Uniqueness, renormalization and smooth approximations for transport equations

Gianluca Crippa - *Scuola Normale Superiore, Pisa*
Joint work with F. Bouchut (Ecole Normale Supérieure, Paris)

The Cauchy problem for the transport equation

$$\begin{cases} \partial_t u + \mathrm{div}\,(bu) = 0 \\ u(0,\cdot) = \bar{u} \end{cases}$$

in the case when the vector field $b : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ is not smooth is of great interest for the applications to conservation laws and to the study of the motion of fluids.

An important concept in this framework is that of renormalized solution, introduced by DiPerna and Lions in 1989: it corresponds (for example in the divergence free case) to the request that, if u is a solution, then also $\beta(u)$ satisfies the equation, for every smooth function β. For smooth solutions this is always true, due to the classical chain rule formula. Out of the smooth framework, it turns out that the renormalization property is a useful tool in showing the well-posedness of the Cauchy problem: it gives in a natural way a comparison principle for (strongly continuous with respect to the time) solutions, and this implies easily uniqueness.

In the same paper, DiPerna and Lions show that if b has Sobolev regularity with respect to the space, then the renormalization property holds. Their proof consists in regularizing the equation convolving it with a smooth convolution kernel and showing that the regularity of the vector field is sufficient to pass to the limit in a strong enough sense. This strong convergence allows the passage to the limit in the approximate equation, for which the renormalization is trivial because we are dealing with smooth functions.The more recent proof of the renormalization property for BV vector field (Ambrosio, 2003) uses a more refined argument, to avoid the lack of strong convergence of the DiPerna-Lions scheme under this hypothesis. The convolution kernel used in the regularization procedure is no more arbitrary and the proof is obtained through a careful optimization of the kernel itself. This weak convergence turns out to be sufficient to pass to the limit in this case.

It is interesting to connect in an "abstract" way all these results. In particular, in a recent paper in collaboration with F. Bouchut, we show that the renormalization property and the strong continuity of all weak solutions are equivalent to the uniqueness for both the forward and the backward Cauchy problems. This provides a new (and in some sense more explicit) characterization of the renormalization property. The proof requires some approximation tools, and a byproduct of this is the equivalence of the two conditions to the existence of a smooth approximation u^ε of the solution such that $u^\varepsilon \to u$ and $\partial_t u^\varepsilon + \mathrm{div}\,(bu^\varepsilon) \to \partial_t u + \mathrm{div}\,(bu)$ strongly. We will show how this approximation is more or less equivalent to the strong convergence of the DiPerna-Lions scheme, and we will observe that (a posteriori) this implies that in Ambrosio scheme it is possible to get strong convergence, but with different types of regularization, different from convolutions. We will also indicate the "sharpness" of our results describing an example by Depauw of an ill-posed transport equation, in which the solution could develop oscillations.