} \in H^1(\Omega) \times L_2(\Omega) \to L_2(\Omega)$), we then obtain $w \equiv 0$ in Q; in fact, in $\mathbb{R}_t \times \Omega$. □

8 Extension of estimates to finite energy solutions

So far our estimates have been stated and proved only for $C^2(\mathbb{R}_t \times \mathbb{R}^n_x)$-solutions, hence $H^{2,2}(Q)$-solutions (Theorem 5.1(ii)), of Eqn. (1.1.1a), with $f \in L_2(Q)$ as in (1.1.3). In this section, we extend all our previous estimates to finite energy solutions of Eqn. (1.1.1a) in the following class

$$\begin{cases} w \in H^{1,1}(Q) = L_2(0,T; H^1(\Omega)) \cap H^1(0,T; L_2(\Omega)); \\ w_t, \quad \dfrac{\partial w}{\partial \nu} \in L_2(\Sigma) = L_2(0,T; L_2(\Gamma)). \end{cases} \tag{8.1}$$

In order to achieve this goal, it suffices to extend the validity of estimate (5.1) of Theorem 5.1(i) from $H^{2,2}(Q)$-solutions to finite energy solutions defined by the class in (8.1). Here, the main difficulty is the fact that finite energy solutions subject to Neumann B.C. do not produce (in dimension ≥ 2) H^1-traces on the boundary [L-T.5]. To overcome this difficulty, we shall invoke a regularizing procedure inspired from [La-Ta.1].

To this end, we shall make use of the following result.

Lemma 8.1. Let w be a solution of Eqn. (1.1a) in the class (8.1), with $f \in L_2(Q)$ as in (1.1.3). Then, in fact,

(i)

$$\{w, w_t\} \in C([0,T]; H^1(\Omega) \times L_2(\Omega)); \tag{8.2}$$

(ii) there is a constant $C_T > 0$, such that

$$\int_0^T \int_\Gamma |\nabla_{\tan} w|^2 d\Gamma\, dt \le C_T \left\{ \int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu} \right)^2 + w_t^2 \right] d\Sigma + \mathbb{E}(0) \right\}, \tag{8.3}$$

where

$$\mathbb{E}(0) = \| \{ w(0,\ \cdot\ ), w_t(0,\ \cdot\ ) \} \|^2_{H^1(\Omega) \times L_2(\Omega)}. \tag{8.4}$$

Proof. (i) To Eqn. (1.1.1a) with $f \equiv 0$, we associate the B.C.

$$\frac{\partial w}{\partial \nu} + w_t \equiv g \in L_2(\Sigma), \tag{8.5}$$

derived from (8.1b). Then, as is well-known [Las.1], [L-T.6, Chapter 7], for problem (1.1.1a) with $f \equiv 0$ and (8.5), the following regularity property holds true:

$$\text{the map} : f \equiv 0,\ \ g \to \{w, w_t\} \text{ is continuous} :$$
$$L_2(\Sigma) \to C([0,T]; H^1(\Omega) \times L_2(\Omega)). \tag{8.6}$$

(This is so, because the presence of the boundary damping term w_t in the B.C. (8.5), which increases, when $\dim \Omega \ge 2$, the interior regularity of $\{w, w_t\}$ to $H^1(\Omega) \times L_2(\Omega)$, over the case when such w_t is absent. [By (1.1.2), F facts as a bounded linear operator $\{w_1, w_2\} \in H^1(\Omega) \times L_2(\Omega) \to L_2(\Omega)$]. On the other hand, for Eqn. (1.1.1a) with $g = 0$ in (8.5), we have:

$$\text{the map} : g = 0,\ \ f \in L_2(Q)$$
$$\to \{w, w_t\} \in C([0,T]; H^1(\Omega) \times L_2(\Omega)) \text{ is continuous} \tag{8.7}$$

(as seen, e.g., by using the variation of parameter formula, based on the corresponding s.c. contraction semigroup on $H^1(\Omega) \times L_2(\Omega)$, which describes the evolution of (1.1.1a) and (8.5) with $f = 0$, $g = 0$). Thus, (8.6), (8.7) yield (8.2).

(ii) Conclusion (8.3) follows from the by now classical identity, e.g. [Tr.1], [L-T.2], [Lio.1–2], obtained by the multiplier $m \cdot \nabla w$, where m is a $C^2(\bar{\Omega})$-vector field such that $m = \nu$ on Γ, so that $m \cdot \nu \equiv 1$, on Γ; i.e., by

$$\int_\Sigma \frac{\partial w}{\partial \nu} m \cdot \nabla w\, d\Sigma + \frac{1}{2} \int_\Sigma \left\{ w_t^2 - \left[|\nabla_{\tan} w|^2 + \left(\frac{\partial w}{\partial \nu} \right)^2 \right] \right\} m \cdot \nu\, d\Sigma = \mathcal{O}(\mathbb{E}(0)). \tag{8.8}$$

Since $m = (m \cdot \nu)\nu + (m \cdot s)s$, s being a unit tangent vector on Γ, then

$$\nabla w \cdot m = (m \cdot \nu)\frac{\partial w}{\partial \nu} + (m \cdot s)\frac{\partial w}{\partial s}, \quad \nabla_{\tan} w = \frac{\partial w}{\partial s}\, s, \quad |\nabla_{\tan} w|^2 = \left(\frac{\partial w}{\partial s}\right)^2,$$

and (8.8) becomes with $m \cdot \nu = 1$,

$$\int_\Sigma \left(\frac{\partial w}{\partial s}\right)^2 d\Sigma = 2\int_\Sigma \left(\frac{\partial w}{\partial \nu}\right)\left(\frac{\partial w}{\partial s}\right) m \cdot s\, d\Sigma + \int_\Sigma \left[\left(\frac{\partial w}{\partial \nu}\right)^2 + w_t^2\right] d\Sigma + \mathcal{O}(\mathbb{E}(0)), \quad (8.9)$$

and (8.3) follows from (8.9) via $2ab \le \epsilon a^2 + \frac{1}{\epsilon}b^2$, with $a = \frac{\partial w}{\partial s}$ and $b = \frac{\partial w}{\partial \nu}$. $\Box$

Theorem 8.2. Let $f \in L_2(Q)$. Let $w \in H^{2,2}(Q)$ be a solution of Eqn. (1.1.1a) for which inequality (5.1) holds true, at least as guaranteed by Theorem 5.1(iii). Let u be a solution of Eqn. (1.1.1a) in the class defined by (8.1): i.e., $u \in H^{1,1}(Q)$; $\frac{\partial u}{\partial \nu}$, $u_t \in L_2(\Sigma)$. Then, estimate (5.1) is satisfied by such solution u as well.

Proof. Step 1. Let u be a solution of Eqn. (1.1a) in the class (8.1), and define accordingly the boundary function $g = \frac{\partial u}{\partial \nu} + u_t \in L_2(\Sigma)$ as in (8.5) and the interior function $\hat{f} = Fu + f \in L_2(Q)$. Let $u_0 \in H^1(\Omega)$ and $u_1 \in L_2(\Omega)$ be the initial conditions for such solution u. Given these data, there exist sequences $\{\hat{f}^n\}_{n=0}^\infty$, $\{g^n\}_{n=0}^\infty$, $\{u_0^n\}_{n=0}^\infty$, $\{u_1^n\}_{n=0}^\infty$, such that

$$\hat{f}^n \in H^{1,1}(Q), \quad \text{and} \quad \hat{f}^n \to \hat{f} \equiv Fu + f \text{ in } L_2(Q); \qquad (8.10)$$

$$g^n \in H^{1,1}(\Sigma), \quad \text{and} \quad g^n \to g \equiv \frac{\partial u}{\partial \nu} + u_t \text{ in } L_2(\Sigma); \qquad (8.11)$$

$$u_0^n \in H^2(\Omega), \quad \text{and} \quad u_0^n \to u_0 \text{ in } H^1(\Omega); \qquad (8.12)$$

$$u_1^n \in H^1(\Omega), \quad \text{and} \quad u_1^n \to u_1 \text{ in } L_2(\Omega), \qquad (8.13)$$

subject to the Compatibility Relation (C.R.): $\frac{\partial u_0^n}{\partial \nu} + u_1^n = g^n(0)$ on Γ.

Next, we consider the problem corresponding to these smooth data

$$\begin{cases} w_{tt}^n = \Delta w^n + \hat{f}^n & \text{in } Q; & (8.14a) \\ w^n(0, \cdot) = u_0^n;\ w_t^n(0, \cdot) = u_1^n & \text{in } \Omega; & (8.14b) \\ \dfrac{\partial w^n}{\partial \nu} + w_t^n = g^n & \text{in } \Sigma, & (8.14c) \end{cases}$$

Step 2. It follows *a-fortiori* from the given regularity of the data in (8.10)–(8.13) (left), that problem (8.14) admits the following regularity:

$$\{w^n, w_t^n, w_{tt}^n\} \in C([0,T]; H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)). \tag{8.15}$$

Indeed, $g^n \in H^{1,1}(\Sigma)$ and $\hat{f}^n \in H^{1,1}(Q)$ imply *a-fortiori* [L-M.1, p. 19] that

$$g^n \in H^1(0,T; L_2(\Gamma)) \cap C([0,T]; H^{\frac{1}{2}}(\Gamma)), \;\; \hat{f}^n \in H^1(0,T; L_2(\Omega)) \cap C([0,T]; H^{\frac{1}{2}}(\Omega)). \tag{8.16}$$

It will suffice to let $u_0^n = 0$; $u_1^n = 0$. Then, the problem

$$(w_t^n)_{tt} - \Delta w_t^n = \hat{f}_t^n \in L_2(0,T; L_2(\Omega)); \tag{8.17a}$$

$$\frac{\partial (w_t^n)}{\partial \nu} + (w_t^n)_t = g_t^n \in L_2(0,T; L_2(\Gamma)), \tag{8.17b}$$

obtained from differentiating (8.14a,c) in t yields the regularity stated in (8.15) for $\{w_t^n, w_{tt}^n\}$, by simply invoking for problem (8.17) the regularity maps (8.6), (8.7), *mutatis mutandis*. This preliminary regularity of $\{w_t^n, w_{tt}^n\} \in C([0,T]; H^1(\Omega) \times L_2(\Omega))$, hence $w_t^n \in C([0,T]; H^{\frac{1}{2}}(\Gamma))$ by trace theory, is then used in (8.14) for the resulting elliptic problem at each t. Elliptic regularity [L-M.1, p. 188] with $\hat{f}^n$, g^n as in (8.16) (right yields $w^n \in H^2(\Omega)$ at each t, as desired. Thus, (8.15) is established.

Step 3. Next, multiplying problem (8.14a) by w_t^n and integrating by parts yields

$$\begin{aligned} &\int_\Omega [|w_t^n(t)|^2 + |\nabla w^n(t)|^2] d\Omega + \int_0^t \int_\Gamma |w_t^n|^2 d\Gamma\, dr \\ &\quad = \int_\Omega [|u_1^n|^2 + |\nabla u_0^n|^2] d\Omega + \int_0^t \int_\Gamma g^n w_t^n d\Gamma\, dr + \int_0^t \int_\Omega \hat{f}^n w_t^n d\Omega\, dr. \end{aligned} \tag{8.18}$$

This, combined with estimate (8.3) then yields

$$\begin{aligned} &\int_0^t \int_\Omega [|w_t^n(t)|^2 + |\nabla w^n(t)|^2] d\Omega\, dr + \int_0^t \int_\Gamma \left(\frac{\partial w^n}{\partial \nu}\right)^2 d\Gamma\, dr \\ &\quad + \int_0^t \int_\Gamma |\nabla_{\tan} w^n|^2 d\Gamma\, dr + \int_0^t \int_\Gamma |w_t^n|^2 d\Gamma\, dr \\ &\quad \le C_T \left\{ \|\{u_0^n, u_1^n\}\|^2_{H^1(\Omega)\times L_2(\Omega)} + \int_0^T \int_\Gamma |g^n|^2 d\Sigma + \int_0^T \int_\Omega |\hat{f}^n|^2 dQ \right\}, \;\; 0 \le t \le T. \end{aligned} \tag{8.19}$$

In fact, integrating (8.18) over $\int_0^t$ and majorizing its right-hand side yields

$$\left(1-\frac{\epsilon}{2}\right)\int_0^t\int_\Omega (w_t^n)d\Omega\, dr + \int_0^t\int_\Omega |\nabla w^n|^2 d\Omega\, dr + \left(1-\frac{\epsilon}{2}\right)\int_0^t\int_\Gamma (w_t^n)d\Gamma\, dr$$

$$\leq \quad T\left\{\|\{w_0^n, w_1^n\}\|^2_{H^1(\Omega)\times L_2(\Omega)} + \frac{1}{2\epsilon}\int_0^T\int_\Gamma (g^n)^2 d\Sigma + \frac{1}{2\epsilon}\int_0^T\int_\Omega (\hat{f}^n)^2 dQ\right\}. \quad (8.20)$$

Next, adding to both sides of (8.20) the quantity

$$\int_0^t\int_\Gamma |\nabla_{\tan} w^n|^2 d\Gamma\, dr + \int_0^t\int_\Gamma \left(\frac{\partial w^n}{\partial\nu}\right)^2 d\Gamma\, dr,$$

and using first inequality (8.3), next (8.5) for $\frac{\partial w^n}{\partial \nu}$, and finally the bound (8.20) for $\int_0^t\int_\Gamma (w_t^n)^2 d\Gamma\, dr$, yields readily (8.19), as desired.

Step 4. The above estimate (8.19), when applied to Cauchy sequences, allows one to pass to the limit and obtain the following convergence relations at the interior and at the boundary

$$w^n \to w^* \text{ in } H^1(Q). \quad (8.21a)$$

$$\frac{\partial w^n}{\partial\nu} \to \frac{\partial w^*}{\partial\nu} \text{ in } L_2(\Sigma); \quad \frac{\partial w^n}{\partial s} \to \frac{\partial w^*}{\partial s} \text{ in } L_2(\Sigma); \quad w_t^n \to w_t^* \text{ in } L_2(\Sigma). \quad (8.21b)$$

From (8.10)–(8.13) and (8.21), passing to the limit on problem (8.14), we obtain that the limit w^* obtained above in (8.21a) satisfies

$$\begin{cases} w_{tt}^* = \Delta w^* + F(u) + f & \text{on } Q; \quad (8.22a)\\ w^*(0,\ \cdot\ ) = u_0,\ w_t^*(0,\ \cdot\ ) = u_1 & \text{in } \Omega; \quad (8.22b)\\ \dfrac{\partial w^*}{\partial\nu} + w_t^* = g & \text{in } \Sigma. \quad (8.22c)\end{cases}$$

Comparing problem (8.22) with the problem satisfied by u, by its very definition

$$\begin{cases} u_{tt} = \Delta u + F(u) + f & \text{in } Q; \quad (8.23a)\\ u(0,\ \cdot\ ) = u_0,\ u_t(0,\ \cdot\ ) = u_1 & \text{in } \Omega; \quad (8.23b)\\ \dfrac{\partial u}{\partial\nu} + u_t = g & \text{in } \Sigma, \quad (8.23c)\end{cases}$$

we see that the difference $\hat{w} = w^* - u$ satisfies

$$\hat{w}_{tt} = \Delta\hat{w} \text{ in } Q;\ \hat{w}(0,\ \cdot\ ) = 0,\ \hat{w}_t(0,\ \cdot\ ) = 0 \text{ in } \Omega;\ \frac{\partial\hat{w}}{\partial\nu} + \hat{w}_t = 0 \text{ in } \Sigma, \quad (8.24)$$

and hence

$$\hat{w} = w^* - u \equiv 0 \text{ in } Q. \tag{8.25}$$

Step 5. Since the solution w^n of problem (8.14) is in $H^{2,2}(Q)$, by Step 1, then w^n satisfies estimate (5.1) of Theorem 5.1(ii) with f there replaced by $\hat{f} = Fu + f$ now. By the limit properties (8.21), we obtain that the limit $w^* = u$ satisfies estimate (5.1), with f there replaced by $\hat{f} = Fu + f$ now. Finally, recalling estimate (1.1.2) for F, we obtain that estimate (5.1) holds true for the postulated finite energy solution u in the class (8.1) as well. The proof of Theorem 8.2 is complete. □

Remark 8.1. As a consequence of Theorem 8.2, estimate (5.2) of Theorem 5.1(ii) and all subsequent estimates through Section 7 can be extended from $H^{2,2}(Q)$-solutions of Eqn. (1.1.1a) to finite energy solutions the class (8.1).

9 Continuous observability without geometrical conditions on Γ_1. Non-explicit constant

Key to the elimination of geometrical conditions on the (controlled or observed) portion Γ_1 of the boundary Γ, is the following result from [L-T.3, Section 7.2].

Lemma 9.1. Let w be a solution of Eqn. (1.1.1a) in the class (8.1). Given $\epsilon > 0$, $\epsilon_0 > 0$ arbitrary, given $T > 0$ there exists a constant $C_{\epsilon,\epsilon_0,T} > 0$ such that

$$\begin{aligned} &\int_\epsilon^{T-\epsilon} \int_\Gamma |\nabla_{\tan} w|^2 d\Sigma \\ &\leq \; C_{\epsilon,\epsilon_0,T} \left\{ \int_0^T \int_\Gamma \left[\left(\frac{\partial w}{\partial \nu} \right)^2 + w_t^2 \right] d\Sigma + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} + \|f\|^2_{H^{-\frac{1}{2}+\epsilon_0}(Q)} \right\}. \quad \square \end{aligned} \tag{9.1}$$

Using Lemma 9.1, we shall establish the sought-after continuous observability inequality.

Remark 9.1. We remark that estimate (9.1) is much sharper than (8.3) in that—unlike (8.3)—it is given in terms of a lower-order term, *below* energy level, while (8.3) contains an energy term $\mathbb{E}(0)$. The argument needed to prove (9.1) is much more subtle than the argument for (8.3). The 'loss' of ϵ in the time interval in (9.1) is not critical, as seen in the proof of Theorem 9.2.

Theorem 9.2. Assume hypotheses (A.1) and (A.2). Let $w \in H^{1,1}(Q)$ be a solution of problem (1.1.1a-b-c) with $f \equiv 0$. Then, the following continuous observability inequality holds true for $T > T_0$, with T_0 given by (1.1.8b): there exists a constant $C_T > 0$ such that

$$\int_0^T \int_{\Gamma_1} [w^2 + w_t^2] d\Sigma_1 \geq C_T E(0). \tag{9.2}$$

Proof. Step 1. Lemma 9.3. Under hypotheses (A.1) and (A.2), we first establish the weaker conclusion

$$\int_0^T \int_{\Gamma_1} [w^2 + w_t^2] d\Sigma + \|w\|^2_{L_2(0,T;H^{\frac{1}{2}+\epsilon_0}(\Omega))} \geq C_T E(0), \tag{9.3}$$

which is the desired inequality (9.2) polluted by the interior l.o.t $\|w\|$.

Proof of (9.3). To this end, we invoke Theorem 6.1—which holds true under the present assumptions (A.1) and (A.2), also for $H^{1,1}(Q)$-solutions of class (8.1), by virtue of the extension Theorem 8.2 and consequent Remark 8.1. We then apply estimate (6.6), except on the interval $[\epsilon, T-\epsilon]$, rather than on $[0,T]$ as in (6.6). Thus, we obtain since $f \equiv 0$:

$$\overline{BT}|_{[\epsilon,T-\epsilon]\times\Gamma} \geq k_\phi E(\epsilon), \tag{9.4}$$

where $\overline{BT}|_{[\epsilon,T-\epsilon]\times\Gamma}$ is the counterpart of (6.8b) since $\frac{\partial w}{\partial \nu}|_\Sigma \equiv 0$ by (1.1c), with the additional information that $h \cdot \nu = 0$ on Γ_0 by (A.1); i.e., with $\Sigma_1^\epsilon = (\epsilon, T-\epsilon) \times \Gamma_1$:

$$\begin{aligned} \overline{BT}|_{[\epsilon,T-\epsilon]\times\Gamma} &= 2\tau \int_\epsilon^{T-\epsilon} \int_{\Gamma_1} e^{2\tau\phi}(w_t^2 - |\nabla_{\tan} w|^2) h \cdot \nu \, d\Sigma_1 \\ &+ \ 2\tau \int_\epsilon^{T-\epsilon} \int_{\Gamma_1} e^{2\tau\phi} \left[2\tau^2 \left(|h|^2 - 4c^2 \left(t - \frac{T}{2} \right)^2 \right) - (4c+1-k)\tau \right] w^2 h \cdot \nu \, d\Sigma_1^\epsilon \\ &+ \ \text{const}_\phi \left[\int_\epsilon^{T-\epsilon} \int_{\Gamma_1} |w w_t| d\Sigma_1^\epsilon + \int_{t_0}^{t_1} \int_{\Gamma_1} w^2 d\Gamma_1 dt. \right]. \end{aligned} \tag{9.5}$$

Next, by the right side of equivalences (6.4) and (6.18), we obtain

$$E(\epsilon) \geq \frac{\mathcal{E}(\epsilon)}{b} \geq \frac{a}{2b} E(0) e^{-C_T T} - 2 \int_0^T \int_{\Gamma_1} |w w_t| d\Sigma_1, \tag{9.6}$$

recalling $N(T)$ in (6.14) via (1.1.1c). We use (9.6) in (9.4). Finally, we invoke estimate (9.1) of Lemma 9.1 on the first integral term of (9.5) and recall that $\frac{\partial w}{\partial \nu}|_\Sigma = 0$ by (1.1.1c). This way, we readily obtain (9.3). □

Step 2. To eliminate the interior l.o.t. in estimate (9.3), we apply the by now standard compactness/uniqueness argument [Lit.1], [L-T.2], [Lio.1–2]. To this end, we need to invoke the global uniqueness Theorem 7.1, which has been extended to $H^{1,1}(Q)$-solutions by virtue of Theorem 8.2 and consequence Remark 8.1. □

10 Replacement of assumption (A.2) = (1.1.6) by virtue of two vector fields

Orientation. As the examples in Appendix A persuasively illustrate, assumption (A.2) = (1.1.6) introduces undesirable limitations, when enforced along with assumption (A.1) Accordingly, the purpose of this section is to replace this assumption suitably. This will be done by using the following strategy. First, from the given domain $\Omega \in \mathbb{R}^n$, we extract two overlapping subdomains Ω_1 and Ω_2, to which we can apply, separately, the setting of Section 1 (including the counterpart version of assumption (A.2) = (1.1.6)), in correspondence to two postulated strictly convex functions $d_1(x)$ and $d_2(x)$. This leads to the estimate of Corollary 4.3—Eqns. (4.24), (4.26)—to each subproblem on Ω_i: see estimates (10.2.16), (10.2.17) below of Proposition 10.2.1. Next, it is then a delicate matter to combine these estimates (10.2.16) for each separate subproblem to finally obtain the desired global estimate (10.5.1), hence (10.6.3), for the original problem, this time, however, having dispensed with assumption (A.2) = (1.1.6) on all of Ω. This latter step requires the introduction of a suitable cut-off function $\chi_i(t,x)$ [in (10.2.10) below] for each subproblem on Ω_i. In particular, each $\chi_i(t,x)$ is *space-independent* on a small layer of the boundary $\Gamma = \partial\Omega$, so that the cut-off solutions w_i corresponding to each subdomain Ω_i, $i = 1,2$, and the original solution w have same traces on Γ, see (10.2.14) below, except for a multiplicative time-dependent function. In short: we replace assumptions (A.1), (A.2) in (1.1.4), (1.1.5), (1.1.6) of Section 1 with assumptions (A.1i), (A.2i) in (10.1.2), (10.1.3), (10.1.4) below. The latter are weaker than the former in many cases, e.g., when the unobserved boundary Γ_0 is *flat*: see Appendix. The advantages of removing assumption (A.2) = (1.1.6) on all of Ω are multiple. This is illustrated in the Appendix.

10.1 Basic setting using two conservative vector fields of the same class as in Section 1. Statement of main result. Continuous observability. Global uniqueness

Postulated setting. We divide the original Ω into two overlapping subdomains Ω_1 and Ω_2:

$$\Omega = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \text{non-empty}, \tag{10.1.1}$$

chosen (in infinitely many ways) as to fulfill the following conditions (following Section 1): there exist two strictly convex functions $d_i : \bar{\Omega} \to \mathbb{R}$ of class C^3, $i = 1,2$, such that for the corresponding (conservative) vector fields $h_i(x) = [h_{i;1}(x), \dots, h_{i;n}(x)] \equiv \nabla d_i(x), \quad x \in \Omega$, the following three properties hold true for $i = 1,2$:

(A.1i) (a)

$$\frac{\partial d_i}{\partial \nu} = \nabla d_i \cdot \nu = h_i \cdot \nu = 0 \quad \text{on } \Gamma_0,\ h_i = \nabla d_i; \tag{10.1.2}$$

(b) the Hessian matrix $\mathcal{H}_{d_i}$ of $d_i(x)$ [i.e., the Jacobian matrix J_{h_i} of $h_i(x)$] is strictly positive definite on $\bar{\Omega}_i$: there exists a constant $\rho_0 > 0$, such that for all $x \in \bar{\Omega}_i$, we have

$$\mathcal{H}_{d_i}(x) = J_{h_i}(x) = \begin{bmatrix} d_{i;x_1x_1}, & \cdots, & d_{i;x_1x_n} \\ \vdots & & \vdots \\ d_{i;x_nx_1}, & \cdots, & d_{i;x_nx_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial h_{i;1}}{\partial x_1}, & \cdots, & \dfrac{\partial h_{i;1}}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial h_{i;n}}{\partial x_1}, & \cdots, & \dfrac{\partial h_{i;n}}{\partial x_n} \end{bmatrix} \geq \rho_0 I. \quad i = 1, 2. \tag{10.1.3}$$

Indeed, after rescaling if necessary (Section 1.2), we may and shall require that $\rho_0 \geq 2$;

(A.2i)

$$\inf_{x \in \Omega_i} |h_i(x)| \equiv \inf_{x \in \Omega_i} |\nabla d_i(x)| \geq p > 0, \quad i = 1, 2. \tag{10.1.4}$$

The validity of the above setting, comprising hypotheses (A.1i), (A.2i), for large classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ is established in Theorem C.1 of Appendix C. This culminates the analysis in the appendices, carried our by virtue of several different approaches, aimed at verifying the required geometrical assumptions of the setting of Section 1, as well as of the setting of the present Section 10.

Pseudo-convex functions. Let $d_i(x) : \bar{\Omega} \to \mathbb{R}$ be the C^3-functions provided by the above setting, $i = 1, 2$, and satisfying assumptions (A.1i) and (A.2i). Without loss of generality, we shall require—after, possibly, translation as in Section 1—that: $\min_{\bar{\Omega}_i} d_i(x) \equiv m > 0$. We then define

$$T_{0,i}^2 = 4 \max_{x \in \bar{\Omega}_i} d_i(x), \qquad i = 1, 2. \tag{10.1.5}$$

We next define the pseudo-convex functions

$$\phi_i(x,t) = d_i(x) - c\left(t - \frac{T}{2}\right)^2, \quad x \in \Omega, \; 0 \leq t \leq T, \tag{10.1.6a}$$

where $T > T_{0,i}$, $i = 1, 2$, and the constant $0 < c < 1$ is selected as follows. If $T > T_{0,i}$, there exists a constant $\delta > 0$ such that [as in (1.1.8c)]

$$T^2 > 4 \max_{x \in \bar{\Omega}_i} d_i(x) + 4\delta, \quad i = 1, 2. \tag{10.1.6b}$$

For such $\delta > 0$, there exists c, $0 < c < 1$, such that

$$cT^2 > 4 \max_{x \in \bar{\Omega}_i} d_i(x) + 4\delta, \quad i = 1, 2. \tag{10.1.6c}$$

Henceforth, let $\phi_i(x,t)$ be defined by (10.1.6a) with T and c chosen above, unless otherwise explicitly noted. Such constant $0 < c < 1$, close to 1, may be taken independent of rescaling of $d_i(x)$, see Section 1.2. Such functions $\phi_i(x,t)$ have the following properties (as in Section 1):

(a) for the constant $\delta > 0$ fixed in (10.1.6b), we have

$$\phi_i(x,0) = \phi_i(x,T) = d_i(x) - c\frac{T^2}{4} \leq -\delta, \text{ uniformly in } x \in \Omega_i,\ i = 1,2; \tag{10.1.7}$$

(b) there are t_0, t_1, with $0 < t_0 < \frac{T}{2} < t_1 < T$, say symmetric about $\frac{T}{2}$, such that

$$\phi_i(x,t) \geq \sigma > 0, \quad 0 < \sigma < m, \text{ for all } (t,x) \in [t_0,t_1] \times \Omega_i,\ i = 1,2, \tag{10.1.8}$$

since $\phi_i(x, \frac{T}{2}) = d_i(x) \geq m > 0$ in Ω_i: indeed, we take

$$m - c\left(t_1 - \frac{T}{2}\right)^2 \equiv \sigma > 0 \text{ and } t_1 - \frac{T}{2} < \sqrt{\frac{m-\sigma}{c}},$$

as in Section 1.1.

Consequences of above setting. [Assumptions (A.1i), (A.2i)] Let $d_i(x) : \bar{\Omega}_i \to \mathbb{R}$, be the C^3-functions satisfying assumptions (A.1i), (A.2i) and the rescaling choice $\rho_0 \geq 2$. Then, as in Section 1, it follows that: there exist functions $\alpha_i(x) \in C^1(\bar{\Omega})$, in fact, take

$$\alpha_i(x) \equiv \Delta d_i(x) - 2c - 1 + k \in C^1(\bar{\Omega}) \tag{10.1.9}$$

for a constant $1 + 4c - 2\rho_0 < k < 1$, to be selected below, such that the following properties $(\mathrm{p}_{1;i})$, $(\mathrm{p}_{2;i})$ hold true:

$(\mathrm{p}_{1;i})$

$$\Delta d_i(x) - 2c - \alpha_i(x) \equiv 1 - k > 0, \quad x \in \bar{\Omega}, \tag{10.1.10}$$

$(\mathrm{p}_{2;i})$

$$\gamma \equiv \alpha_i(x) - 2c - \Delta d_i(x) \equiv -4c - 1 + k, \tag{10.1.11}$$

and the Hessian matrix $\mathcal{H}_{d_i}$ of $d_i(x)$ satisfies the following inequality for all $x \in \bar{\Omega}_i$:

$$\begin{aligned} &2\mathcal{H}_{d_i}(x) + [\alpha_i(x) - 2c - \Delta d_i(x)]I = 2\mathcal{H}_{d_i}(x) + \gamma I \\ &= \begin{bmatrix} 2d_{i;x_1x_1} + \gamma & 2d_{i;x_1x_2} & \cdots & 2d_{i;x_1x_n} \\ 2d_{i;x_2x_1} & 2d_{i;x_2x_2} + \gamma & \cdots & 2d_{i;x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ 2d_{i;x_nx_1} & 2d_{i;x_nx_2} & \cdots & 2d_{i;x_nx_n} + \gamma \end{bmatrix} \geq \rho I, \quad \forall\ x \in \Omega_i,\ i = 1,2, \end{aligned} \tag{10.1.12}$$

for some constant $\rho = 2\rho_0 + \gamma = 2\rho_0 - 4c - 1 + k > 0$, see (10.1.3), for a constant k chosen as to satisfy: $1 + 4c - 2\rho_0 < k < 1$, which is possible since $\rho_0 > 2c$, $0 < c < 1$, due to the choice $\rho \geq 2$. In addition, by additional rescaling, if necessary [at worst by imposing the rescaling condition: $|\nabla d_i(x)|^2 - 4d_i(x) \geq 0, \quad \forall\ x \in \Omega_i,\ i = 1, 2$ as in (1.2.3) of Section 1.2], we may require that such $d_i(x)$ fulfill, in addition, also the following property $(p_{3;i})$:

$(p_{3;i})$ Noting via (10.1.9) that

$$6c + \Delta d_i(x) - \alpha_i(x) \equiv 8c + 1 - k, \quad x \in \bar{\Omega}, \tag{10.1.13}$$

we have that the following inequalities hold true, by virtue of assumption (A.2i) = (10.1.4):

$$\begin{cases} (2c + \Delta d_i - \alpha_i)|\nabla d_i|^2 + 2\mathcal{H}_{d_i}\nabla d_i \cdot \nabla d_i - (6c + \Delta d_i - \alpha_i)4c^2\left(t - \frac{T}{2}\right)^2 \\ \equiv (4c + 1 - k)|\nabla d_i|^2 + 2\mathcal{H}_{d_i}\nabla d_i \cdot \nabla d_i - (8c + 1 - k)4c^2\left(t - \frac{T}{2}\right)^2 \geq \tilde{\beta} > 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\ (x,t) \in \text{set } Q_i^*(\sigma^*), \end{cases} \tag{10.1.14}$$

for a constant $\tilde{\beta} > 0$, where the set $Q_i^*(\sigma^*)$ is defined (as in Section 1.1) by

$$Q_i^*(\sigma^*) \equiv \{(x,t) : x \in \Omega,\ 0 \leq t \leq T,\ \phi_i^*(x,t) \geq \sigma^* > 0\}, \tag{10.1.15}$$

for a constant σ^* chosen as to satisfy $0 < \sigma^* < \sigma$, see (10.1.8), where the functions ϕ_i^* are in turn defined by

$$\phi_i^*(x,t) \equiv d_i(x) - c^2\left(t - \frac{T}{2}\right)^2, \quad x \in \Omega,\ 0 \leq t \leq T. \tag{10.1.16}$$

Since $0 < c < 1$, we obtain, by (10.1.6a) and (10.1.16), that

$$\phi_i^*(x,t) \geq \phi_i(x,t), \quad x \in \Omega,\ 0 \leq t \leq T. \tag{10.1.17}$$

Thus, if we define, in agreement with (10.1.15), the set $Q_i(\sigma)$ by

$$Q_i(\sigma) = \{(x,t) :\ x \in \Omega,\ 0 \leq t \leq T,\ \phi_i(x,t) \geq \sigma > 0\}, \tag{10.1.18}$$

we see, since $0 < \sigma^* < \sigma$, and by (10.1.8), that:

$$\begin{cases} [t_0, t_1] \times \Omega_i \subset Q_i(\sigma) \subset Q_i^*(\sigma^*) \subset [0,T] \times \Omega, \text{ properly} \\ \text{by (10.1.7), at } t = 0 \text{ and } t = T\text{: no point } x \in \Omega_i \text{ belongs to } Q_i(\sigma). \end{cases} \tag{10.1.19}$$

The main result of this paper is the following Theorem 10.1.1, which in many cases (e.g., when Γ_0 is flat) extends Theorem 2.1.1, by replacing assumptions (A.1), (A.2) with assumptions (A.1i), (A.2i), $i = 1, 2$.

Theorem 10.1.1. Let the above setting of Section 10.1 based on assumptions (A.1i) and (A.2i) be in force for a given triple $\{\Omega, \Gamma_0, \Gamma_1\}$. Let $T_{0,i} > 0$ be the constants defined by (10.1.5) and let $T > T_{0,i}$. Then

(a) for all $T > T_{0,i}$, the following continuous observability inequality holds true for $H^{1,1}(Q)$-solutions of problem (1.1.1), with F satisfying (1.1.2): there is a constant $C_T > 0$ such that

$$C_T E(0) \leq \int_0^T \int_{\Gamma_1} [w_t^2 + w^2] d\Sigma_1 + \int_0^T \int_\Omega f^2 dQ. \tag{10.1.20}$$

The constant C_T is *explicit* if in addition we assume geometrical star-shaped conditions on Γ_1:

$$h_i \cdot \nu \geq 0, \quad i = 1, 2 \qquad \text{on } \Gamma_1. \tag{10.1.21}$$

Such constant C_T is of the order of Ce^{Cr^2}, where C is a generic constant and r is the norm in (1.1.2c) on the involved coefficients.

(b) A-fortiori, the following global uniqueness result holds true: let $T > T_{0,i}$ and let w be an $H^{1,1}(Q)$-solution of problem (1.1.1a) with $f \equiv 0$, along with the B.C.

$$\frac{\partial w}{\partial \nu}\Big|_\Sigma = 0 \quad \text{ and } \quad w|_{\Sigma_1} = 0 \text{ where } h_i \cdot \nu = 0 \text{ on } \Gamma_0,\ i = 1, 2, \tag{10.1.22}$$

as in the assumed Eqn. (10.1.2). Then, in fact, $w \equiv 0$, in Q (in fact, $w \equiv 0$ in $\mathbb{R}_t \times \Omega$). $\square$

The same comments as those below Theorem 2.1.1 apply. In effect we shall first prove the uniqueness statement of part (b) in Section 10.6 [as a direct consequence of the Carleman estimates of Theorem 10.5.1 for $H^{2,2}(Q)$-solutions]. Next, part (b) will be used to establish part (a) in Section 10.7.

A discussion was given in Section 1 on several topics including: (i) the role of the set $Q_i^*(\sigma^*)$ in relationship to the set $Q_i(\sigma)$; (ii) that in the case where Γ_0 is *flat*, whereby then $d_i(x) = |x - z_i|^2$, z_i just outside Ω on the hyperplane containing Γ_0 (radial vector field case), properties ($\text{p}_{1;i}$), ($\text{p}_{2;i}$), and ($\text{p}_{3;i}$) automatically hold true, with no rescaling needed; (iii) the issue of possibly rescaling $d_i(x)$ as to satisfy properties ($\text{p}_{1;i}$), ($\text{p}_{2;i}$), and ($\text{p}_{3;i}$), the latter one even on the cylinder $[0, T] \times \Omega_i$, at the price of deteriorating the minimal time $T_{0,i}$ of observability, by imposing the rescaling condition (as in (1.2.3)):

$$|\nabla d_i(x)|^2 - 4 d_i(x) \geq 0, \quad \forall\, x \in \Omega_i,\ i = 1, 2. \tag{10.1.23}$$

10.2 The cut-off functions $\chi_i(t, x)$ and corresponding subproblems for $w_i \equiv \chi_i w$. Preliminary estimate

Definition of cut-off functions χ_i. Step 1. First, we recall the time-space sets $Q_i(\sigma) \subset Q_i^*(\sigma^*)$ (properly), defined in (10.1.15), (10.1.18) for $0 < \sigma^* < \sigma$, $i = 1, 2$, which are proper subsets of the basic cylinder $Q \equiv (0, T] \times \Omega$. Accordingly, we may introduce C^∞-functions $m_i(t, x)$, $0 \leq m_i(t, x) \leq 1$, on $[0, T] \times \Omega$, such that

$$m_i(t, x) \equiv 1 \text{ on } Q_i(\sigma); \qquad \text{supp } m_i \subset Q_i^*(\sigma^*), \tag{10.2.1}$$

so that $m_i \equiv 0$ on $[Q_i^*(\sigma^*)]^c$, the complement of $Q_i^*(\sigma^*)$ with respect to $[0,T] \times \Omega$.

Step 2. Next, for $x \in \mathbb{R}^n$, we define

$$\rho(x) = \inf_{y \in \Gamma} |x - y|, \qquad \Gamma = \partial\Omega, \tag{10.2.2}$$

so that the set $\{x \in \Omega : \ \rho(x) < \epsilon\}$ is an ϵ-internal layer of the boundary $\Gamma = \partial\Omega$. For any $\epsilon > 0$, let μ_ϵ be the usual mollifier [Kes.1, p. 4]

$$\mu_\epsilon(x) = \begin{cases} k\epsilon^{-n} \exp\left(-\frac{\epsilon^2}{\epsilon^2 - |x|^2}\right), & |x| < \epsilon \\ 0, & |x| \geq \epsilon \end{cases}, \quad k^{-1} = \int_{|x| \leq 1} \exp\left(-\frac{1}{1 - |x|^2}\right) dx, \tag{10.2.3a}$$

and then let

$$\rho^\epsilon(x) = (\mu_\epsilon * \rho)(x) \in C^\infty(\mathbb{R}^n), \qquad x \in \mathbb{R}^n. \tag{10.2.3b}$$

It is well known [Kes.1, p.] that, given $\epsilon_1 > 0$, then for all $\epsilon > 0$ sufficiently small, we have

$$\sup_{\bar{\Omega}} |\rho^\epsilon(x) - \rho(x)| < \frac{\epsilon_1}{4}. \tag{10.2.4}$$

Step 3. Introduce the following two subsets of $[0,T]$ on the t-axis:

$$E_i(\sigma) \equiv \text{orthogonal projection of the set } Q_i(\sigma) \text{ onto the } t\text{-axis}; \tag{10.2.5}$$

$$E_i^*(\sigma^*) \equiv \text{orthogonal projection of the set } Q_i^*(\sigma^*) \text{ onto the } t\text{-axis}, \tag{10.2.6}$$

see Figure 3. Thus, properties (10.1.19) yield

$$[t_0, t_1] \subset E_i(\sigma) \subset E_i^*(\sigma) \subset [0,T] \text{ properly}. \tag{10.2.7}$$

We then introduce a function $b_i(t) \in C_0^\infty(\mathbb{R}; [0,1])$ by setting

$$b_i(t) = \begin{cases} 1 & \text{for } t \in E_i(\sigma); \\ 0 & \text{for } t \in \mathbb{R} \setminus E_i^*(\sigma^*). \end{cases} \qquad i = 1, 2. \tag{10.2.8}$$

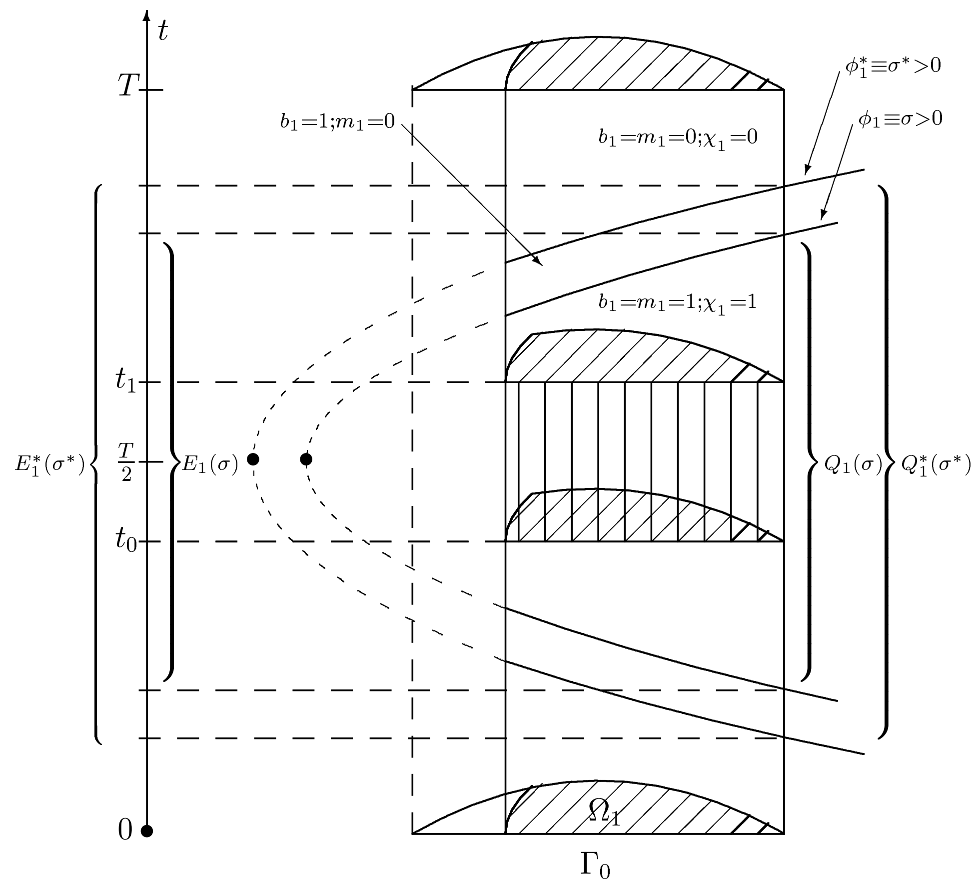

FIGURE 3

Step 4. Let $g(x)$ be a $C^\infty(\mathbb{R};[0,1])$-function such that

$$g(s) = \begin{cases} 1 & \text{for } s > \epsilon_1; \\ 0 & \text{for } s < \dfrac{\epsilon_1}{2}. \end{cases} \tag{10.2.9}$$

Finally, we define the following C^∞-function by setting

$$\chi_i(t,x) = m_i(t,x)g(\rho^\epsilon(x)) + [1 - g(\rho^\epsilon(x))]b_i(t), \quad t, x \in [0,T] \times \Omega, \tag{10.2.10}$$

with $\epsilon, \epsilon_1 > 0$ as in (10.2.4).

Properties of $\chi_i(t,x)$. First, let $x \in \Omega$ satisfy $\rho^\epsilon(x) < \frac{\epsilon_1}{2}$, i.e., $\rho(x) < \frac{\epsilon_1}{4}$ by (10.2.4), so that x lies in the $\frac{\epsilon_1}{4}$-internal layer of Γ. Then, $g(\rho^\epsilon(x)) = 0$ by (10.2.9), and then $\chi_i(t,x) \equiv b_i(t)$ for all $t \in [0,T]$.

Next, let $x \in \Omega$ satisfy $\rho^\epsilon(x) > \epsilon_1$, i.e., $\rho(x) > \frac{3}{4}\epsilon_1$. Then, $g(\rho^\epsilon(x)) = 1$ by (10.2.9), and then $\chi_i(t,x) = m_i(t,x)$ on $[0,T]$.

Also, if $(x,t) \in Q_i(\sigma)$, then: $m_i(t,x) = 1$ by (10.2.1), and $b_i(t) = 1$ by (10.2.8) and (10.2.5), and thus $\chi_i(t,x) \equiv m_i(t,x) \equiv 1$ by (10.2.10).

Let $t \in [0,T] \setminus E_i^*(\sigma^*)$: then $b_i(t) = 0$ by (10.2.8) and $m_i(t,x) = 0$ for all $x \in \Omega$ by (10.2.1) and (10.2.6). Hence $\chi_i(t,x) = 0$. We summarize (see Figure 3):

$$\begin{cases} \chi_i(t,x) = \begin{cases} b_i(t) & \forall\, x \in \Omega \text{ s.t. } \rho^\epsilon(x) < \dfrac{\epsilon_1}{2}, \text{ or } \rho(x) < \dfrac{\epsilon_1}{4}; \\ m_i(t,x) & \forall\, x \in \Omega \text{ s.t. } \rho^\epsilon(x) > \epsilon_1, \text{ or } \rho(x) > \dfrac{3}{4}\epsilon_1; \end{cases} & (10.2.11) \\ \chi_i(t,x) = m_i(t,x) = 1 \text{ on } Q_i(\sigma); \\ \chi_i(t,x) = 0, \quad \forall\, t \in \mathbb{R} \setminus E_i^*(\sigma^*),\ \forall x \in \Omega. & (10.2.12) \end{cases}$$

Remark 10.2.1. We can now explain our goal in this section. We shought cut-off functions which, among other features, are *only* time dependent (but *not* space dependent) on a small interior layer of the boundary Γ. This goal is dictated by the fact that we are dealing with Neumann B.C., and it would not be necessary if we were dealing instead with Dirichlet B.C. Once the above goal is achieved in (10.2.11), then the Neumann B.C. $\frac{\partial w_i}{\partial \nu}$ of the corresponding subproblems are readily expressed in terms of $\frac{\partial w}{\partial \nu}$ as in (10.2.14f). If, by contrast, χ_i were also space dependent near Γ, it would then polute $\frac{\partial w_i}{\partial \nu}$.

Dynamical systems for $w_i \equiv \chi_i w$. Let $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x^n)$ be a solution of Eqn. (1.1.1a). We then introduce new variables on $[0,T] \times \Omega$:

$$w_i(t,x) \equiv \chi_i(t,x) w(t,x); \quad f_i(t,x) \equiv \chi_i(t,x) w(t,x), \quad i = 1,2. \tag{10.2.13}$$

Then, we see that each term w_i satisfies the following problem:

$$\begin{cases} w_{i,tt} - \Delta w_i = F(w_i) + f_i + [D_t^2 - \Delta - F, \chi_i] w, \quad i = 1,2; & (10.2.14a) \\ w_i(0,\ \cdot\ ) = w_{i,0},\ w_{i,t}(0,\ \cdot\ ) = w_{i,1}, \quad \text{in } \Omega; & (10.2.14b) \end{cases}$$

$$\kappa_i \equiv [D_t^2 - \Delta - F, \chi_i] \text{ commutator active only on } (\text{supp } \chi_i); \tag{10.2.14c}$$

$$w_{i,0} = \chi_i(0,\ \cdot\ ) w(0,\ \cdot\ );\ w_{i,1} = \chi_{i,t}(0,\ \cdot\ ) w(0,\ \cdot\ ) + \chi_i(0,\ \cdot\ ) w_t(0,\ \cdot\ ); \tag{10.2.14d}$$

$$\begin{cases} w_i(t,x) = b_i(t) w(t,x),\ \forall\ x \text{ in an } \dfrac{\epsilon_1}{4}\text{-internal layer of } \Gamma, & (10.2.14e) \\ \text{hence} \\ w_i(t,x) = b(t) w(t,x); \quad \dfrac{\partial w_i}{\partial \nu} = b_i(t) \dfrac{\partial w}{\partial \nu} \quad \text{on } [0,T] \times \Gamma. & (10.2.14f) \end{cases}$$

In (10.2.14a), $D_t = \frac{\partial}{\partial t}$, while [,] denote the corresponding commutator of order 1 in time and space. Moreover, the key relation in (10.2.14e), hence the key trace properties in (10.2.14f), are a consequence of (10.2.11a): $\chi_{_i} = b(t)$ on an $\frac{\epsilon_1}{4}$-internal layer of Γ.

Since $\chi_{_i}$ is smooth and the commutator in (10.2.14a) is of order 1 in time and space, we then obtain via (10.2.14a) and (1.2) on F:

$$(w_{i,tt} - \Delta w_i)^2 \leq C_T\{[w_{i,t}^2 + |\nabla w_i|^2 + w_i^2 + f_i^2] + [w_t^2 + |\nabla w|^2 + w^2]\},$$

$$(t,x) \in [0,T] \times \Omega. \qquad (10.2.15)$$

Preliminary estimate: Counterpart of Corollary 4.3. As constructed above, each problem w_i in (10.2.14), $i = 1, 2$, satisfies the setting of assumptions (A.1i) and (A.2i), i.e., the setting of Section 1. As a result, each problem (10.2.14) satisfies the counterpart of Theorem 4.1/Corollary 4.2/Corollary 4.3, Eqns. (4.24)–(4.27), in particular, we recall (4.8). We take this result as our present starting point.

Proposition 10.2.1. Let $w \in C^2(\mathbb{R}_t \times \mathbb{R}_x^n)$ be a solution of Eqn. (1.1.1a). Let the setting of Section 10.1 based on assumptions (A.1i) and (A.2i) be in force. Then, each problem (10.2.14), $i = 1, 2$, satisfies the following pointwise inequality for $\epsilon > 0$ small:

$$\theta_i^2(w_{i,tt} - \Delta w_i)^2 - \frac{\partial M_i}{\partial t} + \text{div } V_i$$

$$\geq \epsilon\tau\theta_i^2\rho[w_{i,t}^2 + |\nabla w_i|^2] + \theta_i^2 B_i w_i^2, \quad 0 \leq t \leq T, \ x \in \Omega, \qquad (10.2.16)$$

see (4.24), where $\rho > 0$ is a constant, and where for $i = 1, 2$ and (10.1.14), we have recalling (4.8), (4.19), (4.25)–(4.27):

$$\begin{aligned} B_i &\equiv \tilde{B}_i - 2\epsilon\rho\tau^3(\phi_{i,t}^2 + |\nabla d_i|^2) \\ &\geq 2\tau^3\left\{[2c + \Delta d_i - \alpha_i]|\nabla d_i|^2 + 2\mathcal{H}_{d_i}\nabla d_i \cdot \nabla d_i - (6c + \Delta d_i - \alpha_i)4c^2\left(t - \frac{T}{2}\right)^2\right\} \\ &\quad + \mathcal{O}(\tau^2) - 2\epsilon\rho\tau^3 r \\ &\geq 2\tau^3\beta + \mathcal{O}(\tau^2), \quad \forall\ (x,t) \in \text{ set } Q_i^*(\sigma^*), \end{aligned} \qquad (10.2.17)$$

$$\beta \equiv \beta_\epsilon \equiv \tilde{\beta} - \epsilon\rho r > 0; \ r = \max_i \max_Q(\phi_{i,t}^2 + |\nabla d_i|^2); \ B_i = \mathcal{O}(\tau^3), \qquad (10.2.18)$$

where the set $Q_i^*(\sigma^*)$ is defined in (10.1.15), and $\beta > 0$ is a constant depending on $\epsilon > 0$. Above, $\theta_i = e^{\tau\phi_i}$; ϕ_i as in (10.1.6). Moreover, M_i, V_i are obtained from M and V in (3.3), (3.4) in the present case: i.e., by replacing $\ell = \tau d$, $\psi = \tau\alpha$, etc., with $\ell_i = \tau d_i$, $\psi_i = \tau\alpha_i$, where d_i is given by assumption (A.1i) and α_i is defined by (10.1.9) so that $\tilde{B}_i$ on the left of (10.2.17) is the counterpart of (4.8), while the estimate in (10.2.17) is due to (10.1.14).
□

10.3 Proof of Theorem: Carleman estimate for the w_i-problem

Building up on Proposition 10.2.1, we obtain the counterpart of Theorem 5.1 (Carleman estimate, first version) for the w_i-problems.

Proposition 10.3.1. Let $w \in C^2(\mathbb{R}_t \times \mathbb{R}^n_x)$ be a solution of Eqn. (1.1.1a). Let the setting of Section 10.1 based on assumptions (A.1i) and (A.2i) be in force. Let $w_i = \chi_i w$ as in (10.2.13). Let $E(t)$ be defined by (5.5), or (2.1.1), as usual. Then, for $\epsilon > 0$ small as in (10.2.18), and for all τ sufficiently large (with $\epsilon\tau$ large with respect to C_T in (1.1.2)), the following estimate holds true:

$$
\begin{aligned}
&(BT)_{w_i}|_\Sigma + C_{1,T} e^{2\tau\sigma} \int_0^T E(t)dt + C_{2,T} \int_0^T \int_\Omega f_i^2 d\Omega\, dt \\
&\qquad \geq \ (\epsilon\tau\rho - 2C_T) \int_0^T \int_\Omega e^{2\tau\phi_i} [w_{i,t}^2 + |\nabla w_i|^2] d\Omega\, dt \\
&\qquad + \ [2\tau^3\beta + \mathcal{O}(\tau^2) - 2C_T] \int_{Q_i(\sigma)} e^{2\tau\phi_i} w^2 dx\, dt - c_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)], \qquad (10.3.1)
\end{aligned}
$$

where $Q_i(\sigma)$ is the subset of $[0,T] \times \Omega \equiv Q$ defined by (10.1.18); moreover, the constants $\sigma > 0$, $\delta > 0$ are defined in (10.1.8) and (10.1.7), while the constant $\beta > 0$ is defined via (10.1.14) for $\tilde{\beta}$ and (10.2.18). In (10.3.1), the boundary term $(BT)_{w_i}|_\Sigma$ is defined (counterpart of (5.3)) by

$$
(BT)_{w_i}|_\Sigma \equiv \int_0^T \int_\Omega \text{div } V_i d\Omega\, dt = \int_0^T \int_\Gamma V_i \cdot \nu\, d\Gamma\, dt, \quad i = 1, 2, \qquad (10.3.2)
$$

and is explicitly given below (as in (5.4)), recalling $w_i = b_i w$, $\frac{\partial w_i}{\partial \nu} = b_i \frac{\partial w}{\partial \nu}$ on Γ, by (10.2.14f); as well as $h_i \cdot \nu = 0$ on Γ_0 by (10.1.21)):

$$
\begin{aligned}
(BT)_{w_i}|_\Sigma \ &\equiv \ 2\tau \int_0^T \int_{\Gamma_1} e^{2\tau\phi_i} (w_{1,t}^2 - |\nabla w|^2 b_i^2(t)) h_i \cdot \nu\, d\Gamma_1 dt \\
&+ \ 8c\tau \int_0^T \int_\Gamma e^{2\tau\phi_i} \left(t - \frac{T}{2}\right) w_{i,t} \frac{\partial w}{\partial \nu} b_i^2(t) d\Gamma\, dt \\
&+ \ 4\tau \int_0^T \int_\Gamma e^{2\tau\phi_i} (h_i \cdot \nabla w) \frac{\partial w}{\partial \nu} b_i^2(t) d\Gamma\, dt \\
&+ \ 4\tau^2 \int_0^T \int_\Gamma e^{2\tau\phi_i} \left[|h_i|^2 - 4c^2 \left(t - \frac{T}{2}\right)^2 + \frac{\alpha_i}{2\tau} \right] w \frac{\partial w}{\partial \nu} b_i^2(t) d\Gamma\, dt
\end{aligned}
$$

$$+ \quad 2\tau \int_0^T \int_{\Gamma_1} e^{2\tau\phi_i} \left[2\tau^2 \left(|h_i|^2 - 4c^2 \left(t - \frac{T}{2} \right)^2 \right) \right.$$

$$\left. + \tau(\alpha_i - \Delta d_i - 2c) \right] w^2 b_i^2(t) h_i \cdot \nu \, d\Gamma_1 \, dt. \tag{10.3.3}$$

(ii) The above inequality (10.3.3) may be extended to $H^{2,2}(Q)$-solutions.

Remark 10.3.1. For τ sufficiently large as to obtain $[2\tau^3\beta + \mathcal{O}(\tau^2) - C_T] > 0$, see (10.2.18), we reach one of our goals and drop the integral term involving w^2, accordingly, from inequality (10.3.1). □

Proof. Step 1. We return to Eqn. (10.2.16) of Proposition 10.2.1, which we now integrate over $[0,T] \times \Omega$. By use of the divergence theorem, we obtain the counterpart of (5.6) for $\epsilon > 0$:

$$\int_0^T \int_\Omega \theta_i^2 (w_{i,tt} - \Delta w_i)^2 d\Omega \, dt - \left[\int_\Omega M_i d\Omega \right]_0^T + \int_0^T \int_\Gamma V_i \cdot \nu \, d\Gamma \, dt$$

$$\geq \quad \epsilon\tau\rho \int_0^T \int_\Omega \theta_i^2 [w_{i,t}^2 + |\nabla w_i|^2] d\Omega \, dt + \int_0^T \int_\Omega \theta_i^2 B_i w_i^2 \, d\Omega \, dt, \tag{10.3.4}$$

with $\rho > 0$. Moreover, M_i and V_i are the counterpart of Eqns. (3.3) and (3.4), as specialized to the two cases $i = 1, 2$. With reference to (10.3.4), we now define $(BT)_{w_i}|_\Sigma$ by (10.3.2), and then obtain, via the counterpart of (5.4), that $(BT)_{w_i}|_\Sigma$ coincides with the expression given by (10.3.3) [which is the counterpart of (5.4)]. In doing so, we use two ingredients: (a) that $h_i \cdot \nu \equiv 0$ on Γ_0, $i = 1, 2$ by assumption (10.1.2), so that integration where $h_i \cdot \nu$ occurs is restricted to Γ_1 only; (b) that by (10.2.14f) $w_i \equiv b_i(t) w$, $\frac{\partial w_i}{\partial \nu} \equiv b_i(t) \frac{\partial w}{\partial \nu}$ on an interior strip of Γ and for all $0 \leq t \leq T$, where $b_i(t)$ is defined by (10.2.8).

Step 2. We next estimate the first term on the left side of inequality (10.3.4). We shall prove that

$$\int_0^T \int_\Omega \theta_i^2 (w_{i,tt} - \Delta w_i)^2 d\Omega \, dt \leq C_T \left\{ \int_0^T \int_\Omega e^{2\tau\phi_i} [w_{i,t}^2 + |\nabla w_i|^2] d\Omega \, dt \right.$$

$$\left. + \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx \, dt + e^{2\tau\sigma} \int_0^T E(t) dt + \int_0^T \int_\Omega f_i^2 d\Omega \, dt \right\}, \tag{10.3.5}$$

where the subset $Q_i(\sigma)$ of $[0,T] \times \Omega$ is defined in (10.1.18). In fact, to prove (10.3.5), we introduce the following simplified notation on $[0,T] \times \Omega$:

$$e_i(t,x) \quad \equiv \quad w_{i,t}^2(t,x) + |\nabla w_i(t,x)|^2 + w_i^2(t,x); \tag{10.3.6}$$

$$e(t,x) \quad \equiv \quad w_t^2(t,x) + |\nabla w(t,x)|^2 + w^2(t,x); \tag{10.3.7}$$

$$E_i(t) \equiv \int_\Omega e_i(t,x)d\Omega; \qquad E(t) \equiv \int_\Omega e(t,x)d\Omega. \tag{10.3.8}$$

As in the proof of Theorem 5.1, Step 3, we further split $Q \equiv [0,T]\times\Omega = Q_i(\sigma)\cup[Q_i(\sigma)]^c$, where $[\quad]^c$ denotes complement with respect to Q, and the set $Q_i(\sigma)$ is defined by (10.1.18). Moreover, we have

$$\begin{cases} \phi_i(t,x) \le \sigma \text{ on } [Q_i(\sigma)]^c \text{ by (10.1.18); } e_i(t,x) \le \text{ const } e(t,x) \\ \qquad \text{in particular on } [Q_i(\sigma)]^c, \text{ by (10.2.10), (10.2.13);} \\ \\ e_i(t,x) \equiv e(t,x) \text{ on } Q_i(\sigma) \text{ by (10.2.12), i.e., } \chi_i \equiv 1 \text{ on } Q_i(\sigma). \end{cases} \tag{10.3.9}$$

We next invoke inequality (10.2.15), as well as (10.3.6)–(10.3.9), recall $\theta_i = e^{\tau\phi_i}$, and estimate

$$\int_0^T \int_\Omega \theta_i^2 (w_{i,tt} - \Delta w_i)^2 d\Omega\, dt$$

$$\le C_T \left\{ \int_0^T \int_\Omega \theta_i^2 [e_i(t,x) + f_i^2 + e(t,x)] d\Omega\, dt \right\} \tag{10.3.10}$$

$$\le C_T \left\{ \int_{Q_i(\sigma)} e^{2\tau\phi_i}[e_i(t,x) + e(t,x)]dx\, dt + \int_{[Q_i(\sigma)]^c} e^{2\tau\phi_i}[e_i(t,x) \right.$$

$$\left. + e(t,x)]dx\, dt + \int_0^T \int_\Omega f_i^2 d\Omega\, dt \right\} \tag{10.3.11}$$

(by (10.3.9))
$$\le C_T \left\{ \int_{Q_i(\sigma)} e^{2\tau\phi_i} 2 e_i(t,x) dx\, dt \right.$$

$$\left. + \text{ const } \int_{[Q_i(\sigma)]^c} e^{2\tau\sigma} e(t,x)dx\, dt + \int_0^T \int_\Omega f_i^2 d\Omega\, dt \right\} \tag{10.3.12}$$

$$\le C_T \left\{ 2 \int_{Q_i(\sigma)} e^{2\tau\phi_i} e_i(t,x) dx\, dt \right.$$

$$\left. + \text{ const } e^{2\tau\sigma} \int_0^T \int_\Omega e(t,x) d\Omega\, dt + \int_0^T \int_\Omega f_i^2 d\Omega\, dt \right\}. \tag{10.3.13}$$

Finally, we estimate, since $Q_i(\sigma)$ is a subset of $[0,T]\times\Omega$, via (10.3.6):

$$\int_{Q_i(\sigma)} e^{2\tau\phi_i} e_i(t,x) dx\, dt = \int_{Q_i(\sigma)} e^{2\tau\phi_i}[w_i^2(t,x) + w_{i,t}^2(x,t) + |\nabla w_i(x,t)|^2] dx\, dt$$

$$\leq \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt + \int_0^T \int_\Omega e^{2\tau\phi_i}[w_{i,t}^2 + |\nabla w_i|^2] d\Omega\, dt, \tag{10.3.14}$$

by majorizing the integral term on $Q_i(\sigma)$ but only for the integrand $[w_{i,t}^2 + |\nabla w_i|^2]$. Finally, we insert (10.3.14) into the right side of (10.3.13) and obtain (10.3.5) by invoking (10.3.8).

Step 3. Next, we use the counterpart of inequality (5.9) in the present case, to obtain, recalling (10.1.7) for ϕ_i at $t = 0$ and $t = T$:

$$\left| \left[\int_{\Omega_i} M_i d\Omega_i \right]_0^T \right| \leq c_T \tau^3 \left[\int_{\Omega_i} e^{2\tau\phi_i} [w_{i,t}^2 + |\nabla w_i|^2 + w_i^2] d\Omega_i \right]_0^T \tag{10.3.15}$$

$$\text{(by (10.1.7))} \quad \leq c_T \tau^3 e^{-2\tau\delta} [E_i(0) + E_i(T)] \tag{10.3.16}$$

$$\text{(by (10.2.13))} \quad \leq c_T \tau^3 e^{-2\tau\delta} [E(0) + E(T)], \tag{10.3.17}$$

where $E_i(t)$ and $E(t)$ are defined by (10.3.8), and where in the last step we have recalled $w_i = \chi_i w$ from (10.2.13).

Step 4. Thus, inequalities (10.3.5) and (10.3.17) used on the left side of estimate (10.3.4), yield

$$C_T e^{2\tau\sigma} \int_0^T E(t) dt + C_T \int_0^T \int_\Omega f_i^2 d\Omega\, dt + (BT)_{w_i}|_\Sigma$$

$$\geq (\epsilon\tau\rho - C_T) \int_0^T \int_\Omega e^{2\tau\phi_i} [w_{i,t}^2 + |\nabla w_i|^2] d\Omega\, dt + \int_0^T \int_\Omega e^{2\tau\phi_i} B_i w_i^2 d\Omega\, dt$$

$$- C_T \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt - c_T \tau^3 e^{-2\tau\delta} [E(0) + E(T)]. \tag{10.3.18}$$

Step 5. In this step, we refine (10.3.18) to arrive at (10.3.1) by estimating the integral term containing B_i in (10.3.18). To this end, we proceed as in the proof of Theorem 5.1, Step 1. We split now $[0, T] \times \Omega = Q_i^*(\sigma^*) \cup [Q_i^*(\sigma^*)]^c$, where $[Q_i^*(\sigma^*)]^c$ is the complement in $[0, T] \times \Omega$ of the set $Q_i^*(\sigma^*)$ defined by (10.1.15). It is here that we use the critical property that B_i is strictly positive on $Q_i^*(\sigma^*)$ noted in (10.2.17) via (10.1.14). We compute

$$\int_0^T \int_\Omega e^{2\tau\phi_i} B_i w_i^2 d\Omega\, dt = \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i} B_i w_i^2 dx\, dt + \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} B_i w_i^2 dx\, dt \tag{10.3.19}$$

$$\text{(by (10.2.17))} \quad \geq \quad [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i} w_i^2 dx\, dt$$

$$+ \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} B_i w_i^2 dx\, dt. \tag{10.3.20}$$

As to the last term on the right of (10.3.20), we estimate as in the proof of Theorem 5.1, Step 2, recalling from its definition in (10.2.18) that B_i is $\mathcal{O}(\tau^3)$, and that $\phi_i \leq \phi_i^* \leq \sigma^*$ in $[Q_i^*(\sigma^*)]^c$, by the very definition of (10.1.15) and by (10.1.17),

$$-\int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} B_i w_i^2 dx\, dt \leq C\tau^3 \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\phi_i} w_i^2 dx\, dt \tag{10.3.21}$$

$$\leq \quad C\tau^3 \int_{[Q_i^*(\sigma^*)]^c} e^{2\tau\sigma^*} w_i^2 dx\, dt \leq C\tau^3 e^{2\tau\sigma^*} \int_0^T \int_\Omega w^2 dx\, dt, \tag{10.3.22}$$

majorizing w_i^2 by const w^2, and $[Q_i^*(\sigma^*)]^c$ by $[0,T] \times \Omega$. Next, we insert (10.3.20) in the right side of (10.3.18), move the last term of (10.3.20) on $[Q_i^*(\sigma^*)]^c$ to the left side of (10.3.18), and apply for it estimate (10.3.22). We obtain

$$C_1\tau^3 e^{2\tau\sigma^*} \int_0^T \int_\Omega w^2 d\Omega\, dt + C_T e^{2\tau\sigma} \int_0^T E(t)dt + C_T \int_0^T \int_\Omega f_i^2 d\Omega\, dt + (BT)_{w_i}|_\Sigma$$

$$\geq \quad (\epsilon\tau\rho - C_T) \int_0^T \int_\Omega e^{2\tau\phi_i} [w_{i,t}^2 + |\nabla w_i|^2] d\Omega\, dt$$

$$+ [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i} w_i^2 dx\, dt$$

$$- C_T \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt - c_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)]. \tag{10.3.23}$$

Finally, we use the following estimates:

(i) Regarding the first two terms on the left of (10.3.23), we use that, by selection, $0 < \sigma^* < \sigma$, see (10.1.15)–(10.1.19). Thus, for all τ sufficiently large, the first term is absorbed by the second, and we obtain

$$C\tau^3 e^{2\tau\sigma^*} \int_0^T \int_\Omega w^2 d\Omega\, dt \leq C_{1,T} e^{2\tau\sigma} \int_0^T E(t)dt. \tag{10.3.24}$$

(ii) Regarding the integral terms on $Q_i^*(\sigma^*)$, $Q_i(\sigma)$, we see that since $Q_i^*(\sigma^*) \supset Q_i(\sigma)$, we have that for all τ sufficiently large, as in Step 3 in the proof of Theorem 5.1,

$$[2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q_i^*(\sigma^*)} e^{2\tau\phi_i} w_i^2 dx\, dt - C_T \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt$$

$$\geq \quad [2\tau^3\beta + \mathcal{O}(\tau^2)] \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt - C_T \int_{Q_i(\sigma)} e^{2\tau\phi_i} w_i^2 dx\, dt \tag{10.3.25}$$

$$= \quad [2\tau^3\beta + \mathcal{O}(\tau^2) - C_T] \int_{Q_i(\sigma)} e^{2\tau\phi_i} w^2 dx\, dt, \tag{10.3.26}$$

since, on $Q_i(\sigma)$, we have that $\chi_i \equiv 1$, by (10.2.12), hence $w_i \equiv w$ by (10.2.13). Thus, using (10.3.24) and (10.3.25) in (10.3.23), we arrive at the derived estimate (10.3.1). The proof of Proposition 11.3.1 is complete.

10.4 Carleman estimate, first version, for the w-problem

The counterpart of Theorem 5.1(ii) is now:

Theorem 10.4.1. Let $w \in H^{2,2}(Q)$ be a solution of Eqn. (1.1.1a). Let the setting of Section 10.1 based on assumptions (A.1i) and (A.2i) be in force. Then

(i) for all $\tau > 0$ sufficiently large (with $\epsilon\tau$ large, as compared to the constant C_T in (1.1.2)), the following one-parameter family of inequalities holds true:

$$BT|_\Sigma + 2C_T \int_0^T \int_\Omega f^2 d\Omega\, dt \quad \geq \quad 2(\epsilon\tau\rho - 2C_T)e^{2\tau\sigma} \int_{t_0}^{t_1} \mathcal{E}(t)dt - 2C_{1,T}e^{2\tau\sigma} \int_0^T E(t)dt$$

$$- \; 2c_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)], \tag{10.4.1}$$

where

$$BT|_\Sigma = \sum_{i=1}^{2} (BT)_{w_i}|_\Sigma + 2(\epsilon\tau\rho - 2C_T)e^{2\tau\sigma} \int_{t_0}^{t_1} \int_{\Gamma_1} w^2 d\Gamma_1 dt, \tag{10.4.2}$$

with $(BT)_{w_i}|_\Sigma$ is defined in (10.3.2), (10.3.3); while $\mathcal{E}(t)$ and $E(t)$ are defined by (6.3) and (5.5), respectively.

(ii) By virtue of Theorem 8.2 and Remark 8.1, the above estimate (10.4.1) may be extended to $H^{1,1}(Q)$-solutions of (1.1.1a) in the class (8.1).

Proof. Step 1. We recall property (10.1.8): $\phi_i \geq \sigma > 0$ on $[t_0, t_1] \times \Omega_i$; finally, that $[t_0, t_1] \times \Omega_i \subset Q_i(\sigma)$ by (10.1.19), where $w_i \equiv w$ on $Q_i(\sigma)$ by $\chi_i \equiv 1$ on $Q_i(\sigma)$, see (10.2.12) and (10.2.13). Thus, we estimate the first terms on the right side of (10.3.1) for $i = 1, 2$:

$$\sum_{i=1}^{2} \int_0^T \int_\Omega e^{2\tau\phi_i}[w_{i,t}^2 + |\nabla w_i|^2] d\Omega\, dt$$

$$\text{(by (10.1.8))} \qquad \geq \quad \sum_{i=1}^{2} \int_{t_0}^{t_1} \int_{\Omega_i} e^{2\tau\sigma}[w_{i,t}^2 + |\nabla w_i|^2] d\Omega_i\, dt$$

$$
\text{(as } w_i \equiv w \text{ on } Q_i(\sigma) \supset [t_0,t_1]\times\Omega_i) \qquad = \quad e^{2\tau\sigma}\sum_{i=1}^{2}\int_{t_0}^{t_1}\int_{\Omega_i}[w_t^2+|\nabla w|^2]d\Omega_i\,dt
$$

$$
\text{(by (10.1.1))} \qquad \geq \quad e^{2\tau\sigma}\int_{t_0}^{t_1}\int_{\Omega}[w_t^2+|\nabla w|^2]d\Omega\,dt, \quad (10.4.3)
$$

where the last step follows, since the integral terms over Ω_i, $i=1,2$, collects also contributions on the non-empty portion $\Omega_1\cap\Omega_2$, see (10.1.1).

Step 2. With τ sufficiently large as to have the coefficients $[2\tau^3\beta+\mathcal{O}(\tau^2)-C_T]>0$, so that Remark 10.3.1 applies, we sum up Eqn. (10.3.1) of Proposition 10.3.1 for $i=1,2$, and obtain also by virtue of (10.4.3),

$$
2C_{1,T}e^{2\tau\sigma}\int_0^T E(t)dt + C_{2,T}\sum_i\int_0^T\int_\Omega f_i^2 d\Omega\,dt + \sum_i (BT)_{w_i}|_\Sigma
$$

$$
\geq \; 2(\epsilon\tau\rho-2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_\Omega[w_t^2+|\nabla w|^2]d\Omega\,dt - 2c_T\tau^3e^{-2\tau\delta}[E(0)+E(T)] \quad (10.4.4)
$$

Step 3. Finally, adding to both sides of inequality (10.4.4) the term

$$
\left[2(\epsilon\tau\rho-2C_T)e^{2\tau\sigma}\int_{t_0}^{t_1}\int_{\Gamma_1} w^2 d\Gamma_1 dt\right],
$$

and invoking the definition of $\mathcal{E}(t)$ in (6.3), we readily obtain the desired inequality (10.4.1), with $BT|_\Sigma$ as in (10.4.2). $\square$

10.5 Carleman estimate, second version, for the w-problem

Our main final step in establishing Theorem 10.1.1 is the following counterpart of Theorem 6.1, Eqn. (6.6).

Theorem 10.5.1. Let w be a $H^{2,2}(Q)$-solution of problem (1.1.1), including the Neumann B.C. on Σ. Let the setting of Section 10.1, based on assumptions (A.1i) and (A.2i), be in force. Then:

(i) for $\epsilon>0$ small as in (10.2.18), and for all $\tau>0$ sufficiently large (with $\epsilon\tau$ large as compared to the constant C_T in (1.1.2)), the following one-parameter family of inequality holds true: there is a constant $k_{1\phi}>0$, depending on the pseudo-convex functions, such that

$$
\overline{BT}|_\Sigma + \int_0^T\int_\Omega f^2 d\Omega\,dt \geq k_{1\phi}[E(0)+E(T)], \quad (10.5.1)
$$

where, recalling (10.4.2) for $BT|_\Sigma$, we have

$$\begin{aligned}\overline{BT}|_\Sigma &= BT|_\Sigma + C_\phi \int_0^T \int_{\Gamma_1} |ww_t| d\Gamma_1 dt & \text{(10.5.2a)}\\ &\leq \text{const} \left\{ \int_0^T \int_{\Gamma_1} [w^2 + w_t^2] d\Gamma_1 dt \right. \\ &\quad + \left. \sum_i \int_0^T \int_{\Gamma_1} e^{2\tau\phi_i} [w_t^2 - |\nabla_{\tan} w|^2] b_i^2 h_i \cdot \nu \, d\Gamma_1 dt \right\}. & \text{(10.5.2b)}\end{aligned}$$

Proof. Same as the proof of Theorem 6.1. First we note that, with the Neumann B.C. (1.1c), $\frac{\partial w}{\partial \nu} = 0$ on Σ in force now, we have that the term $N(T)$ in (6.14) specializes to

$$N(T) = \int_0^T \int_\Omega f^2 dQ + 2 \int_0^T \int_{\Gamma_1} |ww_t| d\Sigma_1. \tag{10.5.3}$$

Thus, with reference to estimate (10.4.1) of Theorem 10.4.1, and recalling (6.18), we obtain

$$\begin{aligned}&2(\epsilon\tau\rho - 2C_T) e^{2\tau\sigma} \int_{t_0}^{t_1} \mathcal{E}(t) dt \\ &\qquad \geq \; 2(\epsilon\tau\rho - 2C_T) e^{2\tau\sigma} (t_1 - t_0) \left\{ \frac{a}{2} [E(T) + E(0)] e^{-C_T T} - N(T) \right\}. \qquad \text{(10.5.4)}\end{aligned}$$

Moreover, recalling (6.20), we obtain

$$-2C_{1,T} e^{2\tau\sigma} \int_0^T E(t) dt \geq -C_{1,T} e^{2\tau\sigma} \frac{T e^{C_T T}}{a} \left\{ b[E(T) + E(0)] - 2N(T) \right\}. \tag{10.5.5}$$

Adding up (10.5.4)–(10.5.6), we can estimate the right-hand side (RHS) of (10.4.1) as follows:

$$\begin{aligned}\text{RHS of (10.4.1)} &= 2(\epsilon\tau\rho - 2C_T) e^{2\tau\sigma} \int_{t_0}^{t_1} \mathcal{E}(t) dt \\ &\quad - 2C_{1,T} e^{2\tau\sigma} \int_0^T E(t) dt - 2c_T \tau^3 e^{-2\tau\delta} [E(0) + E(T)] \quad \text{(10.5.6)} \\ &\geq \left\{ \left[(\epsilon\tau\rho - 2C_T)(t_1 - t_0) a e^{-C_T T} - C_{1,T} \frac{b}{a} T e^{C_T T} \right] e^{2\tau\sigma} \right. \\ &\quad \left. - 2c_T \tau^3 e^{-2\tau\delta} \right\} [E(T) + E(0)] - \text{const}_\phi N(T). \qquad \text{(10.5.7)}\end{aligned}$$

We now use critically that $0 < \sigma$, $\delta > 0$, see (10.1.8), (10.1.6), that, for large τ, $[\tau e^{2\tau\sigma} - \tau^3 e^{-2\tau\delta}]$ is positive. We then obtain that: there exists a critically positive constant $k_{1\phi} > 0$, and a constant $k_{2\phi}$ (also positive, but this is not critical) depending on the pseudo-convex functions ϕ_i, such that for τ sufficiently large we have:

$$\text{RHS of (10.4.1)} \geq k_{1,\phi}[E(T) + E(0)] - k_{2,\phi} N(T). \tag{10.5.8}$$

Then, using (10.5.8) in (10.4.1), moving $N(T)$ on the left side of (10.4.1) and invoking (10.5.3) for it, we readily find estimate (10.5.1). Moreover, $\overline{BT}|_\Sigma$ is given by (10.5.2a).

We now establish (10.5.2b): for this, we return to (10.4.2) for $BT|_\Sigma$ and (10.3.3) for $(BT)_{w_i}|_\Sigma$, with $\frac{\partial w}{\partial \nu} \equiv 0$ on $(0, T] \times \Gamma$: thus, in (10.3.3), we see that the 3 terms (2nd, 3rd, and 4th) which involve $\frac{\partial w}{\partial \nu}$ on $[0, T] \times \Gamma$ vanish. That is, for $\frac{\partial w}{\partial \nu} \equiv 0$ on Σ, we obtain from (10.3.3),

$$\begin{aligned} (BT)_{w_i}|_\Sigma &= 2\tau \int_0^T \int_{\Gamma_1} e^{2\tau\phi_i} (w_{i,t}^2 - b_i^2(t)|\nabla w|^2) h_i \cdot \nu \, d\Gamma_1 \, dt \\ &\quad + 2\tau \int_0^T \int_{\Gamma_1} e^{2\tau\phi_i} \left[2\tau^2 \left(|h_i|^2 - 4c^2 \left(t - \frac{T}{2} \right)^2 \right) \right. \\ &\quad \left. + \tau(\alpha_i - \Delta d_i - 2c) \right] w^2 b_i^2(t) h_i \cdot \nu \, d\Gamma_1 \, dt, \end{aligned} \tag{10.5.9}$$

where $|\nabla_{\tan} w| = |\nabla w|$ since $\frac{\partial w}{\partial \nu} \equiv 0$ on Γ, and where, by (10.2.14e), near Γ we have

$$w_{i,t}^2 = \left[\frac{\partial}{\partial t}(b_i(t) w) \right]^2 = (b_i'(t))^2 w^2 + b_i^2(t) w_t^2 + 2 b_i(t) b_i'(t) w w_t, \tag{10.5.10}$$

in an $\frac{\epsilon_1}{4}$-internal layer of Γ. Thus, substituting (10.5.10) into the first integral of (10.5.9) readily yields (10.5.2b) by recalling also (10.4.2). The proof of Theorem 10.5.1 is complete. □

10.6 Global uniqueness: Theorem 10.1.1(b)

Let $w \in H^{1,1}(Q)$ be a solution of Eqn. (1.1.1a) with $f \equiv 0$, satisfying $\frac{\partial w}{\partial \nu} \equiv 0$ on Σ, as in (1.1.1c), and, in addition, $w|_{\Sigma_1} \equiv 0$, where $h_i \cdot \nu = 0$ in Γ_0, $i = 1, 2$, as in (10.1.22). Then, $\nabla_{\tan} w \equiv 0$ on Σ_1 as well, and thus $\overline{BT}|_\Sigma = 0$ by (10.5.2b) of Theorem 10.5.1(ii). Thus, estimate (10.5.1) with $f \equiv 0$ yields $E(0) = 0$, hence $w \equiv 0$ in Q, in fact in $\mathbb{R}_t \times \Omega$, as in the proof of Theorem 7.1. □

10.7 Continuous observability: Theorem 10.1.1(a)

To complete the proof of Theorem 10.1.1(a) and thus obtain the final estimate (10.1.20) from estimate (10.5.1) already proved with $\overline{BT}|_\Sigma$ satisfying (10.5.2b), we proceed as in the proof of Theorem 9.2. that is, we invoke Lemma 9.1 on the interval $[\epsilon, T-\epsilon]$, $\epsilon > 0$ small, see (9.4) and ff. In particular, we make use of the global uniqueness already established, in Section 10.6, as stated in Theorem 10.1.1(b), to absorb the interior l.o.t., as in Step 2 in the proof of Theorem 9.2. This way the constant C_T in (10.1.20) is *not* explicit. However, if we assume also (10.1.21) that is

$$h_i \cdot \nu \geq 0 \qquad \text{on } \Gamma_1, \tag{10.7.1}$$

then we may drop the terms

$$\int_0^T \int_{\Gamma_1} e^{2\tau\phi_i}(-|\nabla_{\tan} w|^2) h_i \cdot \nu \, d\Sigma_1 \leq 0, \quad i = 1, 2, \tag{10.7.2}$$

in (10.5.2b) and obtain

$$BT|_{\Sigma} \leq \text{const} \int_0^T \int_{\Gamma_1} [w^2 + w_t^2] d\Gamma_1 dt, \tag{10.7.3}$$

as desired, with an explicit constant C_T of the order Ce^{Cr^2}, where r is defined by (1.1.2c). The proof of Theorem 10.1.1(a) is complete. $\square$

Appendix A: Classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ satisfying assumptions (A.1) and (A.2). Setting of Section 1.

Orientation. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with boundary $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$; Γ_0 is the uncontrolled or unobserved part of the boundary, while Γ_1 is the controlled or observed part of the boundary. The purpose of Appendices A through C is to illustrate the claim that: assumptions (A.1) and (A.2) of Section 1, or their counterpart version (A.1i) and (A.2i) of Section 10, hold true for large classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$. Moreover, in some subcases—typically when Γ_0 is *flat*—it is possible to satisfy the additional assumption (10.1.21): $h_i \cdot \nu \geq 0$ on Γ_1, $i = 1, 2$, in the setting of Section 10, although in many cases this requirement $h \cdot \nu \geq 0$ on Γ_1 in addition to the setting of Section 1 is incompatible with the requirement (A.2) = (1.1.6) that the inf $|h(x)|$ over $\bar{\Omega}$ be positive.

We shall provide a few approaches addressing these assumptions and present a few rather general results illustrated by canonical examples. Several other illustrative examples may be given where the unobserved boundary Γ_0 is given analytically by the common elementary functions. However, an exhaustive analysis of these geometrical assumptions, or even a presentation of the most general results within each approach, is beyond the scope of the present Appendices.

A.1 The case where Γ_0 is flat: Explicit construction of $h(x)$ for the setting of Section 1 and the setting of Section 10

In this special but important case where Γ_0 is flat, we can readily consider the setting of Section 1 and the setting of Section 10 in one shot. Another approach for the setting of Section 10 will be given in Appendix C.

Setting of Section 1. Let the triple $\{\Omega, \Gamma_0, \Gamma_1\}$ be given, $\Omega \subset \mathbb{R}^n$, where Γ_0 is assumed flat. Then, we take any point x_0 on the hyperplane containing Γ_0, with $x_0 \notin \bar{\Omega}$, and define the radial field $h(x) = 2(x-x_0) = 2\nabla d(x)$ where $d(x) = \|x-x_0\|^2$. See Fig. A.1. Then, the Jacobian matrix $J_h(x)$ of h is twice the identity matrix; $h \cdot \nu = 0$ on Γ_0; and $|h(x)| \geq p > 0$, $\forall\, x \in \Omega$. Accordingly, see (1.1.21b), the time $T_0 = 2\times$ (diameter of Ω) is then optimal in this case. Thus, assumptions (A.1) and (A.2) hold true for this large, yet special class. However, the additional requirement $h \cdot \nu \geq 0$ on Γ_1 is incompatible, in this case, with the condition that $\inf|h(x)|$ over Ω be positive.

Setting of Section 10. Now we decompose $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let x_1 and x_2 be two points in common to $\bar{\Gamma}_0$ and $\bar{\Gamma}_1$, with x_1 at a finite distance from Ω_1 and x_2 at a finite distance from Ω_2. See Fig. A.2. We then define $h_i(x) = 2(x - x_i) = 2\nabla d_i(x)$, $d_i(x) = \|x - x_i\|^2$, $i = 1, 2$. Then, assumptions (A.1i) and (A.2i) of Section 10 are satisfied: $h_i \cdot \nu = 0$ on Γ_0, since each x_i lies on the hyperplane containing the flat Γ_0; $J_{h_i} = 2(\text{Identity})$, and $\|h_i(x)\| \geq p > 0$, $\forall\, x \in \bar{\Omega}_i$. Moreover, in this case, we may also satisfy assumption (10.1.21): $h_i \cdot \nu \geq 0$ on Γ_1, $i = 1, 2$.

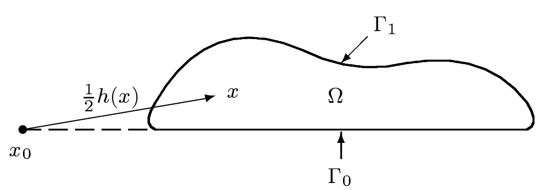

Fig. A.1: Setting of Section 1: The required vector field $h = 2(x - x_0)$ satisfying (A.1), (A.2) when Γ_0 is flat

Fig. A.2: Setting of Section 10: The Required Vector Fields $h_i = 2(x - x_i)$ Satisfying (A.1i), (A.2i), and (10.1.21): $h_i \cdot \nu \geq 0$ on Γ_1

A.2 A first approach where Γ_0 is curved: Explicit construction of $h(x,y)$, 2-dimensional case. Setting of Section 1

Orientation. In this section we consider the two-dimensional case, $\dim \Omega = 2$, where the construction of the required vector field $h(x)$ is most transparent. However, the given treatment admits a natural generalization to higher dimensions as well. See Appendix B.2.

In the two-dimensional case, our approach will, *in particular*, encompass the following result.

Theorem A.2.1. Let the uncontrolled (or unobserved) boundary Γ_0 satisfy the following assumptions:

(i) Γ_0 is described by the graph

$$y = \begin{cases} f_1(x) & x_0 \le x \le x_1,\ y \ge 0; \quad \text{(A.2.1)} \\ f_2(x) & x_0 \le x \le x_1,\ y < 0, \quad \text{(A.2.2)} \end{cases}$$

where the functions $f_i(x)$ are of class C^3 and satisfy

$$\begin{cases} f_1(x_1) = f_2(x_1) = 0;\ f_1'(x_1) = -\infty;\ f_2'(x_1) = \infty; \\ f_2''(x_1) = -\infty;\ f_2''(x_1) = +\infty;\ f_i'(x) \neq 0,\ x_0 \le x \le x_1,\ i = 1, 2; \end{cases} \tag{A.2.3}$$

so that the graph of Γ_0 does nowhere have horizontal tangent, while it has vertical tangent at x_1.

(ii) Both $f_1(x)$ and $f_2(x)$ are *logarithmic concave* on $x_0 < x < x_1$; equivalently (Remark A.2.1), they satisfy the following conditions:

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_1'(x)}\right) > 0 \quad \text{and} \quad \frac{d}{dx}\left(\frac{f_2(x)}{f_2'(x)}\right) > 0, \text{ on } x_0 \le x < x_1. \tag{A.2.4}$$

Then, there exists a conservative vector field $h(x,y) = \{h_1, h_2\}$, constructively defined in a neighborhood of $\Gamma_0 \setminus \{x_1, 0\}$ [the boundary Γ_0 with the point $(x_1, 0)$ removed] by:

$$\begin{cases} h_1(x) = \dfrac{f_1(x)}{f_1'(x)}, \quad h_2(y) = y, \text{ for } x_0 \le x \le x_1;\ y \ge -\epsilon(x - x_1); \quad \text{(A.2.5a)} \\ h_1(x) = \dfrac{f_2(x)}{f_2'(x)}, \quad h_2(y) = y, \text{ for } x_0 \le x \le x_1;\ y \le \epsilon(x - x_1), \quad \text{(A.2.5b)} \end{cases}$$

for a sufficiently small $\epsilon > 0$, such that on its domain of definition specified above, we have:

(a)

$$h \cdot \nu = 0 \text{ on } \Gamma_0;\ J_h(x,y) = \begin{bmatrix} \dfrac{d}{dx}\left(\dfrac{f_i(x)}{f_i'(x)}\right) & 0 \\ 0 & 1 \end{bmatrix} > 0, \tag{A.2.6}$$

where $i = 1$ for $y \geq 0$, and $i = 2$ for $y < 0$; and moreover,

(b)

$$h = \nabla d, \text{ where } d(x, y) = \int^x h_1(\xi) d\xi + \frac{y^2}{2} + C. \tag{A.2.7}$$

If the graph of Γ_0 is symmetric with respect to the x-axis, i.e., if $f_1(x) = -f_2(x)$, then we may take $\epsilon = 0$ in (A.2.5).

(c) Finally, the above vector field $h(x, y)$ can be extended by continuity in the triangular region: $\epsilon(x - x_1) < y < -\epsilon(x - x_1)$; $x_0 \leq x < x_1$, as well as for $x < x_0$, so that the condition $J_h(x, y) > 0$ is preserved. $\square$

Remark A.2.1. Let $f \in C^2$, either $f(x) > 0$ or $f(x) < 0$. Define

$$G(x) \equiv \ln |f(x)|, \text{ so that } G'(x) = \frac{f'(x)}{f(x)} \equiv \frac{1}{F(x)}. \tag{A.2.8}$$

Then [concerning assumptions (A.2.4)] we have that

$$G''(x) = \frac{f''(x)f(x) - [f'(x)]^2}{f^2(x)} = -\frac{F'(x)}{F^2(x)} < 0 \Longleftrightarrow F'(x) = \frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right) > 0, \tag{A.2.9}$$

in which case $G(x)$ is strictly concave and so $f(x)$ is logarithmic concave.

If $f''(x) \leq 0$ (f is concave) and, without loss of generality modulo a translation, $f(x) \geq 0$, then condition (A.2.9) is satisfied. Similarly, if $f''(x) \geq 0$ (f is convex) and, without loss of generality modulo a translation, $f(x) \leq 0$, then condition (A.2.9) is satisfied as well.

For the purposes of Figure A.3, which illustrates a possible graph Γ_0 covered by Theorem A.2.1, we note that $f(x) = \sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $f(x) = \cos x$, $0 < x < \pi$ are logarithmic concave; i.e., satisfy (A.2.9). But they are neither convex nor concave. $\square$

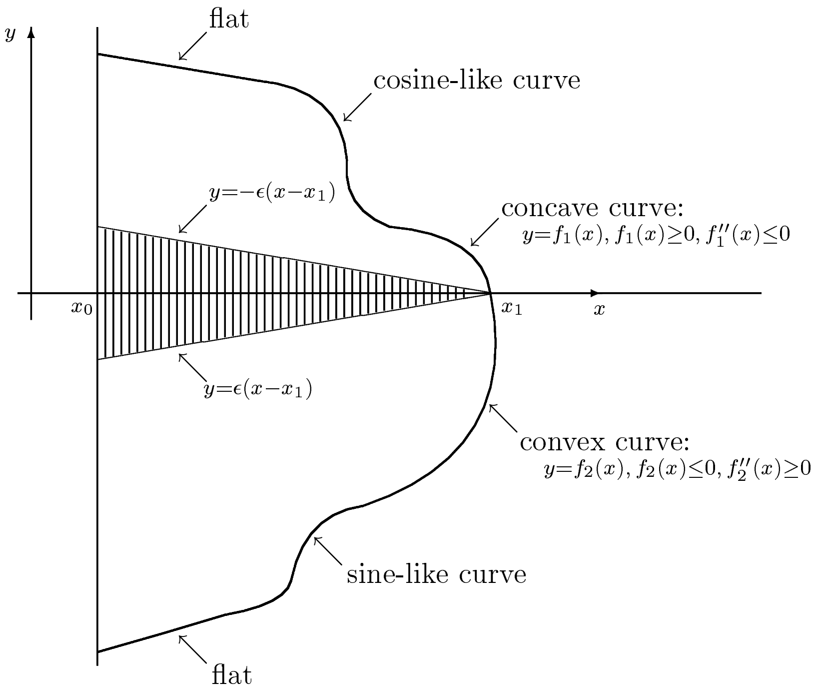

FIG. A.3. A POSSIBLE CONFIGURATION OF Γ_0 COVERED BY THEOREM A.2.1. Ω MAY BE ON EITHER SIDE OF Γ_0.

Geometrical description of the approach encompassing Theorem A.2.1. Let $\{\Omega, \Gamma_0, \Gamma_1\}$, $\partial\Omega = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, Γ_0 and Γ_1 relatively open in Γ, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let the (non-intersecting) curve Γ_0 be given explicitly as a level set by the equation

$$\Gamma_0 = \{(x,y) \in \mathbb{R}^2 : \ \ell(x,y) = 0\}, \quad \ell \in C^3. \tag{A.2.10}$$

with $|\nabla \ell| \neq 0$ on Γ_0, for a suitable domain in (x,y).

Assumption (A.1) of Section 1. As exemplified by the statement of Theorem A.2.1, we shall explicitly construct a conservative vector field $h = \{h_1, h_2\}$, such that the following conditions are satisfied:

$$h \cdot \nu \equiv 0 \text{ on } \Gamma_0; \quad J_h(x,y) > 0 \text{ near } \Gamma_0; \tag{A.2.11}$$

J_h being the Jacobian matrix of h. Moreover, it will be possible to extend smoothly such $h(x,y)$ so that $J_h(x,y) > 0$ on $\bar{\Omega}$, as well.

This way, *condition (A.1) of Section 1 is satisfied*. In fact, the constructed vector field will be, near Γ_0, of the form $h_1 = h_1(x)$, $h_2 = h_2(y)$, i.e., with first (second) component depending only on the first (second) coordinate, as in the statement of Theorem A.2.1.

Assumption (A.2) of Section 1. Depending upon the given unobserved/uncontrolled boundary Γ_0, the constructed vector field h may vanish at one point P_0 of Γ_0. This is the case if Γ_0 has a U-turn, as in the case of Theorem A.2.1, where $h(x_1, 0) = 0$. Then, the present setting of Section 1, assumption (A.2) can be satisfied only by restricting Γ_0; that is, by removing from Γ_0 a small neighborhood, on Γ_0, of the pathological point $P_0 \in \Gamma_0$, and assigning it instead to the observed/controlled boundary Γ_1. This way, neither the resulting, new Γ_0 nor the resulting, new observed boundary Γ_1 are connected. See also Remark 1.1.3. In Appendix C below, we will be able, in many cases, to avoid the above situation and overcome the related difficulty. To do this it will be necessary to rely on the 2-vector field setting of Section 10. This way, we will be able to keep, as unobserved boundary, the original portion Γ_0, even if it has a U-turn, and still fit the setting of Section 10. See Theorem C.1 in Appendix C.

First setting: Nowhere horizontal and nowhere vertical tangent to Γ_0. We shall at first introduce our geometrical construction of h in the case where Γ_0, the graph of ℓ, *has nowhere horizontal and nowhere vertical tangent* in the (x, y)-plane. Thus, the slope of Γ_0 is finite and either always positive, or else always negative. Locally the domain Ω may lie on either side of Γ_0. In this case, the construction of the required vector field h goes through the following two steps:

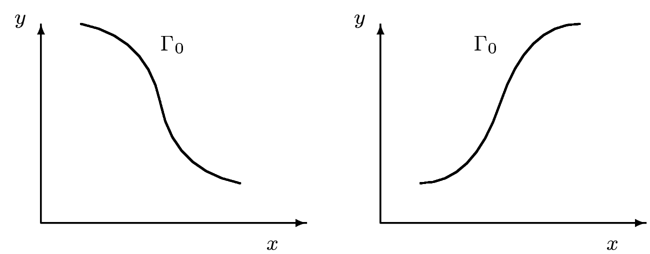

FIG. A.4: FIRST SETTING FOR Γ_0

Step 1: Definition of h on Γ_0. Let $P = (x, y) \in \Gamma_0$. Then, $\{\ell_x, \ell_y\}$ is a vector orthogonal to Γ_0 at P, and thus $\{-\ell_y, \ell_x\}$ is a vector tangent to Γ_0 at P. Multiplying this tangent vector either by $\frac{y}{\ell_x}$, or else by $\frac{-x}{\ell_y}$ respectively, we obtain two vectors

$$h = \overrightarrow{XP} = \left\{ -y\,\frac{\ell_y}{\ell_x},\, y \right\}, \text{ and respectively, } h = \overrightarrow{YP} = \left\{ x, -x\,\frac{\ell_x}{\ell_y} \right\}, \quad (x, y) \in \Gamma_0, \quad \text{(A.2.12)}$$

which are also tangent to Γ_0 at P. Geometrically, the points

$$X = \left\{ x + y\,\frac{\ell_y}{\ell_x},\, 0 \right\}, \text{ and } Y = \left\{ 0,\, y + x\,\frac{\ell_x}{\ell_y} \right\}, \quad \text{(A.2.13)}$$

are the intersection points of the tangent line $\frac{\xi - x}{-\ell_y} = \frac{\eta - y}{\ell_x}$ to Γ_0 at the point P with the x-axis and the y-axis, respectively. In many cases, the following assumption is satisfied by $\ell(x,y)$, and this is surely the case, at least locally, by the Implicit Function Theorem, which is valid under our assumptions:

$$\text{for } (x,y) \in \Gamma_0 : \begin{cases} \text{either the term } \left(-y\,\frac{\ell_y}{\ell_x}\right)(x) \text{ is only a function of } x; \\ \text{or else the term } \left(-x\,\frac{\ell_x}{\ell_y}\right)(y) \text{ is only a function of } y. \end{cases} \tag{A.2.14}$$

Assumption (A.2.14) holds true in particular (but not exclusively, see examples A.2.1 and A.2.2 below) when

$$\text{either } \Gamma_0 : \ell(x,y) = y - f(x) \equiv 0,\ a \le x \le b; \text{ or else } \Gamma_0 : \ \ell(x,y) = x - g(y) \equiv 0,\ c \le y \le d, \tag{A.2.15}$$

in which case, we obtain, respectively,

$$\left(-y\,\frac{\ell_y}{\ell_x}\right)(x) = \frac{f(x)}{f'(x)}; \qquad \left(-x\,\frac{\ell_x}{\ell_y}\right)(y) = \frac{g(y)}{g'(y)}, \qquad (x,y) \in \Gamma_0, \tag{A.2.16}$$

with finite $f'(x) \neq 0$, or finite $g'(y) \neq 0$, respectively, as assumed.

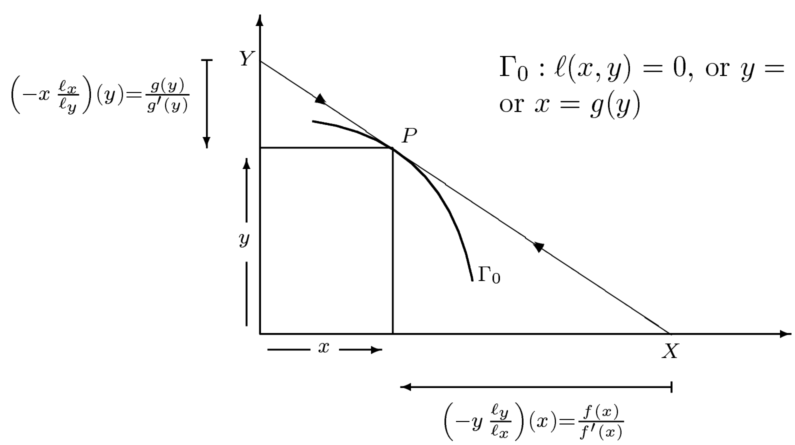

FIG. A.5: DEFINITION OF $h = \overrightarrow{XP}$ OR $h = \overrightarrow{YP}$ FOR $P \in \Gamma_0$

Step 2: Extension of h near Γ_0. We now extend h, defined by (A.2.12) on Γ_0, to any point $P = (x,y) \in \Omega$ near the graph of Γ_0 of ℓ. The horizontal and vertical lines through P meet Γ_0 at points P_h and P_v, respectively, on Γ_0. Let X_h and X_v [respectively, Y_h and Y_v] be the intersections of the tangent lines to Γ_0 at P_h and P_v with the x-axis [respectively, with the y-axis]

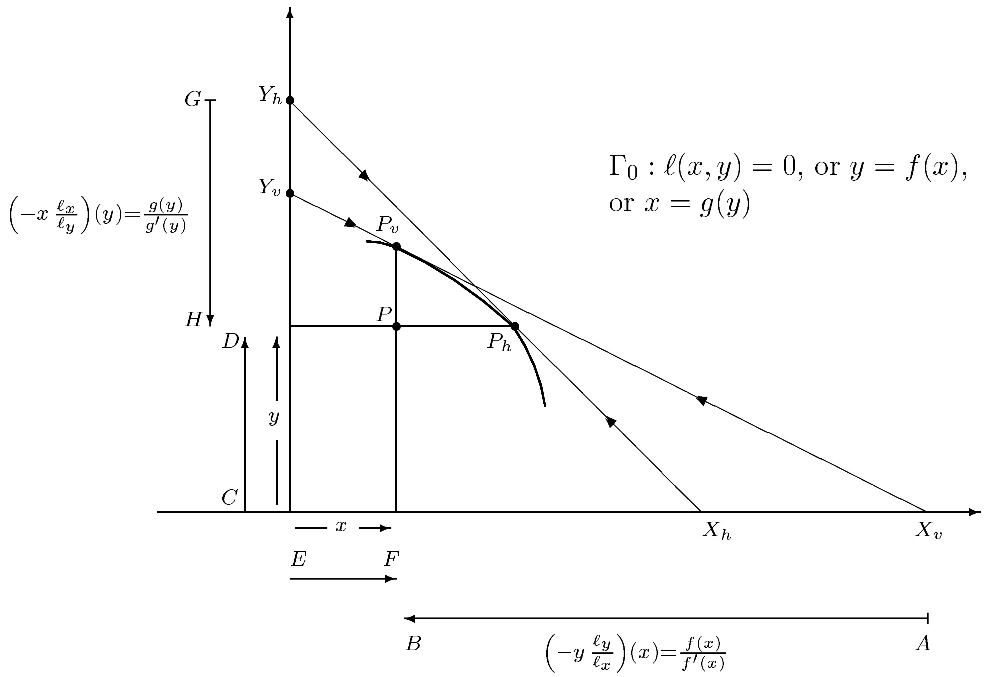

FIG. A.6: DEFINITION OF $h = \{h_1 = \overrightarrow{AB},\ h_2 = \overrightarrow{CD}\}$ FOR P NEAR Γ_0: $\overrightarrow{AB} =$ HORIZONTAL COMPONENT OF $\overrightarrow{X_vP_v}$; $\overrightarrow{CD}$ = VERTICAL COMPONENT OF $\overrightarrow{X_hP_h}$. OR ELSE, DEFINITION OF $h = \{h_1 = \overrightarrow{EF},\ h_2 = \overrightarrow{GH}\}$, $\overrightarrow{EF} =$ HORIZONTAL COMPONENT OF $\overrightarrow{Y_vP_v}$; $\overrightarrow{GH} =$ VERTICAL COMPONENT OF $\overrightarrow{Y_hP_h}$

At such nearby point $P = (x, y)$, we define the vector field $h = \{h_1, h_2\}$ by either

$$\begin{cases} h_1 = \text{horizontal component of } \overrightarrow{X_vP_v} = \left(-y\,\dfrac{\ell_y}{\ell_x}\right)(x), \quad \text{in particular, } \dfrac{f(x)}{f'(x)},\ a \le x \le b; \\ h_2 = \text{vertical component of } \overrightarrow{X_hP_h} = y; \end{cases} \tag{A.2.17}$$

or, respectively, by

$$\begin{cases} h_1 = \text{horizontal component of } \overrightarrow{Y_vP_v} = x; \\ h_2 = \text{vertical component of } \overrightarrow{Y_hP_h} = \left(-x\,\dfrac{\ell_x}{\ell_y}\right)(y), \quad \text{in particular, } \dfrac{g(y)}{g'(y)},\ c \le y \le d. \end{cases} \tag{A.2.18}$$

The Jacobian matrix of h is then either

$$J_h = \begin{bmatrix} \dfrac{d}{dx}\left(-y\,\dfrac{\ell_y}{\ell_x}\right)(x) & 0 \\ 0 & 1 \end{bmatrix}, \text{ in particular, } J_h = \begin{bmatrix} \dfrac{d}{dx}\left(\dfrac{f(x)}{f'(x)}\right) & 0 \\ 0 & 1 \end{bmatrix}, \tag{A.2.19}$$

for (A.2.17), or else

$$J_h = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{d}{dy}\left(-x\,\dfrac{\ell_x}{\ell_y}\right)(y) \end{bmatrix}, \text{ in particular, } J_h = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{d}{dy}\left(\dfrac{g(y)}{g'(y)}\right) \end{bmatrix}, \tag{A.2.20}$$

for (A.2.18), respectively.

Lemma A.2.2. Let $\ell \in C^3$ and assume hypothesis (A.2.14) with $\ell_x \neq 0$, or $\ell_y \neq 0$, respectively, unless a cancellation occurs, in particular, let either one of the situations in (A.2.15) hold true, with $f \in C^3$, $f'(x) \neq 0$; or $g \in C^3$, $g'(y) \neq 0$, respectively, unless a cancellation occurs.

(a) Assume, moreover, that either

$$\frac{d}{dx}\left(-y\,\frac{\ell_y}{\ell_x}\right)(x) > 0, \text{ in particular } \frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right) \equiv 1 - \frac{f(x)f''(x)}{[f'(x)]^2} > 0,\ a \leq x \leq b, \tag{A.2.21}$$

i.e., $f(x)$ is logarithmic concave, see Remark A.2.1; or else

$$\frac{d}{dy}\left(-x\,\frac{\ell_x}{\ell_y}\right)(y) > 0, \text{ in particular } \frac{d}{dy}\left(\frac{g(y)}{g'(y)}\right) \equiv 1 - \frac{g(y)g''(y)}{[g'(y)]^2} > 0,\ c \leq y \leq d, \tag{A.2.22}$$

respectively, i.e., $g(y)$ is logarithmic concave, see Remark A.2.1. Then:

(i) the Jacobian matrix J_h is positive definite near Γ_0, in either case;

(ii) the vector field $h = \{h_1, h_2\}$ defined in (A.2.17), or respectively (A.2.18), satisfies $h = \overrightarrow{XP}$, or $h = \overrightarrow{YP}$ for $(x, y) = P \in \Gamma_0$, respectively, and thus $h \cdot \nu \equiv 0$ on Γ_0;

(iii) the function $d(x, y) \in C^3$ defined near Γ_0 either by

$$d(x,y) = \int \left(-y\,\frac{\ell_y}{\ell_x}\right)(x)dx + \frac{y^2}{2}, \text{ or else by } d(x,y) = \frac{x^2}{2} + \int \left(-x\,\frac{\ell_x}{\ell_y}\right)(y)dy, \tag{A.2.23}$$

satisfies the first two conditions (i) = (1.1.4) and (ii) = (1.1.5) of assumption (A.1) in Section 1.

(b) The conclusion of part (a) applies, in particular, if either (b_1) $f(x)$ is convex: $f''(x) \geq 0$, $a \leq x \leq b$; or else (b_2) $f(x)$ is concave: $f''(x) \leq 0$, $a \leq x \leq b$. Similarly, for $g(y)$.

Proof. (a) The proof of (a) is contained in the construction of h.

(b) To prove (b), we notice that $f(x)$ can be translated without loss of generality. Thus, if $f(x)$ is convex [respectively, concave] we can always assume that $f(x) \leq 0$ [respectively,

$f(x) \geq 0]$ so that in either case we have: $f(x)f''(x) \leq 0$, $a \leq x \leq b$, and condition (A.2.21) holds true. □

Examples. Lemma A.2.2 can be used to construct many examples, in fact even more general than those admitted by Lemma A.2.2 itself, in the sense that (as in some of the examples below) the slope of Γ_0 may be infinite. These are given in Appendix B.

Second setting: The case of Theorem A.2.1. We next generalize the construction leading to Lemma A.2.2 to the case where the boundary Γ_0 has a U-turn (such as in the situation covered by Theorem A.2.1, and such as in the case of examples B.1.1, B.2.2 below). In particular, Γ_0 (locally) bounds either a convex, or a concave domain. After an appropriate choice of axes, we are let to the situation depicted in Fig. A.7, where Γ_0 has nowhere horizontal tangent.

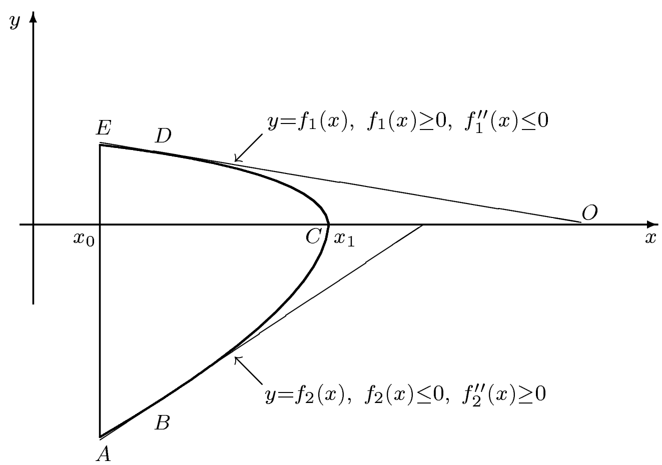

FIG. A.7: Ω MAY BE ON EITHER SIDE OF Γ_0: $ABCDE$, WITH ABC A CONVEX CURVE AND CDE A CONCAVE CURVE. AB AND DE ARE FLAT.

We notice that the key assumption, beside (A.2.4), in particular convexity/concavity, that Γ_0 has nowhere a horizontal tangent can be rephrased as follows: *there exists a point O*—in the setting of (A.2.1), (A.2.3), on the x-axis, in fact, see Fig. A.7—such that the *radial field $\overrightarrow{OP}$ is entering or exiting Ω (locally) through Γ_0: $\overrightarrow{OP} \cdot \nu \leq 0$* on Γ_0, or else $\overrightarrow{OP} \cdot \nu \geq 0$ on Γ_0. The tangent line at x_1 is vertical. In the present case we define a vector field h in Ω locally near Γ_0, following the geometric ideas and construction leading to Lemma A.2.2, suitably adapted, thus establishing Theorem A.2.1.

Proof of Theorem A.2.1. Case 1. (symmetric case) If $f_2(x) = -f_1(x)$, $x_0 \leq x \leq x_1$, then the construction leading to Lemma A.2.2 can be applied, to obtain a vector field

$h = \{h_1, h_2\}$,

$$h_1(x) = \frac{f_1(x)}{f_1'(x)} = \frac{f_2(x)}{f_2'(x)}, \quad x_0 \le x < x_1,\ h_1(x_1) = 0,\ h_2(y) = y. \tag{A.2.24}$$

The vector field h in (A.2.24) is tangent to Γ_0. We next verify that the positivity condition of the Jacobian matrix:

$$J_h > 0 \text{ on } x_0 \le x < x_1, \text{ where } \frac{d}{dx}\left(\frac{f_1(x)}{f_1'(x)}\right) = 1 - \frac{f_1(x) f_1''(x)}{[f_1'(x)]^2} > 0,\ x_0 \le x < x_1, \tag{A.2.25}$$

can be extended up to $x = x_1$. At $x = x_1$, we have $f_1(x_1) = 0$, $f_1'(x_1) = -\infty$, $f_1''(x_1) = -\infty$, by (A.2.3). Near x_1, setting $x = g_1(y)$ to be the inverse of $y = f_1(x)$, so that $x \equiv g_1(f_1(x))$, then the chain rule shows that

$$\frac{f_1''(x)}{[f_1'(x)]^2} = -\frac{g_1''(y)}{g_1'(y)}, \qquad x < x_1,\ x \text{ near } x_1. \tag{A.2.26}$$

From here, one can prove that

$$\lim_{x \to x_1} \frac{f_1(x) f_1''(x)}{[f_1'(x)]^2} = -\lim_{y \to 0} \frac{y g_1''(y)}{g_1'(y)} = -1, \tag{A.2.27}$$

thus extending the validity of the positivity condition $J_h > 0$ in (A.2.25) up to $x = x_1$, as required. To show (A.2.27), we simply use the Taylor formula with remainder in the form of Lagrange to the functions $g_1'(y)$, $g_1''(y)$ where $g_1'(0) = \frac{1}{f_1'(x_1)} = 0$, and for some η_y and ζ_y comprised between 0 and y, we have:

$$g_1'(y) = g_1''(0)y + g_1'''(\eta_y)\frac{y^2}{2}; \qquad g_1''(y) = g_1''(0) + g_1'''(\zeta_y)y. \tag{A.2.28}$$

Using (A.2.28), one readily shows

$$\lim_{y \to 0} \frac{y g_1''(y)}{g_1'(y)} = \frac{g_1''(0)}{g_1''(0)} = 1, \tag{A.2.29}$$

and (A.2.27) (right) follows.

Case 2. (Local symmetry near the vertex point x_1) If $f_1(x) = -f_2(x)$ only near $x = x_1$, say, $x_1 - \delta \le x \le x_1$, for some small $\delta > 0$, we define h on the set $x_1 - \delta \le x \le x_1$, $f_2(x) \le y \le f_1(x)$, as in Case 1. Instead, for $x_0 \le x < x_1 - \delta$, where $f_1(x) \ne -f_2(x)$, we define h, consistently, as follows:

$$\begin{cases} h_1(x) = \dfrac{f_1(x)}{f_1'(x)}, & h_2(y) = y,\ x_0 \le x \le x_1 - \delta,\ y \ge f_1(x_1 - \delta); \quad \text{(A.2.30)} \\[2ex] h_2(x) = \dfrac{f_2(x)}{f_2'(x)}, & h_2(y) = y,\ x_0 \le x \le x_1 - \delta,\ y \le f_2(x_1 - \delta). \quad \text{(A.2.31)} \end{cases}$$

Then, in the strip: $\{x_0 \leq x < x_1 - \delta,\ f_2(x_1 - \delta) < y < f_1(x_1 - \delta)\}$ we extend the original $h_1(x)$ in (A.2.30) and (A.2.31) smoothly (as done in Example B.1.1) to obtain a global C^2-function.

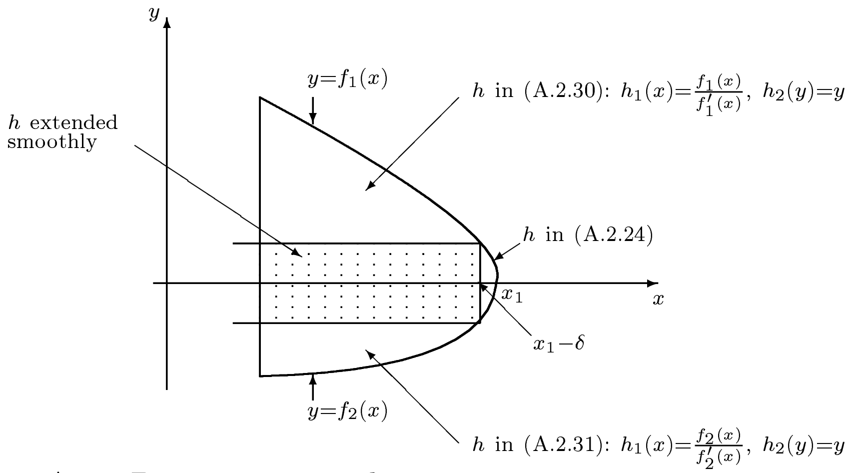

FIG. A.8: DEFINITION OF h IN THE CASE OF LOCALLY SYMMETRIC Γ_0 NEAR x_1: $f_1(x) = -f_2(x),\ x_1 - \delta \leq x \leq x_1$.

Case 3. (general case) If generally $f_1(x) \neq -f_2(x)$, we first consider a small triangular domain $\mathcal{T}$ with one vertex at x_1: $\mathcal{T} = \{x_0 \leq x \leq x_1,\ \epsilon(x - x_1) \leq y \leq -\epsilon(x - x_1)\}$ for a small $\epsilon > 0$.

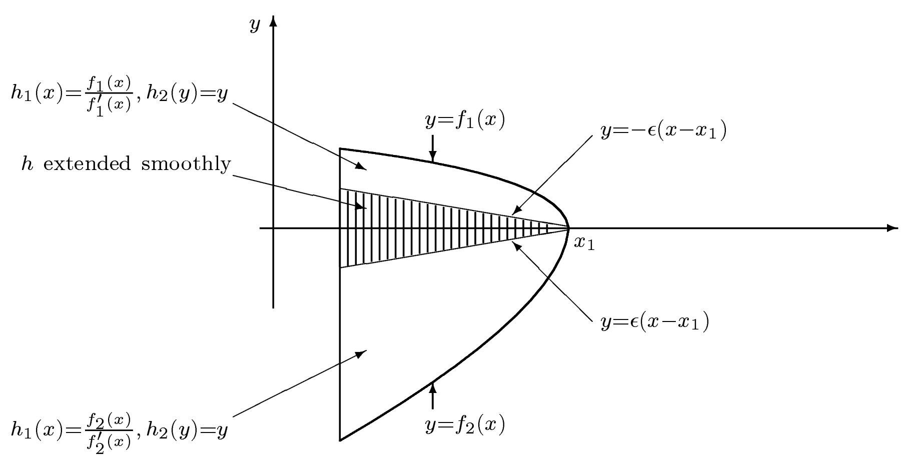

FIG. A.9: DEFINITION OF h IN THE GENERAL CASE.

We then define h first outside the triangular domain that is for: $x_0 \leq x \leq x_1$, and either $y > -\epsilon(x - x_1)$ or else $y < \epsilon(x - x_1)$, as in (A.2.30), (A.2.31). Next, we extend h_1 smoothly across the triangular domain. This is possible since $h_1(x_1) = h_2(0) = 0$ at $(x_1, 0)$. $\square$

A.3 A second approach where Γ_0 is curved: Conformal mapping methods, 2-dimensional case. Setting of Section 1

Assumption (A.1) in (1.1.4), (1.1.5) of Section 1. In this subsection we point out the possibility of using conformal mapping methods to obtain, in the 2-dimensional case, a smooth *conservative* vector vield $h(x,y)$ such that the two conditions in (A.2.11) [i.e., assumption (A.1) of Section 1] are satisfied. The approach presented here is based on the following well-known result.

Theorem A.3.1. [B-C, p. 294] Suppose that a transformation

$$w = f(z) = u(x,y) + iv(x,y), \qquad z = x + iy \tag{A.3.1}$$

is conformal on a smooth arc C_0, and let Γ_0 be the image of C_0 under the transformation: $\Gamma_0 = f(C_0)$. Assume that there exists a scalar function $d(u,v) \in C^3$, $h = \nabla d \neq 0$, which along Γ_0 in the (u,v)-plane, satisfies

$$\left.\frac{\partial d}{\partial \nu}\right|_{\Gamma_0} = \nabla d \cdot \nu = h \cdot \nu = 0 \quad \text{ on } \Gamma_0, \tag{A.3.2}$$

$\nu =$ unit normal vector to Γ_0. Then, the function defined by

$$\delta(x,y) \equiv d(u(x,y), v(x,y)) \in C^3, \tag{A.3.3}$$

along C_0 in the (x,y)-plane, satisfies

$$\left.\frac{\partial \delta}{\partial \bar{n}}\right|_{C_0} = \nabla \delta \cdot \bar{n} = 0 \text{ on } C_0, \tag{A.3.4}$$

$\bar{n} =$ unit normal vector to C_0. □

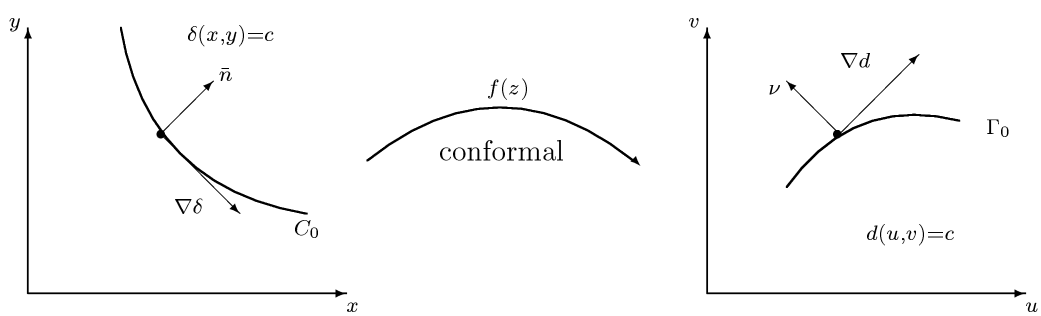

FIG. A.10: CONFORMAL MAPPING FROM C_0 TO Γ_0

Thus, the idea of the present approach is as follows:

Step 1. In the (u, v)-plane, select a curve Γ_0 for which a scalar function $d(u, v) \in C^3$ is known to exist, such that assumption (A.1) of Section 1 is satisfied: this means that such $d(u, v)$ satisfies the boundary condition (A.3.2), as well as the positivity condition

$$\mathcal{H}_d(\Gamma_0) = J_h(\Gamma_0) = \begin{bmatrix} d_{uu} & d_{uv} \\ d_{uv} & d_{vv} \end{bmatrix} (\Gamma_0) > 0. \tag{A.3.5}$$

Here, $\mathcal{H}_d(\Gamma_0)$ is the Hessian matrix of $d(u, v)$, evaluated on Γ_0, $h = \nabla d$. To this end, we may use Sections A.1 and A.2. In particular, we may take, in the simplest case, $\Gamma_0 =$ line segment, the case of Section A.1; or else $\Gamma_0 =$ part of a circumference as in Section A.2 (Example B.1.1), etc.

Step 2. If $f(z)$ is any conformal mapping from C_0 in the (x, y)-plane onto Γ_0 in the (u, v)-plane, Theorem A.3.1 already yields a function $\delta(x, y)$ which fulfills half of assumption (A.1) of Section 1, concerning the arc C_0, namely condition (A.3.4). Therefore, it *remains to select the conformal mapping* $f(z)$ [or the arc C_0] *such that the resulting function* $\delta(x, y)$ *constructed via (A.3.3) satisfies also the positivity condition*

$$\mathcal{H}_\delta(C_0) = J_{\nabla\delta}(C_0) = \begin{bmatrix} \delta_{xx} & \delta_{xy} \\ \delta_{xy} & \delta_{yy} \end{bmatrix} (C_0) > 0 \tag{A.3.6}$$

on C_0. Once this is established, then, by continuity, $\mathcal{H}_\delta > 0$ in a neighborhood of the arc C_0 as well, and $\delta(x, y)$ can then be extended smoothly to all of Ω while preserving positivity of the Hessian matrix $\mathcal{H}_\delta$.

The positivity condition (A.3.6) may, in turn, be tested according to any of the following well-known *equivalent* characterizations:

(a) the principal minors have positive determinant: $\delta_{xx} > 0$ and $\det \mathcal{H}_\delta > 0$ on C_0;

(b) the eigenvalues λ_1 and λ_2 of the matrix $\mathcal{H}_\delta(C_0)$ are both positive: $\lambda_1, \ \lambda_2 > 0$;

(c) the determinant and the trace of the matrix $\mathcal{H}_\delta(C_0)$ are both positive:

$$\det \mathcal{H}_\delta(C_0) = \lambda_1\lambda_2 > 0; \quad \operatorname{tr} \mathcal{H}_\delta(C_0) = \lambda_1 + \lambda_2 = [\delta_{xx} + \delta_{yy}](C_0) > 0. \tag{A.3.7}$$

Test (c) is the most useful here, in view of the well-known identity [B-C.1, p. 298],

$$\delta_{xx}(x, y) + \delta_{yy}(x, y) = [d_{uu}(u, v) + d_{vv}(u, v)]|f'(z)|^2, \tag{A.3.8}$$

which can be easily derived from (A.3.3), where, moreover, $[d_{uu} + d_{vv}] > 0$ on Γ_0, by assumption (A.3.5) in Step 1, via test (c) applied this time to the matrix $\mathcal{H}_d(\Gamma_0)$. Hence, by conformality, the trace $\operatorname{tr} \mathcal{H}_\delta(C_0)$ in (A.3.8) is always positive on C_0, as desired in (A.3.7). In view of this, we see then that the positivity condition (A.3.6) for the matrix $\mathcal{H}_\delta(C_0)$ holds true if and only if $\det \mathcal{H}_\delta(C_0) > 0$. Overall, the above argument has thus shown the following result.

Theorem A.3.2. Let Γ_0 be a curve (in the (u, v)-plane) for which assumption (A.1) of Section 1 holds true [this means that both (A.3.2) and (A.3.5) are fulfilled]. Let $f(z)$ be a conformal mapping of an arc C_0 in the (x, y)-plane onto Γ_0.

Then C_0 satisfies assumption (A.1) of Section 1 as well [i.e., both conditions (A.3.4) and (A.3.6)] if and only if: det $\mathcal{H}_\delta(C_0) > 0$.

A full exploration of the conformal mapping approach here proposed, and related Riemann mapping theorem, remains to be done; in particular, it would be desirable to characterize explicitly classes of conformal mappings as well as classes of arcs C_0 satisfying Theorem A.3.2 and mapping C_0 onto elementary curves Γ_0 (straight segments, portions of circumferences, etc.) for which assumption (A.1) of Section 1 is satisfied [as in Sections A.1 and A.2]. Here, we confine ourselves to analyzing the simplest case. Specific examples in Appendix B.3 show positive features (Example B.3.2) as well as limitations (Example B.3.4) of this approach over the method of Appendix A.2.

The case where Γ_0 is a straight segment. Let Γ_0 be a segment in the (u, v)-plane, say

$$\Gamma_0 : \; u \equiv u_0 > 0; \quad v_0 \leq v \leq v_1. \tag{A.3.9}$$

Step 1. From Section A.1, we know that the function

$$d(u, v) = \frac{1}{2}\|(u, v) - (u_0, 0)\|^2 = \frac{1}{2}[(u - u_0)^2 + v^2] \tag{A.3.10}$$

satisfies assumption (A.1) of Section 1 on Γ_0: i.e., (A.3.2) as well as (A.3.5).

Step 2. Let $f(z)$ as in (A.3.1) be a conformal mapping as required by Theorem A.3.2. According to (A.3.3), we define the function $\delta(x, y)$ via (A.3.8) by

$$2\delta(x, y) = (u(x, y) - u_0)^2 + v^2(x, y). \tag{A.3.11}$$

We obtain the relevant partial derivatives

$$\delta_x = (u - u_0)u_x + vv_x; \quad \delta_y = (u - u_0)u_y + vv_y; \tag{A.3.12}$$

$$\delta_{xx} = u_x^2 + v_x^2 + (u - u_0)u_{xx} + vv_{xx}; \quad \delta_{yy} = u_y^2 + v_y^2 + (u - u_0)u_{yy} + vv_{yy}; \tag{A.3.13}$$

$$\delta_{xy} = \delta_{yx} = u_x u_y + (u - u_0)u_{xy} + v_x v_y + vv_{xy}. \tag{A.3.14}$$

By the Cauchy-Riemann equations:

$$u_x = v_y, \; u_y = -v_x \text{ and hence } |f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 \neq 0, \tag{A.3.15}$$

by conformality. We next evaluate (A.3.13), (A.3.14) on a curve C_0 which is mapped into Γ_0, i.e., such that $u(x, y) \equiv u_0 > 0$. Then (A.3.13), (A.3.14) specialize to:

on C_0:

$$\begin{cases} \delta_{xx} = |f'(z)|^2 + vv_{xx}; \qquad \delta_{yy} = |f'(z)|^2 + vv_{yy} & \text{(A.3.16)} \\ \delta_{xy} = \delta_{yx} = vv_{xy}; & \text{(A.3.17)} \\ \delta_{xx}\delta_{yy} - \delta_{xy}^2 = |f'(z)|^4 - v^2[v_{xx}^2 + v_{xy}^2] & \text{(A.3.18)} \end{cases}$$

Eqn. (A.3.17) is obtained from (A.3.14) after a cancellation $u_x u_y + v_x v_y \equiv 0$ by the Cauchy-Riemann equations. Moreover, Eqn. (A.3.18) is obtained after using (twice) that v is harmonic: $v_{xx} + v_{yy} \equiv 0$.

A specialization of the general Theorem A.3.2 to this case where Γ_0 is given by (A.3.9) is given next.

Corollary A.3.3. Assume that $f(z)$ is a conformal mapping of C_0 onto Γ_0, where Γ_0 is the straight segment in (A.3.9). With reference to (A.3.18), we have that the curve C_0 satisfies assumption (A.1) of Section 1 [i.e., both Eqns. (A.3.4) and (A.3.6)] if and only if

$$\det \mathcal{H}_\delta(C_0) = \delta_{xx}\delta_{yy} - \delta_{xy}^2 = |f'(z)|^4 - v^2(v_{xx}^2 + v_{xy}^2) > 0 \text{ on } C_0. \quad \Box \tag{A.3.19}$$

With reference to test (a) above, we remark in passing that condition (A.3.19) clearly implies $\delta_{xx} > 0$, see (A.3.16), via

$$[|f'(z)|^2 - vv_{xx}][|f'(z)|^2 + vv_{xx}] > v^2 v_{xy}^2 \geq 0, \tag{A.3.20}$$

where the only option possible is for both terms in the square brackets to be positive. Illustrations are given in Appendix B.3.

Assumption (A.2) = (1.1.6) of Section 1. In preparation for our discussion on the setting of Section 10, we now identify the point(s) c on the boundary C_0, if any, where the conservative vector field $\nabla\delta$ constructed by (A.3.3) in Theorem A.3.1 [or its specialization (A.3.11) of Corollary A.3.3] vanishes: $\nabla\delta(c) = 0$, and hence violates assumption (A.2) = (1.1.6). This will then allow us, in the subsequent Appendix C, to remedy the situation by falling into the setting of Section 10. To this end, we recall the following well-known relationship [B-C.1, p. 296],

$$|\nabla\delta(x, y)| = |\nabla d(u, v)|\ |f'(z)| \tag{A.3.21}$$

between the gradients of the function $\delta(x, y)$ and $d(u, v)$ respectively [which can be readily shown from (A.3.3) via the Cauchy-Riemann equations]. From (A.3.21) since $f(z)$ is conformal, we obtain

Corollary A.3.4. With $f(z)$ a conformal mapping of the curve C_0 in the (x, y)-plane onto the curve Γ_0 in the (u, v)-plane, we have that the boundary point $z = (x, y) \in C_0$ is a critical point of the function δ, i.e., $\nabla\delta(x, y) = 0$ if and only if the point $w = f(z) = (u, v) \in \Gamma_0$ is a critical point of the function d, i.e., $\nabla d(u, v) = 0$. $\Box$

In the setting of Section A.2 [as well as its 3-dimensional generalization B.2] and of the subsequent Section A.4 (see Corollary A.4.2), we have that: *there is at most one critical point for* $d(u, v)$ *on* Γ_0.

A.4 A third approach where Γ_0 is convex or concave: Multidimensional case. Setting of Section 1

Assumption (A.1) in (1.1.4), (1.1.5) of Section 1. The present section gives a result which shows, in the n-dimensional space, that assumption (A.1) of Section 1 holds true, in the case where: (i) the set Ω is convex (respectively, concave) near Γ_0, and (ii) a radial vector field exists which is entering (respectively, exiting) through Γ_0.

From our inquiries within geometric circles at the conference at the University of Colorado, Boulder, and elsewhere, it appears that a version of the statement of Theorem A.4.1 may perhaps be known. However, we were neither given, nor were we able to find, specific references to it. In a related context, the perturbation formula (A.4.6), (A.4.7) was communicated to the first author by D. Tataru, and it appears in his unpublished manuscript, where an argument is given based on Poisson brackets to verify the pseudo-convex property. By contrast, our direct proof and computations below verify the positivity condition on the Hessian matrix under perturbation.

Theorem A.4.1. Consider the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, $\Omega \subset \mathbb{R}^n$, $\Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \phi$, where the surface Γ_0 is given explicitly, as a level set, as in (A.2.10), by the equation,

$$\Gamma_0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ \ell(x) = 0\}, \ \ell \in C^3, \tag{A.4.1}$$

with $\nabla \ell \neq 0$ on Γ_0 (for a suitable domain in $(x_1, \dots, x_n)$). Assume that:

(i) the Hessian matrix $\mathcal{H}_\ell$ of ℓ is non-negative definite on Γ_0:

$$\mathcal{H}_\ell(\Gamma_0) \equiv \begin{bmatrix} \ell_{x_1x_1}, \dots, \ell_{x_1x_n} \\ \vdots \qquad \vdots \\ \ell_{x_nx_1}, \dots, \ell_{x_nx_n} \end{bmatrix} (\Gamma_0) \geq 0 \tag{A.4.2}$$

which is a characterization for the surface $z = \ell(x)$ to be convex [Fl.1, Theorem 36, p. 114] or having convex epigraph, or for the set Ω being a convex set near Γ_0 [Fl.1, Proposition 3.5, p. 108], so that $\ell(x) \leq 0$ for $x \in \Omega$ near Γ_0. Moreover, the gradient $\nabla \ell$ points toward the exterior of Ω.

(ii) there exists a point $x_0 \in \mathbb{R}^n$, outside of Ω, such that

$$\left.\frac{\partial d_0}{\partial \nu}\right|_{\Gamma_0} = \nabla d_0 \cdot \nu = (x - x_0) \cdot \nu \leq 0 \text{ on } \Gamma_0; \text{ where } d_0(x) = \frac{1}{2}\|x - x_0\|^2, \ \nabla d_0(x) = x - x_0; \ \mathcal{H}_{d_0} = I, \tag{A.4.3}$$

where ν is the unit outward normal vector to Γ_0, thus pointing in the same direction of $\nabla \ell(x)$.

Alternatively, in place of (A.4.2) and (A.4.3), respectively, assume $\mathcal{H}_\ell(\Gamma_0) \leq 0$ and $\nabla d_0 \cdot \nu \geq 0$ on Γ_0 with $\nu(x)$ still outward in the same direction of $\nabla \ell(x)$, but now with $(x - x_0)$ making an acute or right angle with $\nu(x)$, for $x \in \Gamma_0$.

[The proof below works for the product $(\nabla d_0 \cdot \nu)\mathcal{H}_\ell(\Gamma_0) \geq 0$, see (A.4.31) below.]

Then, there exists a scalar function $d(x)$, defined explicitly below in (A.4.6)–(A.4.7) in a layer (collar) of Γ_0, such that the following two conditions are satisfied:

(a)

$$\left.\frac{\partial d}{\partial \nu}\right|_{\Gamma_0} = \nabla d \cdot \nu = 0 \text{ on } \Gamma_0; \tag{A.4.4}$$

(b) the Hessian matrix of d, evaluated on Γ_0, $\mathcal{H}_d(\Gamma_0)$, is positive definite:

$$\mathcal{H}_d(\Gamma_0) = \begin{bmatrix} d_{x_1x_1}, \dots, d_{x_1x_n} \\ \vdots \qquad \vdots \\ d_{x_nx_1}, \dots, d_{x_nx_n} \end{bmatrix} (\Gamma_0) > 0; \text{ in fact, } \mathcal{H}_d(\Gamma_0) \geq (1-\epsilon)I. \tag{A.4.5}$$

Thus, assumption (A.1) of Section 1 holds true for $d(x)$. The scalar function $d(x)$ is defined on Ω, near Γ_0, as a 'perturbation' of $d_0(x)$ as follows:

$$\begin{cases} d(x) \equiv d_0(x) + z(x), & \text{(A.4.6)} \\ z(x) \equiv -\left(\frac{\partial d_0}{\partial \nu}\right) \ell k + \lambda \ell^2, \quad k \equiv \frac{1}{|\nabla \ell|}, & \text{(A.4.7)} \end{cases}$$

where λ is a sufficiently large parameter, while $\left(\frac{\partial d_0}{\partial \nu}\right)$ denotes an extension of $\left.\left(\frac{\partial d_0}{\partial \nu}\right)\right|_{\Gamma_0}$ from $\Gamma_0 : \ \ell = 0$ to a layer (collar) of Γ_0 within Ω, defined by:

$$\text{near } \Gamma_0 : \ \frac{\partial d}{\partial \nu} \equiv \nabla d \cdot \nu = \nabla d \cdot (k \nabla \ell), \quad \text{where } \nu \equiv \frac{\nabla \ell}{|\nabla \ell|}, \ k = \frac{1}{|\nabla \ell|}. \tag{A.4.8}$$

Remark A.4.1. Figure A.10 illustrates two typical cases covered under Theorem A.4.1. On the other hand, the assumptions of Theorem A.4.1 exclude the case where Ω is an annulus, say $1 \leq \|x\| \leq 2$, with Γ_0 the internal sphere $\|x\| = 1$, and x_0 the origin. $\square$

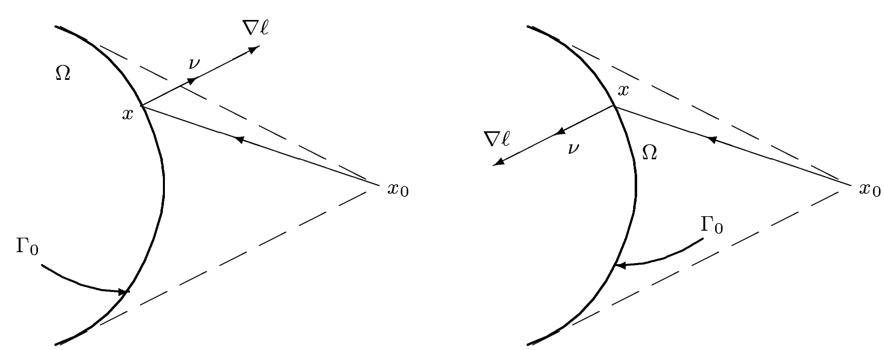

FIG. A.10: THEOREM A.4.1: CASE 1: $\ell = x^2 + y^2 - 1$; $\mathcal{H}_\ell(\Gamma_0) \geq 0$; $(x - x_0) \cdot \nu(x) \leq 0$; CASE 2: $\ell = 1 - x^2 - y^2$; $\mathcal{H}_\ell(\Gamma_0) \leq 0$; $(x - x_0) \cdot \nu(x) \geq 0$

Proof. (a) We first establish (A.4.4). Direct computations show that:
on Γ_0:

$$\begin{cases} \left.\dfrac{\partial(\ell k)}{\partial \nu}\right|_{\Gamma_0} = \cancel{\ell \frac{\partial k}{\partial \nu}} + k\,\frac{\partial \ell}{\partial \nu} = k\nabla \ell \cdot k\nabla \ell = 1, & \text{(A.4.9)} \\ \left.\dfrac{\partial(\ell^2)}{\partial \nu}\right|_{\Gamma_0} = \cancel{2\ell\,\frac{\partial \ell}{\partial \nu}} = 0, & \text{(A.4.10)} \end{cases}$$

since, on $\Gamma_0 : \nu = k\nabla\ell$ (by (A.4.8); and $\ell = 0$. Then, returning to (A.4.6), (A.4.7), we obtain, by virtue of (A.4.9), (A.4.10):

$$\text{on } \Gamma_0 : \left.\frac{\partial d}{\partial \nu}\right|_{\Gamma_0} = \left[\frac{\partial d_0}{\partial \nu} + \frac{\partial z}{\partial \nu}\right]_{\Gamma_0} = \frac{\partial d_0}{\partial \nu} - \cancel{\left(\frac{\partial^2 d_0}{\partial \nu^2}\right)(\ell k)} - \frac{\partial d_0}{\partial \nu}\,\frac{\partial(\ell k)}{\partial \nu} + \lambda\,\cancel{\frac{\partial(\ell^2)}{\partial \nu}} \tag{A.4.11}$$

$$= \left[\frac{\partial d_0}{\partial \nu} - \frac{\partial d_0}{\partial \nu}\right]_{\Gamma_0} = 0, \tag{A.4.12}$$

and (A.4.4) is established.

(ii) We now prove (A.4.5). First, for convenience, set

$$\text{near } \Gamma_0 : p \equiv -\frac{\partial d_0}{\partial \nu}\,k, \text{ so that, by (A.4.7), } z \equiv p\ell + \lambda\ell^2 \text{ near } \Gamma_0. \tag{A.4.13}$$

We shall now use the diadatic product notation

$$A \otimes B \equiv \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} [b_1, \dots, b_n] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\ \cdot & \cdot & & \cdot \\ a_n b_1 & a_n b_2 & \cdots & a_n b_n \end{bmatrix}. \tag{A.4.14}$$

Step 1. Proposition A.4.2. In the notation of (A.4.14), and with reference to z in (A.4.13), the Hessian matrix $\mathcal{H}_z$ of z is given by

(a)

$$\text{near } \Gamma_0 : \mathcal{H}_z = p\mathcal{H}_\ell + \ell\mathcal{H}_p + \nabla\ell \otimes \nabla p + \nabla p \otimes \nabla \ell + 2\lambda\ell\mathcal{H}_\ell + 2\lambda\nabla\ell \otimes \nabla\ell; \tag{A.4.15}$$

(b)

$$\text{on } \Gamma_0 : \mathcal{H}_z = p\mathcal{H}_\ell + \nabla\ell \otimes \nabla p + \nabla p \otimes \nabla \ell + 2\lambda\nabla\ell \otimes \nabla\ell. \tag{A.4.16}$$

Proof of Proposition A.4.2. In the notation of (A.4.14), we can rewrite the Hessian matrix $\mathcal{H}_z$ of z as follows:

$$\mathcal{H}_z = \nabla \otimes \nabla z = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{bmatrix} \left[\frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n}\right] = \begin{bmatrix} z_{x_1 x_1}, \dots z_{x_1 x_n} \\ \vdots \qquad \vdots \\ z_{x_n x_1}, \dots z_{x_n x_n} \end{bmatrix}. \tag{A.4.17}$$

From (A.4.13) we have

$$\nabla z = \nabla(p\ell + \lambda\ell^2) = p\nabla\ell + \ell\nabla p + 2\lambda\ell\nabla\ell, \tag{A.4.18}$$

and hence, using (A.4.18) in (A.4.17), we find

$$\mathcal{H}_z = \nabla \otimes [\ell\nabla p + p\nabla\ell + 2\lambda\ell\nabla\ell]. \tag{A.4.19}$$

Next, we verify that

$$\begin{cases} \nabla \otimes (\ell\nabla p) = \nabla\ell \otimes \nabla p + \ell\mathcal{H}_p, & \text{(A.4.20)} \\ \text{and hence that} & \\ \nabla \otimes (p\nabla\ell) = \nabla p \otimes \nabla\ell + p\mathcal{H}_\ell; \quad \nabla \otimes (\ell\nabla\ell) = \nabla\ell \otimes \nabla\ell + \ell\mathcal{H}_\ell. & \text{(A.4.21)} \end{cases}$$

Then (A.4.20), (A.4.21), used in (A.4.19), yield (A.4.15), from which (A.4.16) follows upon setting $\ell = 0$ on Γ_0. We now verify (A.4.20): we compute

$$\nabla \otimes (\ell\nabla p) = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{bmatrix} [\ell p_{x_1}, \dots, \ell p_{x_n}] = \begin{bmatrix} \ell_{x_1}p_{x_1} + \ell p_{x_1x_1}, \dots, \ell_{x_1}p_{x_n} + \ell p_{x_1x_n} \\ \vdots \qquad\qquad \vdots \\ \ell_{x_n}p_{x_1} + \ell p_{x_1x_n}, \dots, \ell_{x_n}p_{x_n} + \ell p_{x_nx_n} \end{bmatrix} \tag{A.4.22}$$

$$= \begin{bmatrix} \ell_{x_1} \\ \vdots \\ \ell_{x_n} \end{bmatrix} [p_{x_1}, \dots, p_{x_n}] + \ell\mathcal{H}_p, \tag{A.4.23}$$

and (A.4.20) follows from (A.4.23). Proposition A.4.2 is proved. □

Step 2. Lemma A.4.3. For $x \in \mathbb{R}^n$, and with reference to (A.4.16), we have

$$\text{on } \Gamma_0 : (\mathcal{H}_z x, x)_{R^n} = p(\mathcal{H}_\ell x, x)_{R^n} + 2(\nabla\ell \cdot x)(\nabla p \cdot x) + 2\lambda(\nabla\ell \cdot x)^2, \tag{A.4.24}$$

where $(\quad,\quad)_{R^n}$ is the inner product in $\mathbb{R}^n$, and "$\cdot$" denotes the usual dot product.

Proof. We first note that for any two vectors A and B in R^n, we have

$$((A \otimes B)x, x)_{R^n} = (A \cdot x)(B \cdot x), \qquad \forall\, x \in R^n. \tag{A.4.25}$$

This can be readily verified from the definition (A.4.14). Thus, specializing (A.4.25), we obtain

$$\begin{cases} ((\nabla\ell \otimes \nabla p)x, x)_{R^n} = ((\nabla p \otimes \nabla\ell)x, x)_{R^n} = (\nabla\ell \cdot x)(\nabla p \cdot x) & \text{(A.4.26)} \\ ((\nabla\ell \otimes \nabla\ell)x, x)_{R^n} = (\nabla\ell \cdot x)^2. & \text{(A.4.27)} \end{cases}$$

Then, using (A.4.26), (A.4.27) in (A.4.16) yields (A.4.24), as desired. □

Step 3. Proposition A.4.4. Assume hypotheses (A.4.2) and (A.4.3) on Γ_0: $\frac{\partial d_0}{\partial \nu} \leq 0$ and $\mathcal{H}_d \geq 0$ on Γ_0 (hence $p \geq 0$ on Γ_0 by (A.4.13) since $k > 0$ by (A.4.8)). Then, with reference to (A.4.28), for any $\epsilon_0 > 0$, there exists $\lambda_{\epsilon_0} > 0$, in fact $\lambda_{\epsilon_0} = \frac{1}{2\epsilon_0}$, such that, for all $\lambda > \lambda_{\epsilon_0}$ and all $x \in R^n$, we have:

$$\text{on } \Gamma_0: \ (\mathcal{H}_z x, x)_{R^n} \geq -\epsilon_0 \| \, |\nabla p| \, \|^2 \|x\|^2 + \left[2\lambda - \frac{1}{\epsilon_0} \right] (\nabla \ell \cdot x)^2 \geq -\epsilon \|x\|^2,$$

$$\epsilon = \epsilon_0 \max_{\Gamma_0} \| \, |\nabla p| \, \|. \quad \text{(A.4.28)}$$

Proof. We preliminarily have

$$2(\nabla \ell \cdot x)(\nabla p \cdot x) \geq -\epsilon_0 (\nabla p \cdot x)^2 - \frac{1}{\epsilon_0} (\nabla \ell \cdot x)^2, \quad \text{(A.4.29)}$$

and hence

$$2(\nabla \ell \cdot x)(\nabla p \cdot x) + 2\lambda (\nabla \ell \cdot x)^2 \geq \left[2\lambda - \frac{1}{\epsilon_0} \right] (\nabla \ell \cdot x)^2 - \epsilon_0 (\nabla p \cdot x)^2. \quad \text{(A.4.30)}$$

Returning to (A.4.24) on Γ_0, where $p \geq 0$ and $\mathcal{H}_\ell \geq 0$ on Γ_0, or else $p \leq 0$ and $\mathcal{H}_\ell \leq 0$, respectively; or more generally, $p\mathcal{H}_\ell \geq 0$, by assumption, we finally obtain via (A.4.30),

$$\text{on } \Gamma_0 : (\mathcal{H}_z x, x)_{R^n} \geq -\epsilon_0 (\nabla p \cdot x)^2 + \left[2\lambda - \frac{1}{\epsilon_0} \right] (\nabla \ell \cdot x)^2 \quad \text{(A.4.31)}$$

$$\geq -\epsilon_0 \| \, |\nabla p| \, \|^2 \|x\|^2 \geq -\epsilon \|x\|^2, \quad \text{(A.4.32)}$$

by the Schwarz inequality, for all $\lambda > \frac{1}{2\epsilon_0}$. Thus, (A.4.32) establishes (A.4.28), as desired. □

Step 4. We return to (A.4.6) and obtain that, under the given assumptions (A.4.2), (A.4.3), Proposition A.4.4 holds true, and then (A.4.28) yields, since $\mathcal{H}_{d_0} \equiv I$ (identity):

$$\text{on } \Gamma_0 : \mathcal{H}_d = \mathcal{H}_{d_0} + \mathcal{H}_z \geq I - \epsilon I. \quad \text{(A.4.33)}$$

Then, (A.4.32) proves property (ii) in (A.4.5). Theorem A.4.1 is established. □

Assumption (A.2) = (1.1.6) of Section 1. In preparation for our discussion on the setting of Section 10, we now identify the point(s) on the boundary Γ_0, if any, where the conservative vector field ∇d constructed in Theorem A.4.1 violates assumption (A.2) = (1.1.6). This will then allow us, in the subsequent Appendix C, to remedy the situation by devising a strategy [based on splitting Ω as the union of two overlapping subdomains Ω_1, Ω_2: $\Omega = \Omega_1 \cup \Omega_2$ and the employment of two vector fields], which will then fit the setting of Section 10.

Corollary A.4.2. Under the assumptions of Theorem A.4.1, the scalar function $d(x)$ there constructed has the following third property: its gradient $\nabla d|_{\Gamma_0}$, once restricted on the boundary Γ_0, vanishes at the unique point $x \in \Gamma_0$, if such exists on Γ_0, where the vector field $\nabla d_0(x) = x - x_0$ (see (A.4.2)) is orthogonal to Γ_0. In symbols: for $x \in \Gamma_0$,

$$\nabla d(x) = 0 \Longleftrightarrow (x - x_0) \text{ parallel to normal } \nu(x) \text{ at } x \in \Gamma_0,$$

$$\text{i.e., orthogonal to } \Gamma_0 \text{ at } x, \tag{A.4.34}$$

and such point $x \in \Gamma_0$ is unique, if it exists on Γ_0.

Proof. The proof is a direct computation, starting from the definition of the function $d(x)$ in (A.4.6), (A.4.7). The gradient ∇d of d is:

$$\nabla d = \nabla d_0 + \nabla z = \nabla d_0 - \left(\frac{\partial d_0}{\partial \nu}\right) k \nabla \ell - \ell \nabla \left(-\frac{\partial d_0}{\partial \nu} k\right) - 2\lambda \ell \nabla \ell. \tag{A.4.35}$$

Its restriction on Γ_0, where $\ell \equiv 0$, is then, recalling k from (A.4.7):

$$\text{on } \Gamma_0: \ \nabla d|_{\Gamma_0} = \nabla d_0|_{\Gamma_0} - \frac{\partial d_0}{\partial \nu}|_{\Gamma_0} \nu, \qquad \left(\frac{\nabla \ell}{|\nabla \ell|}\right)_{\Gamma_0} = \nu. \tag{A.4.36}$$

Thus, if $x \in \Gamma_0$, since $\nabla d_0(x) = x - x_0$ (see (A.4.2)), we obtain

$$\text{for } x \in \Gamma_0: \ \nabla d(x) = (x - x_0) - [(x - x_0) \cdot \nu(x)] \nu(x). \tag{A.4.37}$$

Thus, for $x \in \Gamma_0$, we obtain $\nabla d(x) = 0$ in (A.4.36) if and only if (A.4.34) holds true. Such point $x \in \Gamma_0$ in (A.4.34) is unique, if it exists on Γ_0, since Γ_0 is convex (concave).

In fact, if Γ_0 is described, say, by the level set $\ell(x, y) = f(x) - y = 0$, with $f(x)$ a convex function defined on a convex set K of $\mathbb{R}^{n-1}$, choose the axes so that: the origin 0 is the point $f(0) = 0$; $f(x) \geq 0$; on Ω near Γ_0 we have $\ell(x, y) = f(x) - y \leq 0$. Let $Y_0 : (0, y_0)$, with $y_0 < 0$, be a point outside of Ω such that $\overrightarrow{Y_0 O}$ is orthogonal to Γ_0. Pick any other point $P : (x, f(x))$ on Γ_0, with normal $\bar{n} = \{\nabla f(x), -1\}$. We claim that $\overrightarrow{Y_0 P} = \{x, f(x) - y_0\}$ cannot be orthogonal to Γ_0; i.e., that $\overrightarrow{Y_0 P}$ cannot be parallel to $\bar{n}$. This is so since the last coordinates, -1 and $[f(x) - y_0]$, of $\bar{n}$ and $\overrightarrow{Y_0 P}$ have opposite sign, while $\nabla f(x) \cdot x \geq f(x) \geq 0$,s ince, by convexity of $f(x)$, we have: $f(0) - f(x) \geq \nabla f(x) \cdot (0 - x)$ [Fl.1, Prop. 3.6a, p. 111]. □

Appendix B: Illustrations satisfying the setting of Section 1

B.1 Illustrations of approach of Section A.2

In this subsection, we provide illustrations of the approach of Section A.2 and its extension to the 3-dimensional case.

Two-dimensional illustrations. Example B.1.1. Let Ω be the two-dimensional ellipsoidal region: $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$ in the (x,y)-plane, surrounded by the ellipse $\Gamma:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Define

$$\Gamma_0=\{(x,y)\in\Gamma:\ 0<x_1\leq x\leq x_2<a\};\ \Gamma_1=\Gamma\setminus\Gamma_0. \tag{B.1.1}$$

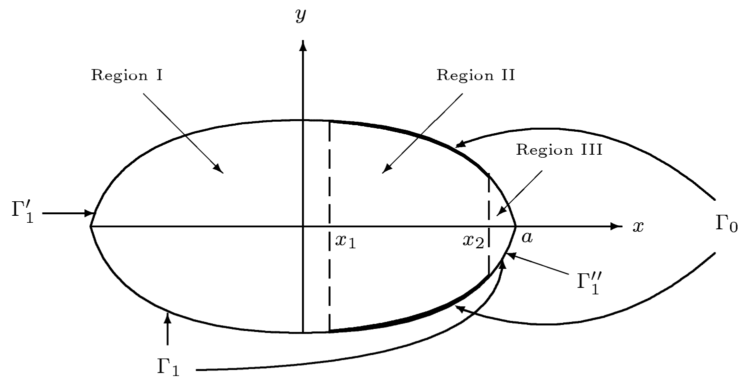

FIG. B.1: UNCONTROLLED PART Γ_0 AND CONTROLLED PART Γ_1, $0<x_1<x_2<a$

Here, x_1 and x_2 are fixed points, arbitrarily close to 0 and a, respectively. It is assumption (A.2) = (1.1.6) of Section 1 that imposes the constraint $x_2<a$: indeed, the vector field h constructed below in (B.1.5) vanishes at the point $\{x=a,y=0\}$, thus violating (A.2) =(1.1.6) of Section 1. It is then necessary (under the present or similar constructions) based on the one-vector setting of Section 1 to 'cut off' an arbitrarily small portion of the boundary Γ around the point $\{a,0\}$ to assign to Γ_1, in order to achieve condition (A.2) = (1.1.6): see (B.1.9) below. Thus, Γ_0 is still 'almost' $\frac{1}{2}$ of the boundary Γ, in line with known control theoretic results for second-order hyperbolic equations, but Γ_0 must miss a small arc around $(a,0)$ in our present construction, based only on Section 1.

In Section C below we shall refine the present analysis by having Γ_0 connected and almost $\frac{1}{2}$ of the boundary Γ; i.e., the present portion Γ_1'' will be dispensed with. Here we have, recalling (A.2.12).

$$-y\,\frac{\ell_y}{\ell_x}=\frac{-y^2}{b^2}\,\frac{a^2}{x}=\frac{a^2}{x}\left(\frac{x^2}{a^2}-1\right)=x-\frac{a^2}{x},\ 0\leq x\leq a,\ (x,y)\in\Gamma_0. \tag{B.1.2}$$

This yields the expression $h_1^{II}(x)$ of the vector field h in (B.1.6b) below, where the additional assumption (A.2) = (1.1.6) of Section 1 [not guaranteed by Lemma A.2.1] forces the constraint: $x_2 < a$; thus, the need to extend smoothly and suitably $h_1^{II}(x)$. This is done below.

Lemma B.1.1. The following function $d(x,y) \in C^3$

$$d(x,y) = \begin{cases} d^I(x) \;+\; \dfrac{y^2}{2}, & -a \le x \le x_1; \quad \text{(B.1.3a)} \\ d^{II}(x) \;+\; \dfrac{y^2}{2}, & 0 < x_1 \le x \le x_2 < a; \quad \text{(B.1.3b)} \\ d^{III}(x) \;+\; \dfrac{y^2}{2}, & x_2 \le x \le a, \quad \text{(B.1.3c)} \end{cases}$$

satisfies assumption (A.1) of Section 1, where

$$\begin{cases} d^I(x) = -\dfrac{a^2(x-x_1)^3}{3x_1^3} + \left(1 + \dfrac{a^2}{x_1^2}\right)\dfrac{(x-x_1)^2}{2} \\ \qquad + \left(x_1 - \dfrac{a^2}{x_1}\right)x + 1 - \dfrac{x_1^2}{2} - a^2 \ln x_1, \quad -1 \le x \le x_1; & \text{(B.1.4a)} \\ d^{II}(x) = \dfrac{x^2}{2} - a^2 \ln x \qquad x_1 \le x \le x_2; & \text{(B.1.4b)} \\ d^{III}(x) = \dfrac{x^2}{2} - a^2 \ln x - \dfrac{\epsilon(x-x_2)^4}{4} \qquad x_2 \le x \le a, & \text{(B.1.4c)} \end{cases}$$

for $0 < \epsilon$ sufficiently small, say $3\epsilon(a-x_2)^2 < \frac{1}{2}$, as to obtain (B.1.8c) below.

Proof. We use the ideas leading to Lemma A.2.1 to construct $d(x,y)$, starting from (B.1.2) to obtain the component $h_1^I(x)$ below and then extend smoothly. Here we verify the requirements of assumptions (A.1) and (A.2):

(a) $d(x,y) \in C^3$: using also the formulas below in (B.1.6), (B.1.8), one checks directly that

$$d^I(x_1) = d^{II}(x_1);\; (d^I)'(x_1) = (d^{II})'(x_1);\; (d^I)''(x_1) = (d^{II})''(x_2);\; d^{II}(x_2) = d^{III}(x_2);$$
$$(d^{II})'(x_2) = (d^{III})'(x_2);\; (d^{II})''(x_2) = (d^{III})''(x_2).$$

(b) The gradient $h(x,y) = (h_1, h_2) = \nabla d(x,y)$ is given by

$$h_1(x) = \begin{cases} h_1^I(x) & -a \le x \le x_1 & \text{(B.1.5a)} \\ h_1^{II}(x) & x_1 \le x \le x_2; \qquad h_2(y) = y; & \text{(B.1.5b)} \\ h_1^{III}(x) & x_2 \le x \le a & \text{(B.1.5c)} \end{cases}$$

$$\begin{cases} h_1^I(x) &= -\dfrac{a^2}{x_1^3}(x-x_1)^2 \\ &\quad + \left(1+\dfrac{a^2}{x_1^2}\right)(x-x_1)+x_1-\dfrac{a^2}{x_1}<0, \quad -a \le x \le x_1; & \text{(B.1.6a)} \\ h_1^{II}(x) &= x-\dfrac{a^2}{x}<0 \qquad x_1 \le x \le x_2; & \text{(B.1.6b)} \\ h_1^{III}(x) &= x-\dfrac{a^2}{x}-\epsilon(x-x_2)^3<0 \qquad x_2 \le x \le a. & \text{(B.1.6c)} \end{cases}$$

The Jacobian matrix J_h of h is given by

$$J_h(x,y) = \begin{bmatrix} h_1'(x) & 0 \\ 0 & 1 \end{bmatrix} = \text{positive definite on } \bar{\Omega}, \tag{B.1.7}$$

where

$$h_1'(x) = \begin{cases} (h_1^I)'(x) &= -\dfrac{2a^2}{x_1^3}(x-x_1)+\left(1+\dfrac{a^2}{x_1^2}\right)>0, \quad -a \le x \le x_1; & \text{(B.1.8a)} \\ (h_1^{II})'(x) &= 1+\dfrac{a^2}{x^2}>0, \qquad x_1 \le x \le x_2; & \text{(B.1.8b)} \\ (h_1^{III})'(x) &= 1+\dfrac{a^2}{x^2}-3\epsilon(x-x_2)^2>0, \qquad x_2 \le x \le a, & \text{(B.1.8c)} \end{cases}$$

for $3\epsilon(a-x_2)^2 < \frac{1}{2}$, as desired.

(c) $h \cdot \nu \equiv 0$ on Γ_0: indeed, for $(x,y) \in \Gamma_0$, see (B.1.2), we have that $h_1^{II}(x)$ in (B.1.6b) is the active component and then $\left(x-\frac{a^2}{x}, y\right) \cdot \left(\frac{x}{a^2}, \frac{y}{b^2}\right) \equiv \frac{x^2}{a^2} - 1 + \frac{y^2}{b^2} \equiv 1$, as required.

(d) For all $(x,y) \in \bar{\Omega}$, we have by (B.1.6),

$$|h(x,y)| \ge |h_1(x)| \ge p > 0. \tag{B.1.9}$$

[Notice that each term in (B.1.6a-b-c) is non-positive on its specific range.]

Remark B.1.1. As pointed out at the outset, the first two requirements (i) = (1.1.4) and (ii) = (1.1.5) of assumption (A.1) are satisfied also by taking $x_2 = a$ in which case the region III vanishes and Γ_0 is connected.

Remark B.1.2. We remark explicitly that, in the present example, we have $h \cdot \nu \ge 0$ on $\Gamma_1' = \{(x,y) \in \Gamma_1 : -a \le x \le x_1\}$, but not on $\Gamma_1'' \equiv \Gamma_1 \setminus \Gamma_1' \equiv \{(x,y) \in \Gamma_1 : x_2 \le x \le a\}$. Indeed, condition $h \cdot \nu \ge 0$ on all of Γ_1 and condition (A.2) = (1.1.6) appear to be incompatible. □

Example B.1.2. (non-convex Ω) We now let Ω be a region exterior to the critical part of the boundary $\Gamma_0 \cup \Gamma_1''$ of Example B.1.1: see Fig. B.2.

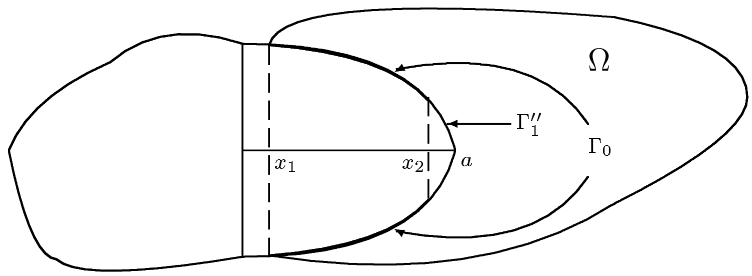

FIG. B.2: EXAMPLE B.1.2

This case is actually simpler than Example B.1.1: here the counterpart of Lemma B.1.1 is that: the function $d(x,y) \in C^3$, with gradient $\nabla d = h = (h_1, h_2)$:

$$d(x,y) = \begin{cases} d^{II}(x) + \dfrac{y^2}{2} \\ d^{III}(x) + \dfrac{y^2}{2} \end{cases} ; \quad h_1(x) = \begin{cases} h_1^{II}(x) & 0 < x_1 \le x \le x_2 < a; \quad \text{(B.1.10a)} \\ h_1^{III}(x) & x_2 \le x \le x_3. \quad \text{(B.1.10b)} \end{cases}$$

$h_2(y) = y$, with $d^{II}(x)$, $d^{III}(x)$ defined by (B.1.4b-c), hence $h_1^{II}(x)$, $h_1^{III}(x)$ defined by (B.1.6b-c), except that now $3\epsilon(x_3 - x_2)^2 < \frac{1}{2}$, *satisfies assumption (A.1) of Section 1.* Again, the first two requirements (i) = (1.1.4) and (ii) = (1.1.5) of assumption (A.1) are fulfilled also with $x_2 = a$, in which case $h_1(x) = x - \frac{a^2}{x}$, $0 < x_1 \le x \le x_3$; $h_2(y) = y \in \mathbb{R}$ does the job. $\square$

Example B.1.3. (parabola) Here $\ell(x,y) = x - 1 + y^2 \equiv 0$ and then $-y\frac{\ell_y}{\ell_x} = -2y^2 = 2x - 2$, $(x,y) \in \Gamma_0$. Then

$$h_1(x) = 2x - 2, \quad h_2(y) = y \tag{B.1.11}$$

provides the required vector field $h(x,y)$, $h_1'(x) = 2 > 0$, which satisfies assumption (A.1) everywhere within, or without the parabolic sector. However, to satisfy also assumption (A.2) we must exclude the vertex point $\{x = 1, y = 0\}$, as in Example B.1.1.

Example B.1.4. (hyperbola) Here $\ell(x,y) = x^2 - y^2 - k^2 \equiv 0$ and then $-y\,\frac{\ell_y}{\ell_x} = x - \frac{k^2}{x}$, $(x,y) \in \Gamma_0$. Then

$$h_1(x) = x - \frac{k^2}{x}, \quad h_2(y) = y, \quad |x| \ge \epsilon > 0 \tag{B.1.12}$$

provides the required vector field $h(x,y)$, $h_1'(x) = 1 + k^2/x^2 > 0$, which satisifies assumption (A.1) everywhere within, or without, the hyperbolic sector, away from the y-axis.

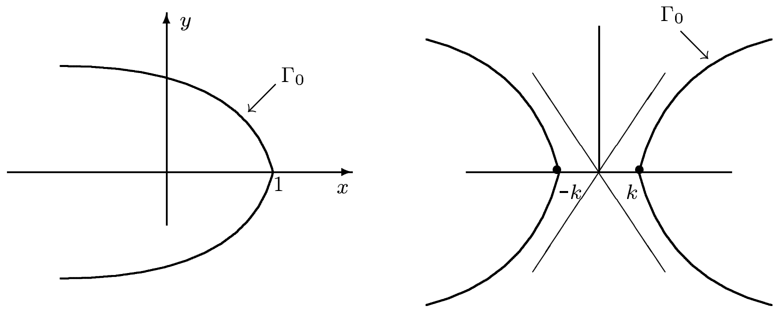

FIG. B.3: EXAMPLES B.1.3 (PARABOLA) AND B.1.4 (HYPERBOLA)

Example B.1.5. (logarithm) Here $f(x) = \ln x$, so that

$$h_1(x) = \frac{f(x)}{f'(x)} = x \ln x, \quad h_2(y) = y \tag{B.1.13}$$

provides the required vector field, however, under the constraint $x > \frac{1}{e}$, whereby then $h_1'(x) = 1 + \ln x > 0$, as required. We note, however, that if the *same* curve is viewed as $y = f(x) = e^x$, then the test of Lemma A.2.2 [which requires a suitable choice of the coordinate axes] fails in x [but the test of Lemma A.2.2 works in y, as seen from the analysis above for $x = \ln y$], as now the vector $\{1, y\}$, $\frac{f(x)}{f'(x)} \equiv 1$, suggested by Lemma A.2.2 does not satisfy the positivity condition of its Jacobian matrix. However, that test can be easily modified to yield a positive conclusion. For $(x, y) \in \Gamma_0$, described by $y = e^x$, we take the tangential vector $h = \{e^x \cdot 1, e^x \cdot y\}$ [instead of $\{1, y\}$], which we then extend to all (x, y), $y > 0$, as $h = \{e^x, y^2\}$. Then this vector field h satisfies assumption (A.1) of Section 1 for the exponential curve $y = e^x$ on any finite interval in x.

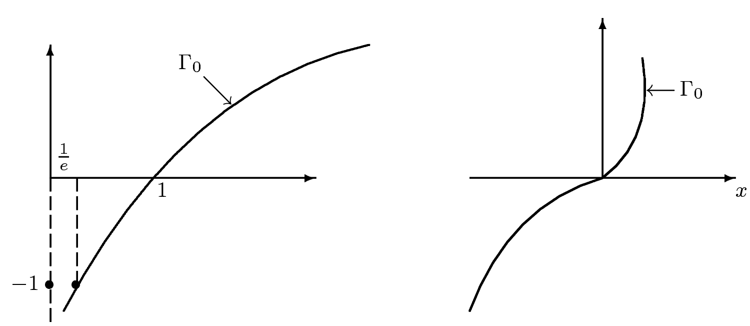

FIG. B.4: EXAMPLE B.1.5 (LOGARITHM) AND B.1.6 (CUBIC)

Thus, $h(x)$ satisfies assumption (A.1) for $x > \frac{1}{e}$.

Example B.1.6. (cubic) Here $f(x) = x^3$ so that

$$h_1(x) = \frac{f(x)}{f'(x)} = \frac{1}{3}x, \quad h_2(y) = y \tag{B.1.14}$$

provides the required vector field $h(x, y)$, as $h_1'(x) \equiv \frac{1}{3} > 0$.

Example B.1.7. (sine and cosine) Here $f(x) = \sin x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$; or else $f(x) = \cos x$, $0 < x < \pi$. Then the vector field

$$h_1(x) = \frac{f(x)}{f'(x)} = \begin{cases} \tan x \\ -\cot x \end{cases} \qquad h_2(y) = y \tag{B.1.15}$$

provides the required vector field so that $h_1'(x) = 1/\cos^2 x$, or else $h_1'(x) = 1/\sin^2 x$, respectively, are positive in the indicated intervals

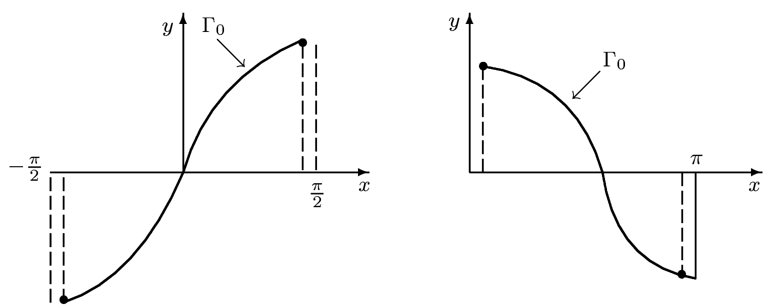

FIG. B.5: EXAMPLE B.1.7 (SINE AND COSINE)

B.2 Three-dimensional analysis: Extension of Section A.2

In this subsection we extend the analysis of Section A.2 from the 2-dimensional to the 3-dimensional case. Let Γ_0 be a 3-dimensional surface described as a level set:

$$\Gamma_0 = \{(x, y, z) \in R^3 : \ell(x, y, z) = 0\}, \quad \ell \in C^3, \tag{B.2.1}$$

with $|\nabla \ell| \neq 0$ on Γ_0. Let $P = (x, y, z) \in \Gamma_0$. Assume that Γ_0 displays symmetry with respect to the x-axis [or else to the y-axis; or else to the z-axis, respectively]. Then, we seek the point X [or else the point Y; or else the point Z, respectively] of intersection between the tangent plane at P and the x-axis [or else the y-axis; or else the z-axis, respectively]. We then consider the vector fields

$$\overrightarrow{XP} = \left\{ -\frac{y\ell_y + z\ell_z}{\ell_x}, y, z \right\}, \quad P = (x, y, z) \in \Gamma_0, \ \ell_x \neq 0; \tag{B.2.2}$$

$$\overrightarrow{YP} = \left\{x, -\frac{x\ell_x + z\ell_z}{\ell_y}, z\right\}, \quad P = (x,y,z) \in \Gamma_0,\ \ell_y \neq 0; \tag{B.2.3}$$

$$\overrightarrow{ZP} = \left\{x, y, -\frac{x\ell_x + y\ell_y}{\ell_z}\right\}, \quad P = (x,y,z) \in \Gamma_0,\ \ell_x \neq 0, \tag{B.2.4}$$

respectively. Thus, in each case, the corresponding vector $\overrightarrow{XP}$, $\overrightarrow{YP}$, $\overrightarrow{ZP}$ is tangent to Γ_0 at P.

In many cases, at least one of the following conditions is satisfied by $\ell(x,y,z)$: for $(x,y,z) \in \Gamma_0$, then

$$\begin{cases} \text{either the term } \left(-\frac{y\ell_y+z\ell_z}{\ell_x}\right)(x) \text{ is only a function of } x;\ \ell_x \neq 0; \\ \text{or else the term } \left(-\frac{x\ell_x+z\ell_z}{\ell_y}\right)(y) \text{ is only a function of } y;\ \ell_y \neq 0; \\ \text{or else the term } \left(-\frac{x\ell_x+y\ell_y}{\ell_z}\right)(z) \text{ is only a function of } z;\ \ell_z \neq 0. \end{cases} \tag{B.2.5}$$

Then, in either case, we extend the boundary vectors $\overrightarrow{XP}$, $\overrightarrow{YP}$, $\overrightarrow{ZP}$ in (B.2.2)–(B.2.4) to all points (x,y,z) near Γ_0 as follows: define a vector field $h(x,y,z) = \{h_1, h_2, h_3\}$, where

$$h_1(x) = \left(-\frac{y\ell_y + z\ell_z}{\ell_x}\right)(x),\ h_2(y) = y;\ h_3(z) = z, \tag{B.2.6}$$

or else

$$h_1(x) = x;\ h_2(y) = \left(-\frac{x\ell_x + z\ell_z}{\ell_y}\right)(y),\ h_3(z) = z, \tag{B.2.7}$$

or else

$$h_1(x) = x;\ h_2(y) = y;\ h_3(z) = \left(-\frac{x\ell_x + y\ell_y}{\ell_z}\right)(z), \tag{B.2.8}$$

respectively. An extension of Lemma A.2.2 is then:

Lemma B.2.1. Let $\ell \in C^3$ and assume hypothesis (B.2.5) with either $\ell_x \neq 0$, or $\ell_y \neq 0$, or $\ell_z \neq 0$, respectively, unless a cancellation occurs. Assume, moreover, that on Γ_0, either

$$\begin{cases} \frac{d}{dx}\left[-\frac{y\ell_x + z\ell_z}{\ell_x}\right](x) > 0; \quad \text{or } \frac{d}{dy}\left[-\frac{x\ell_x + z\ell_z}{\ell_y}\right](y) > 0; \\ \text{or } \frac{d}{dz}\left[-\frac{x\ell_x + y\ell_y}{\ell_z}\right](z) > 0, \end{cases} \tag{B.2.9}$$

respectively. Then:

(i) the Jacobian matrix J_h of the vector field h defined in (B.2.6)–(B.2.8), respectively, is positive definite on, and near, Γ_0;

(ii) the vector field $h = \{h_1, h_2, h_3\}$ defined in (B.2.6)–(B.2.8) respectively, satisfies $h = \overrightarrow{XP}$, or $h = \overrightarrow{YP}$, or $h = \overrightarrow{ZP}$ for $P \in \Gamma_0$, respectively, and thus $h \cdot \nu = 0$ on Γ_0;

(iii) the vector field $h(x, y, z)$ is conservative, $h = \nabla d$, where the function $d(x, y, z) \in \ell^3$ defined near Γ_0 by

$$d(x,y,z) = \begin{cases} \displaystyle\int \left(-\frac{y\ell_y + z\ell_z}{\ell_x} \right)(x)dx + \frac{y^2}{2} + \frac{z^2}{2}, & \text{for } h \text{ in (B.2.6);} \quad \text{(B.2.10)} \\ \displaystyle\frac{x^2}{2} + \int \left(-\frac{x\ell_x + z\ell_z}{\ell_x} \right)(y)dy + \frac{z^2}{2}, & \text{for } h \text{ in (B.2.7);} \quad \text{(B.2.11)} \\ \displaystyle\frac{x^2}{2} + \frac{y^2}{2} + \int \left(-\frac{x\ell_x + y\ell_y}{\ell_z} \right)(z)dz, & \text{for } h \text{ in (B.2.8);} \quad \text{(B.2.12)} \end{cases}$$

satisfies the first two conditions (i) = (1.1.4) and (ii) = (1.1.5) of assumption (A.1) of Section 1. □

Illustrations. Example B.2.1. (ellipsoid) Here $\Gamma_0 : \ell(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{z^2} - 1 = 0$. Then, for $(x, y, z) \in \Gamma_0$, we find

$$-\frac{y\ell_y + z\ell_z}{\ell_x} = -\frac{a^2}{x}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = x - \frac{a^2}{x}, \quad 0 < x \le a, \tag{B.2.13}$$

a function only of x, and the first condition in (B.2.5) is satisfied. Then, according to Lemma B.2.1,

$$h_1(x) = x - \frac{a^2}{x}, \quad h_2(y) = y, \quad h_3(z) = z, \quad J_h = \begin{bmatrix} 1 + \frac{a^2}{x^2} & & \\ & 1 & \\ & & 1 \end{bmatrix} \tag{B.2.14}$$

provides the required vector field, $h_1'(x) = 1 + \frac{a^2}{x^2} > 0$, $0 < x \le a$, which satisfies assumption (A.1) of Section 1, for any $0 < \rho \le x \le a$. An analysis as in Example B.1.1 could be carried out, extending the corresponding functionn $d(x, y, z)$ to the entire ellipsoidal region.

Example B.2.2. (elliptic paraboloid) Here $\ell(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$ with symmetry with respect to the z-axis. Accordingly, we test the third condition in (B.2.5). For $(x, y, z) \in \Gamma_0$, we find

$$-\frac{x\ell_x + y\ell_y}{\ell_z} = 2c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 2z, \tag{B.2.15}$$

a function only of z. Thus, the first condition of (B.2.5) is satisfied. Then, according to Lemma B.2.1,

$$h_1(x) = x, \quad h_2(y) = y, \quad h_3(z) = 2z, \quad J_h = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix} \tag{B.2.16}$$

provides the required vector field, $h_3'(z) = 2 > 0$, which satisfies assumption (A.1) of Section 1.

Example B.2.3. (hyperbolic paraboloid) Here $\ell(x, y, z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0$, with symmetry with respect to the z-axis. Accordingly, we test the third condition in (B.2.5). For $(x, y, z) \in \Gamma_0$, we find

$$-\frac{x\ell_x + y\ell_y}{\ell_z} = 2c\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 2z, \tag{B.2.17}$$

a function only of z. Thus, the same definition as in (B.2.16) provides the required vector field, which satisfies assumption (A.1) of Section 1.

Example B.2.4. (hyperboloid of one sheet) Here, $\ell(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$, with symmetry with respect to the z-axis. Accordingly,

$$-\frac{x\ell_x + y\ell_y}{\ell_z} = \frac{c^2}{z}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = z + \frac{c^2}{z}, \qquad 0 < z, \tag{B.2.18}$$

a function only of z. The third condition of (B.2.5) is satisfied. Then, according to Lemma B.2.1,

$$h_1(x) = x, \quad h_2(y) = y, \quad h_3(z) = z + \frac{c^2}{z}; \quad J_h = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 - \frac{c^2}{z^2} \end{bmatrix} \tag{B.2.19}$$

provides the required vector field, which satisfies assumption (A.1) of Section 1, for $|z| > |c|$.

Example B.2.5. (hyperboloid of two sheets) Here $\ell(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0$, with symmetry with respect to the z-axis. Accordingly,

$$-\frac{x\ell_x + y\ell_y}{\ell_z} = \frac{c^2}{z}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = z - \frac{c^2}{z}, \quad 0 < z, \tag{B.2.20}$$

a function only of z. Thus, the third condition of (B.2.5) is satisfied. Then, according to Lemma B.2.1,

$$h_1(x) = x, \quad h_2(y) = y, \quad h_3(z) = z - \frac{c^2}{z}; \quad J_h = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 + \frac{c^2}{z^2} \end{bmatrix} \tag{B.2.21}$$

provides the required vector field, which satisfies assumption (A.1) of Section 1, for $z > 0$.
□

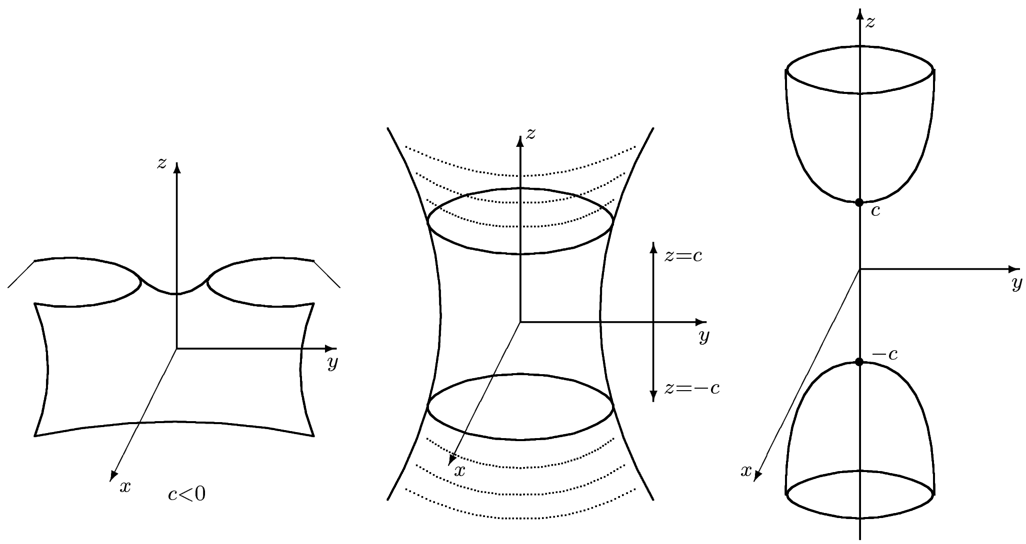

FIG. B.6: EXAMPLES B.2.3 (HYPERBOLIC PARABOLOID), B.2.4 (HYPERBOLOID OF ONE SHEET), B.2.5 (HYPERBOLOID OF TWO SHEETS)

B.3 Illustrations of the conformal mapping approach of Section A.3

In this subsection we provide a few illustrations of the conformal mapping approach presented in Section A.3 to obtain 2-dimensional curves C_0, where assumption (A.1) of Section 1 holds true.

Example B.3.1. We take the conformal mapping

$$f(z) = z^2 = u + iv, \quad u(x,y) = x^2 - y^2; \quad v(x,y) = 2xy, \tag{B.3.1}$$

which maps the hyperbola C_0:

$$C_0 : x^2 - y^2 = u_0, \quad u_0 > 0 \text{ onto the line } \Gamma_0 : u \equiv u_0 > 0. \tag{B.3.2}$$

We test condition (A.3.19) of Corollary A.3.3, with $u_x = 2x,\ v_x = 2y,\ v_{xx} \equiv 0, v_{xy} = 2$, on C_0:
(i)

$$\begin{aligned} |f'(z)|^4 - v^2(v_{xx}^2 + v_{xy}^2) &= [4(x^2+y^2)]^2 - 4xy \cdot 4 = 16(x^4 + y^4 + 16x^2y^2) \\ &= 16[(u_0+y^2)^2 + y^4 + 16(u_0+y^2)y^2 > 0. \end{aligned} \tag{B.3.3}$$

Thus, the assumption of Corollary A.3.3 is satisfied. Then, Corollary A.3.3 yields the following: any finite portion of the hyperbola $x^2 - y^2 = u_0$, $u_0 > 0$ *satisfies assumption*

(A.1) of Section 1, thus re-proving by conformal mapping methods the result of Example B.1.4. More precisely, the function

$$\delta(x,y) = \frac{1}{2}[(u(x,y)-u_0)^2 + v^2(x,y)] = \frac{1}{2}[(x^2-y^2-u_0)^2 + 4x^2y^2] \tag{B.3.4}$$

[see (A.3.11) plus (B.3.1)] satisfies assumption (A.1) of Section 1 for any portion C_0 of the hyperbola, with corresponding vector field $h = \nabla d$ given by (see (A.3.12) in a different notation $\delta(x,y)$]

$$\begin{cases} h_1 &= \delta_x = (u-u_0)u_x + vv_x = 2(x^3+2xy^2-xy^2-u_0x); \\ h_2 &= \delta_y = (u-u_0)u_y + vv_y = 2(2x^2y - x^2y + y^3). \end{cases} \tag{B.3.5}$$

Example B.3.2. Here we take the conformal mapping

$$f(x) = e^z = u + iv, \quad u(x,y) = e^x\cos y, \quad v(x,y) = e^x \sin y, \tag{B.3.6}$$

which maps the curve

$$C_0: \ e^x\cos y \equiv u_0 \neq 0 \text{ onto the line } \Gamma_0: \ u \equiv u_0. \tag{B.3.7}$$

We test condition (A.3.19) of Corollary A.3.3, with $u_x = u$, $v_x = v_{xx} = v$, $v_{xy} = e^x\cos y$, on C_0:

(i)

$$\begin{aligned} |f'(z)|^4 - v^2(v_{xx}+v_{xy}^2) &= e^{4x} - e^{2x}\sin^2 y(e^{2x}) \\ &= e^{4x}\cos^2 y = e^{2x}u_0^2 > 0, \quad u_0 \neq 0. \end{aligned} \tag{B.3.8}$$

Thus, the assumption of Corollary A.3.3 is satisfied. Then, Corollary A.3.3 yields the following: *any finite portion of the curve $C_0: e^x\cos y = u_0$, or $x = \ln(u_0/\cos y)$ satisfies assumption (A.1) of Section 1,* with function (see (A.3.11)),

$$\delta(x,y) = \frac{1}{2}[(u(x,y)-u_0)^2 + v^2(x,y)] = \frac{1}{2}(e^{2x} - 2u_0e^x\cos y + u_0^2), \tag{B.3.9}$$

and vector field $h = \nabla\delta$ given by (see (A.3.12)),

$$h_1 = \delta_x = e^{2x} - u_0e^x\cos y; \quad h_2 = \delta_y = u_0e^x\sin y. \tag{B.3.10}$$

Remark B.3.1. By contrast, if we apply the test of Lemma A.2.2 to the curve $C_0: \ \ell(x,y) = e^x\cos y - u_0 = 0$, or $x = -\ln\cos y = g(y)$ for say, $u_0 = 1$, we find a worse conclusion. Indeed, we find that on the curve C_0, where $\cos y = e^{-x}$, we have

$$\frac{d}{dy}\left(\frac{g(y)}{g'(y)}\right)\Big|_{C_0} = 1 - x - e^{-2x} = 1 + \ln(\cos y) - \cos^2 y, \tag{B.3.11}$$

and thus

$$\frac{d}{dy}\left(\frac{g(y)}{g'(y)}\right)\Big|_{C_0}\begin{cases} > 0 & \text{for } 0 \le |y| < y_1; \quad \text{(B.3.12)} \\ \le 0 & \text{for } y_1 \le |y| \le \frac{\pi}{2}, \quad \text{(B.3.13)} \end{cases}$$

for some point y_1, where $e^{-x_1} = \cos y_1 < \frac{1}{e}$, $x_1 = 0.79681\ldots$.

This is a result *less precise* than the one obtained above by conformal mapping.

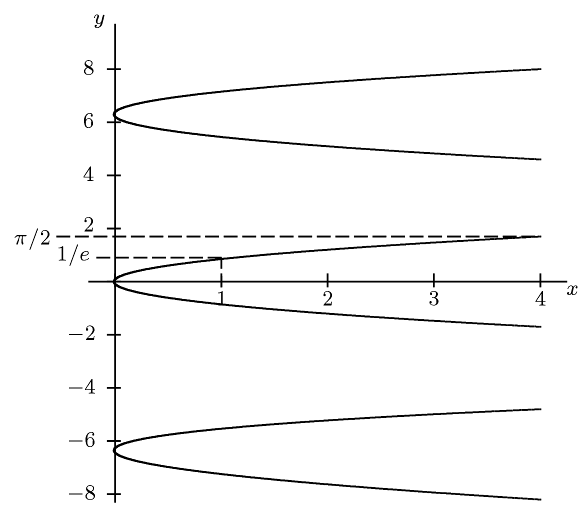

FIG. B.7: PLOT OF $e^x \cos y = 1$

Example B.3.3. Again we take the conformal mapping $f(z) = e^z$ in (B.3.7), which this time we view as a mapping from the family of vertical lines in the (x, y)-plane onto the family of circles centered at the origin in the (u, v)-plane:

$$C_0 : x \equiv x_0 \qquad \text{onto } \Gamma_0 : u^2 + v^2 = e^{2x_0}. \tag{B.3.14}$$

Specialize to $x_0 = 0$, so that for the unit circle Γ_0 we make take, according to Example B.1.1, Eqn. (B.1.2),

$$d(u, v) = \frac{1}{2}(u^2 + v^2) - \ln u, \quad \nabla d = h = \left\{u - \frac{1}{u}, v\right\} \tag{B.3.15}$$

to satisfy assumption (A.1) of Section 1, for $0 < u \le 1$. Then, according to Section A.3, Theorem A.3.2, we take

$$\delta(x, y) = \frac{1}{2}[u^2(x, y) + v^2(x, y)] - \ln u(x, y) = \frac{1}{2}e^{2x} - \ln(e^x \cos y), \tag{B.3.16}$$

whose gradient vector field is then

$$\tilde{h} = \nabla\delta = \{\delta_x, \delta_y\} = \{e^{2x} - 1, \tan y\}. \tag{B.3.17}$$

On C_0, where $x \equiv 0$, this vector field satisfies (in agreement with Example B.1.1 and Theorem A.3.2, the orthogonality condition

$$\text{on } C_0 : \tilde{h}|_{C_0} = \{0, \tan y\}, \text{ so that } \tilde{h} \cdot \nu = 0 \text{ on } C_0. \tag{B.3.18}$$

Moreover, the Hessian matrix $\mathcal{H}_\delta$ is (since $\delta_{xx} = 2e^{2x}$, $\delta_{xy} \equiv 0$, $\delta_{yy} = 1/\cos^2 y$,

$$\mathcal{H}_\delta(C_0) = \begin{bmatrix} 2e^{2x} & 0 \\ 0 & \dfrac{1}{\cos^2 y} \end{bmatrix} (C_0) = \begin{bmatrix} 2 & 0 \\ 0 & \dfrac{1}{\cos^2 y} \end{bmatrix} > 0, \tag{B.3.19}$$

provided $-\frac{\pi}{2} < y < \frac{\pi}{2}$ in which case Γ_0 is arbitrarily closed to a half-circumference (in agreement with Example B.1.1, while C_0 is arbitrarily close to the straight segment: $x = 0$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Example B.3.4. Here, we take the conformal mapping

$$f(z) = \frac{1}{z} = u + iv, \quad u(x,y) = \frac{x}{x^2+y^2}, \quad v(x,y) = \frac{-y}{x^2+y^2}, \tag{B.3.20}$$

which maps the unit circle C_0

$$C_0 : \ (x-1)^2 + y^2 = 1 \text{ onto the straight line } \Gamma_0 : \ u \equiv \frac{1}{2}. \tag{B.3.21}$$

We test condition (A.3.19) of Corollary A.3.3 and after straightforward computations we obtain that on C_0:

$$|f'(z)|^4 - v(v_{xx}^2 + v_{xy}^2 > 0 \quad \text{iff} \quad 16x^4(2x-3) > 0, \text{ i.e.,} \quad \text{iff } x > \frac{3}{2}. \tag{B.3.22}$$

Thus, only the portion of the circle C_0 : $(x,y) \in C_0$ with $\frac{3}{2} < x \leq 2$ satisfies the test. Thus, in this case, this mapping $f(z)$ does *not* provide an optimal result. As we know, "the optimal case" is when C_0 is arbitrarily close to half-circle, which would require $x > 1$ on C_0. Thus, in this example, we may take in the (x,y)-plane *any* finite segment $u \equiv \frac{1}{2}$, $v_0 \leq v \leq v_1$, while in the (u,v)-plane the arc C_0 is *limited* by $\frac{3}{2} < x \leq 2$.

Appendix C: Illustrations of the setting of Section 10: Assumptions (A.1i) and (A.2i)

Sections A.1 through B.2 have provided large classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ where assumption (A.1) of Section 1 is satisfied. Throughout those illustrations, we have also noted, however, that the additional requirement of fulfilling also assumption (A.2) = (1.1.6) of

Section 1 imposes geometrical limitations on the allowed triples $\{\Omega, \Gamma_0, \Gamma_1\}$, which are covered by Theorem 2.1.1, under hypotheses (A.1) and (A.2). The setting of Section 10 is meant to relax these geometrical restrictions, by mitigating the impact of assumption (A.2).

The multi-faced treatment of the entire Appendix (A, B, C) then culminates with the following result, which shows that the setting of Section 10 does apply to large classes of triples $\{\Omega, \Gamma_0, \Gamma_1\}$, constructed by various techniques.

Theorem C.1. Let $\{\Omega, \Gamma_0, \Gamma_1\}$ be given with $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Assume that it satisfies: (i) either the setting of Sections A.1 and A.2 (or its 3-dimensional generalization in Section B.2); in particular, Theorem A.2.1, Lemma A.2.2, or Lemma B.2.1; (ii) or else the setting of Section A.4, in particular, Theorem A.4.1; (iii) or else the setting of Section A.3, in particular, Theorem A.3.2. Thus, in all these cases there exists (constructively) a scalar function $d \in C^3$ (called δ in Section A.3) such that the conservative vector field $h = \nabla d$ satisfies assumption (A.1) [i.e., both conditions (1.1.4) and (1.1.5)] of Section 1. Moreover, in all these cases, there is at most one point $P \in \Gamma_0$ (called C_0 in Section A.3) such that: $h(P) = \nabla d(P) = 0$; so that assumption (A.2) = (1.1.6) of Section 1 fails.

Then, it is possible to split Ω as the union of two overlapping subsets Ω_1, Ω_2 as in (10.1.1) of Section 10, such that both assumptions (A.1i) [i.e., both conditions (10.1.2) and (10.1.3)] and (A.2i) = (10.1.4) hold true.

Proof. The case where Γ_0 is flat. This case was already treated (by the simplest method) and illustrated in Section A.1, thus providing a special but important subclass of triples $\{\Omega, \Gamma_0, \Gamma_1\}$ where the setting of Section 10 is fulfilled, and thus Theorem 10.1.1 holds true.

Another method which employs the technique of Section A.4 is illustrated in Fig. C.1.

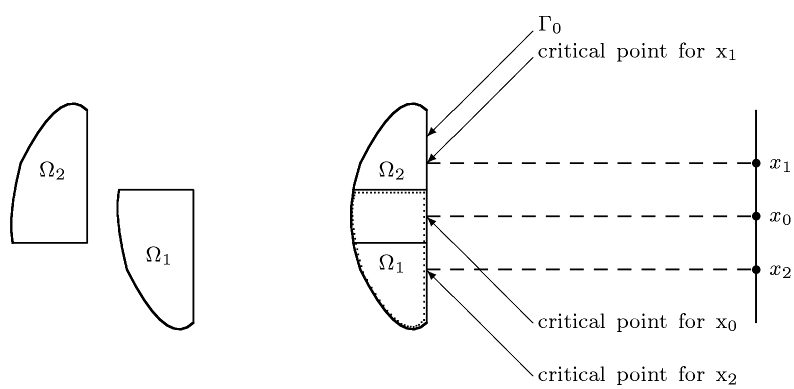

Fig. C.1: Decomposition of Ω with flat Γ_0, as required by the setting of Section 10, according to the construction of two vector fields provided by the method of Section A.4.

The case where Γ_0 has a U-turn with one critical point for $d(x)$ on Γ_0 (according to Section A.2 paragraph below (A.2.11); Corollary A.3.4; Corollary A.4.2). We begin by illustrating the idea in the case where Γ_0 is arbitrarily close to a half-circumference.

The case where Γ_0 is arbitrarily close to a half-circumference. We now show how it is possible to take the unobserved (or uncontrolled) portion Γ_0 of the boundary of the unit disk to be arbitrarily close to a half-circumference. To this end, we need the setting of Section 10. This analysis, therefore, improves upon that of Example B.1.1, which was only based on the setting of Section 1, whereby the sub-boundary Γ_0 obtained there was non-connected.

Sections A.2 and A.4 have provided two distinct geometrically intrinsic ways of constructing conservative vector fields $h = \nabla d$, which both satisfy Assumption (A.1) of Section 1, but have a critical point for d on the mid-point of Γ_0 (point D in Fig C.3).

Method of Section A.2. With reference to Fig. C.2, let Ω_1 be the circular sector OAB, and let Ω_2 be the circular sector $OA'B'$. Thus, Ω_2 is the mirror image of Ω_1 with respect to the x-axis, and we shall then confine on the description of Ω_1. The axis OA makes an angle ϵ with the positive y-axis. The positive η-axis makes an angle $\frac{\epsilon}{2}$ with the positive y-axis, and an angle $\frac{\pi}{2}$ with the positive ξ-axis. The axis OB makes an angle $\frac{\epsilon}{4}$ with the x-axis and an angle $\frac{\epsilon}{4}$ with the positive ξ-axis. We have that $\Omega = \Omega_1 \cup \Omega_2$, the circular sector $OAB'BA'$, which is arbitrarily close to the half-disk, while $\Omega_1 \cap \Omega_2 \neq \emptyset$, as required. We claim that we can take $\Gamma_0 =$ arc $AB'BA'$ as the unobserved or uncontrolled portion of the boundary. We now verify assumptions (A.1i) in (10.1.2), (10.1.3) and (A.2i) = (10.1.4), $i = 1, 2$.

Regarding Ω_1. Given a point $P = (\xi, \eta)$ [with respect to the orthogonal system (ξ, η)] on the arc AB (= boundary of Ω_1), by the analysis of Section A.2 and its specialization in Example B.1.1, we begin by taking the boundary vector field $h_1 = \{h_{1,1}, h_{1,2}\} =\overrightarrow{XP}$, where X is the point of intersection of the tangent line to Γ_0 at P with the ξ-axis. Next, as in Section A.2, Eqn. (A.2.17), we extend h_1 to all of the circular sector OAN containing Ω_1, as follows: let $P' = (\xi', \eta')$ [with respect to the orthogonal system (ξ, η)] be in Ω_1. Then, take $h_1 = \{h_{1,1}, h_{1,2}\}$, where [still with respect to the (ξ, η) axes]:

$$\begin{cases} h_{1,1} &= \text{horizontal component } XF \text{ of } \overrightarrow{XP}; \\ h_{1,2} &= \text{vertical component } FP' = \eta'. \end{cases}$$

Then, as seen in Section A.2, this vector field h_1, which is defined geometrically, satisfies assumption $(A.1_1)$ in (10.1.2), (10.1.3). Moreover, such a vector field $\overrightarrow{XP}= h_1$ vanishes only at the boundary point N of intersection between the circumference and the ξ-axis; which is not a point in Ω_1. Thus, for all $P \in$ arc $AB =$ boundary of Ω_1, the vector field

$\overrightarrow{XP}$ has a length which is bounded away from zero. Thus, h_1 satisfies assumption (A.2_1) = (10.1.4), as well.

The analysis for Ω_2 is symmetric. In conclusion: *Theorem C.1 and hence Theorem 10.1.1 hold true with* $\Gamma_0 =$ *arc* $AB'BA'$ *arbitrarily close to the half-circumference.*

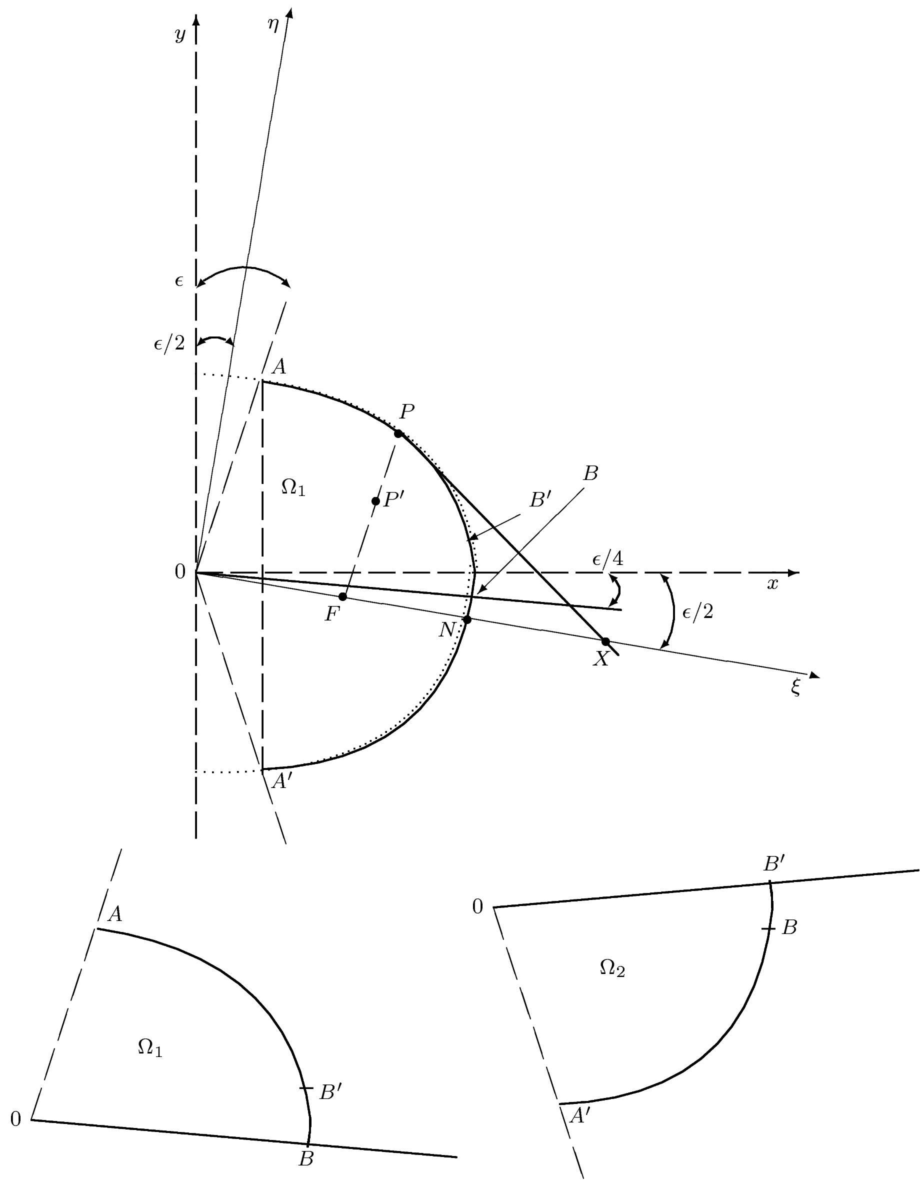

Fig. C.2: Decomposition of a circular region with Γ_0 arbitrarily close to a half-circle, as required by the setting of Section 10, according to the construction of two vector fields provided by the method of Section A.2.

Method of Section A.4. This is illustrated in Fig. C.3. The circular region $\Omega = ABCDEFGHLA$ with boundary $\Gamma_0 = ABCDEFG$ is split as the union of two overlapping subdomains $\Omega_1 = ABCDEHA$ and $\Omega_2 = CDEFGLC$, as required by (10.1.1). The mid-point D of Γ_0 is the original critical point for the scalar function $d(x)$, *constructed through Eqns. (A.4.6), (A.4.7), with respect to the original external point* x_0 (see Corollary A.4.2).

Next, we replace x_0 with x_1 [respectively, x_0 with x_2] and produce, accordingly, a function $d_1(x)$ [respectively, a function $d_2(x)$], defined by the counterpart of Eqns. (A.4.6), (A.4.7), where this time $d_0(x)$ is defined as $d_{0,1}(x) = \frac{1}{2}\|x - x_1\|^2$ in the first case, and as $d_{0,2}(x) = \frac{1}{2}\|x - x_2\|^2$ in the second case. We claim that:

> *the vectors required by the setting of Section 10 are* $h_1(x) = \nabla d_1(x)$ *and* $h_2(x) = \nabla d_2(x)$.

Indeed, they satisfy assumption (A.1i) [i.e., both conditions (10.1.2) and (10.1.3) by Theorem A.4.1.

Moreover, the critical point for $d_1(x)$ is $F \in \Gamma_0$ (see Corollary A.4.2), where $F \notin \Omega_1$, and similarly the critical point for $d_2(x)$ is $B \in \Gamma_0$, where $B \notin \Omega_2$. Thus, $|h_1(x)| = |\nabla d_1(x)| \geq p_1 > 0$ for all $x \in \Omega_1$; and similarly, $|h_2(x)| = |\nabla d_2(x)| \geq p_2 > 0$ for all $x \in \Omega_2$. Thus, $h_i(x)$ satisfy also assumption (A.2i) = (10.1.4). Our claim is established. Thus, Theorem C.1 is proved in this case, and then Theorem 10.1.1 holds true.

Thus, *Theorem C.1 and Theorem 10.1.1 hold true with* $\Gamma_0 = ABFG$ *arbitrarily close to a half-circle.*

The general case where Γ_0 has a U-turn and satisfies either Theorem A.2.1 or Lemma B.2.1, or else Theorem A.4.1 requires only minor modifications over the case of Γ_0 being close to a half-circumference. □

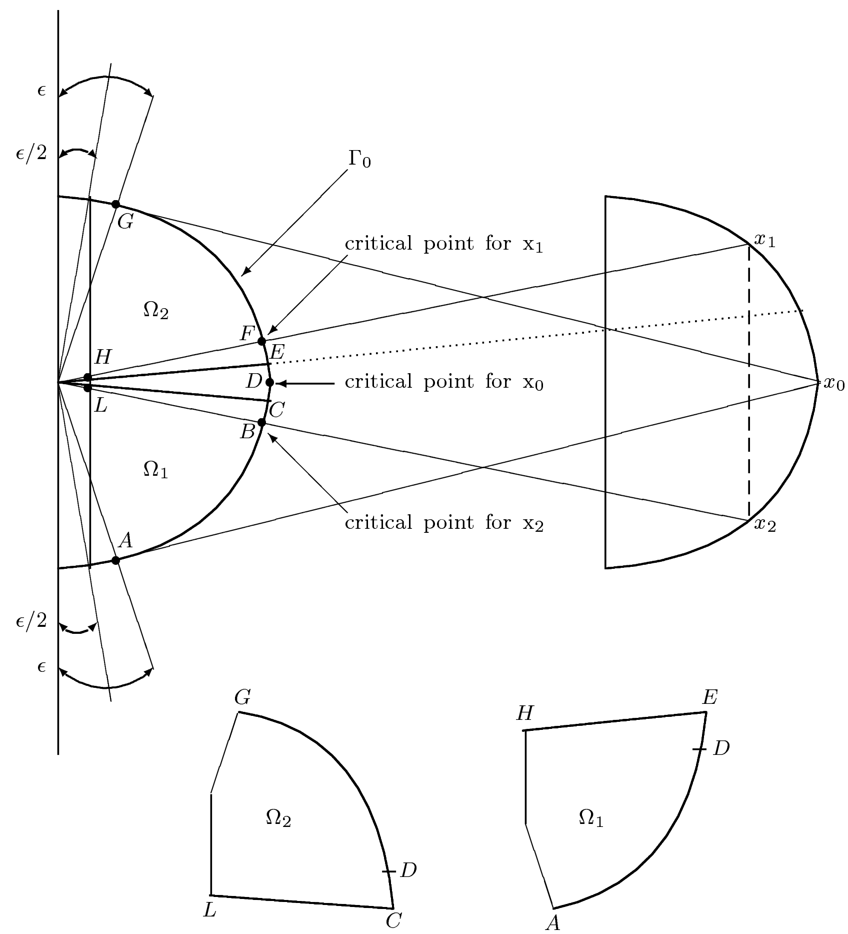

FIG. C.3: DECOMPOSITION OF A CIRCULAR REGION WITH Γ_0 ARBITRARILY CLOSE TO A HALF-CIRCLE, AS REQUIRED BY THE SETTING OF SECTION 10, ACCORDING TO THE CONSTRUCTION OF TWO VECTOR FIELDS PROVIDED BY THE METHOD OF SECTION A.4.

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