A non-isothermal phase-field approach to the second-sound transition in solids

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The phase-field model is applied to describe phase transitions which occur in a finite or unbounded region rather than at a sharp surface. The order parameter, or phase field φ, is then allowed to vary smoothly within the pertinent region between two extreme values: $\varphi = 0$, the less ordered phase, and $\varphi = 1$, the more ordered one. The literature on the subject shows how the phase field–model is widely applied in different contexts (for instance, see [1-4] and references therein). The order parameter can be regarded quite naturally as an internal variable which is governed by an evolution equation to be characterized within the whole set of constitutive assumptions and the thermodynamic restrictions. This viewpoint was successfully applied to a general model of phase transition including both first and second order transitions, as like as solid-fluid transition in water and conducting-superconducting state transition in metals [5].

Depending on the temperature, heat conduction in solids may be well modelled as a stationary phenomenon, through the Fourier law

$$\mathbf{q} = -k(\theta)\nabla\theta\,, \quad k > 0\,, \tag{1}$$

and a non-stationary phenomenon through one of the variants of the Maxwell-Cattaneo equation [6], where the heat flux is described by a vector internal variable

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -k(\theta)\nabla\theta\,, \quad \tau, \qquad (2) k > 0\,.$$

As well known, the stationary model leads to a parabolic energy balance equation. On the contrary, in the Maxwell-Cattaneo regime the energy balance yields a hyperbolic equation, so that a thermal wave has finite speed and is named *second sound*. Second sound, and hence the Maxwell-Cattaneo regime, occurs below some critical temperature θ_0. We regard the second sound as a phenomenon strictly related to a state of matter and that is why we model the two regimes as different phases of heat conduction.

We present here some ideas from [7], in order to frame the passage between the two regimes as a phase transition of the second kind, which spatially occurs in a finite region and is induced by temperature variations. This is suggested e.g. by the phase transition of Helium II: below the critical temperature, $\theta_0 = 2.2°$ K, Helium II is a superfluid with a small value of heat conductivity and viscosity coefficient [9]. The scheme developed here applies to superfluids provided the heat flux is taken as proportional to the velocity of the superfluid phase [8].

To obtain a description of the phase transition we ascribe a phase field $\varphi(x,t)$ to the heat conduction, where $\varphi = 0$ and $\varphi = 1$ in the stationary and non-stationary state, respectively. Adopting the viewpoint of [5], we develop a thermodynamic theory with evolution equations to be characterized for both the (scalar) order parameter φ and the pseudo-heat flux vector

$$\boldsymbol{\omega} = \alpha_0(\theta)\beta(\varphi)\mathbf{q}\,.$$

It is worth noting that, consistent with the nonlocal character of the model, the second law of thermodynamics allows for an entropy extra flux. As a result, we set up a general thermodynamic scheme which provides the modeling evolution equations

$$\dot{\varphi} = -\frac{a}{\theta}\,\varphi(\varphi^2 - 1 + \theta/\theta_0) - b\Delta\varphi - \varphi d_0|\boldsymbol{\omega}|^2\,, \tag{3}$$

$$\varphi^4 d_0\dot{\boldsymbol{\omega}} = -\frac{\gamma_0^2(\theta)[1-\gamma_1^2(\theta)\varphi^4]}{k_0\theta\varphi^2}\boldsymbol{\omega} - \frac{1}{\theta^2\alpha_0(\theta)}\nabla\theta\,. \tag{4}$$

Equation (4) simplifies to stationary (Fourier-like) models or unstationary models depending on whether the order parameter approaches the limit values associated with the two phases. In particular, as $\varphi \to 0$ the Fourier law (1)) is recovered from (4) assuming $k(\theta) = k_0/\theta\alpha_0^2(\theta)\gamma_0^2(\theta)$.

The evolution equation (3) shows that the transition can be induced by θ or $|\mathbf{q}|^2$. Indeed, for a fixed value $\hat{\theta}$, $0 < \hat{\theta} < \theta_0$, the Cattaneo-Maxwell regime occurs if $\mathbf{q}$ satisfies the bound

$$|\mathbf{q}|^2 \leq \frac{a}{d_0}\left(\frac{1}{\hat{\theta}} - \frac{1}{\theta_0}\right).$$

The fact that the transition occurs for suitably small values of $\mathbf{q}$ is the analogue of phase transition in superfluids in which the small valuedness is required for the velocity.

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