Hyperbolic balance laws with a non local source

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We consider systems of conservation laws with non local sources, i.e. equations of the form

$$\partial_t u + \partial_x f(u) = G(u) \tag{1}$$

where f is the flow of a nonlinear hyperbolic system of conservation laws and $G\colon L^1 \mapsto L^1$ is a (possibly) *non local* operator. As an example, we consider below the case $G(u) = g(u) + Q * u$ that enters the Rosenau regularization of Chapman-Enskog expansion of the Boltzmann equation, see [1,2]. We establish the well posedness in L^1, locally in time, for data having sufficiently small total variation. To this aim, we require those assumptions that separately guarantee the well posedness of the convective part

$$\partial_t u + \partial_x f(u) = 0 \tag{2}$$

and of the source part

$$\partial_t u = G(u)\,. \tag{3}$$

These two parts generate two semigroups of solutions, say S and Σ defined in a suitable space of sunctions, say X. To obtain our results we exploit the techniques in [3,4], essentially based on the *fractional step* algorithm, see [5,6,7]. Its core idea is to get a solution of the original equation as a limit of approximations obtained suitably merging S and Σ.

On the two semigroups one needs these two key conditions with respect to a metric $d(\cdot,\cdot)$ on X (see [4]):

i) A Grönwall type estimate: i.e. for a positive C

$$\begin{aligned} d(S_t u, S_t v) &\le e^{Ct} d(u,v) \\ d(\Sigma_t u, \Sigma_t v) &\le e^{Ct} d(u,v) \end{aligned} \qquad \text{for all } u,\ v \in X \text{ and } t \ge 0. \tag{4}$$

ii) A commutativity relation

$$d(\Sigma_t S_t u, S_t \Sigma_t u) \le K\, t^2 \qquad \text{as } t \to 0 \text{ for all } u \in X. \tag{5}$$

The former assumption will be used to prove the uniformly continuous dependence of the approximations from the initial data. The latter condition yields the convergence of the approximations and ensures the uniqueness of the limit.

Now assume that *i)* holds and d is a reasonable metric equivalent to the L^1 distance. Then, the invariance under the hyperbolic rescaling $(t,x) \to (\lambda t, \lambda x)$ of solutions to system of conservation laws, implies that $C = 0$. Hence, to apply the operator splitting tecniques, we need a contractive metric for the conservation law. This role is naturally played by the functional introduced by Bressan, Liu and Yang (see [8]). Note, however, that this functional is *not* a metric, for it may lack to satisfy the triangle inequality. The proof, then, consists in showing that the semigroup generated by the source part satisfies the Grönwall type estimate with respect to the functional and commutes with the conservation law part up to a second order error.

References

[1] P. Rosenau. Extending hydrodynamics via the regularization of the Chapman-Enskog expansion. *Phys. Rev. A (3)*, **40** (1989), pp. 7193-7196.

[2] S. Schochet, E. Tadmor. The regularized Chapman-Enskog expansion for scalar conservation laws. *Arch. Rational Mech. Anal.*, **119** (1992), pp. 95-107.

[3] D. Amadori, G. Guerra. Uniqueness and continuous dependence for systems of balance laws with dissipation. *Nonlinear Anal.*, **49** (Ser. A: Theory Methods) (2002), pp. 987-1014.

[4] R.M. Colombo, A. Corli. A semilinear structure on semigroups in a metric space. *Semigroup Forum*, **68** (2004), pp. 419-444.

[5] M. Crandall, A. Majda. The method of fractional steps for conservation laws. *Numer. Math.*, **34** (1980), pp. 285-314.

[6] C.M. Dafermos, L. Hsiao. Hyperbolic systems and balance laws with inhomogeneity and dissipation. *Indiana Univ. Math. J.*, **31** (1982), pp. 471-491.

[7] H.F. Trotter. On the product of semi-groups of operators. *Proc. Amer. Math. Soc.*, **10** (1969), pp. 545-551.

[8] A. Bressan, T.P. Liu, T. Yang. L^1 stability estimates for $n \times n$ conservation laws. *Arch. Ration. Mech. Anal.*, **149** (1999), pp. 1-22.