G1-234-er1

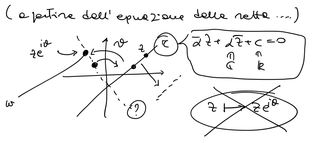
martedì 14 novembre 2023 09:39

⊛ Questioni generali / numeri $\mathbb{C}$

Come trovare l'equazione di una retta ruotata di ϑ attorno all'0 ?

Come trovare l'equazione di una retta traslata di z_0 ?

(a partire dall'equazione della retta ...)



$r = \{ z : \bar{\alpha} z + \alpha \bar{z} + c = 0 \}$ $\qquad \rho : z \mapsto e^{i\vartheta} z$, $\quad e^{-i\vartheta} w \leftarrow\!\shortmid w$

$\rho(r) = \{ \rho(z) : \bar{\alpha} z + \alpha \bar{z} + c = 0 \}$ ($\rho(z) = w$)

rotazione di angolo $-\vartheta$ attorno a 0

$= \{ w : \bar{\alpha}(\rho^{-1}(w)) + \alpha \overline{(\rho^{-1}(w))} + c = 0 \}$ $\qquad \rho^{-1}(w) = e^{-i\vartheta} w$

$= \{ z : \bar{\alpha}(ze^{-i\vartheta}) + \alpha \overline{(ze^{-i\vartheta})} + c = 0 \}$

$= \{ z : \bar{\alpha} e^{-i\vartheta} z + \alpha e^{i\vartheta} \bar{z} + c = 0 \}$

$\{x : f(x) = 0\}$

Nota : se $f(x) = 0$ descrive $S \subseteq V$ e abbiamo biiezione $\rho : V \to V$ come descriviamo $\rho(S)$?

$\rho(S) = \{ \rho(x) : f(x) = 0 \}$

$= \{ y = \rho(x) : f(x) = 0 \}$ $\qquad x = \sigma(y)$ se è inversa di ρ

$= \{ y : f(\sigma(y)) = 0 \}$

$= \{ y : f(\rho^{-1}(y)) = 0 \}$

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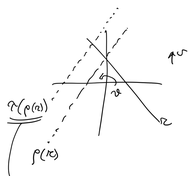


ρ = rotazione di angolo ϑ

r = traslazione di v

Se r avea equazione $\bar{\alpha} z + \alpha \bar{z} + c = 0$ e ρ = rotazione di ϑ, τ = traslazione di v, qual è l'equazione di $\tau(\rho(r))$?

dove sostituire z con $ze^{-i\vartheta} - v$ $\qquad$ NO!

$(z - v)e^{-i\vartheta}$

$z \mapsto \tau\rho(z) \mapsto (z - v)e^{-i\vartheta}$

equazione di $\tau(\rho(r))$ è $\quad (\tau\rho)(r)$

$f((\tau\rho)^{-1}(z)) = 0$

$f(\rho^{-1}\tau^{-1}(z)) = 0$

equazione $\rho(r)$

$f(\rho^{-1}(z)) = 0$

τ, ρ biiezioni

$\tau\rho$ biiezione

inversa di $\tau\rho$ è

$(\tau\rho)^{-1} = \rho^{-1}\tau^{-1}$

perché $(\rho^{-1}\tau^{-1})(\tau\rho) = \mathrm{id}$.

equazioni rette :

$\alpha, \beta \in \mathbb{C}$

$\frac{z-\alpha}{\beta-\alpha} \in \mathbb{R}$ $\qquad \boxed{\frac{z-\alpha}{\beta-\alpha} = \frac{\bar{z}-\bar{\alpha}}{\bar{\beta}-\bar{\alpha}}}$

$z(\bar{\beta} - \bar{\alpha}) - \bar{z}(\beta - \alpha) - \alpha(\bar{\beta} - \bar{\alpha}) + \bar{\alpha}(\beta - \alpha) = 0$

$(\bar{\beta} - \bar{\alpha})\, z + (-(\beta - \alpha))\, \bar{z} + \underbrace{(\alpha\bar{\beta} + \bar{\alpha}\beta)}_{\text{immaginario } i\,2\,\mathrm{Im}(\bar{\alpha}\beta)} = 0$

$\cdot i$

$\underbrace{i(\bar{\beta} - \bar{\alpha})}_{\gamma} z + \underbrace{(-i(\beta - \alpha))}_{\gamma} \bar{z} + i(\bar{\alpha}\beta - \alpha\bar{\beta}) - 2\,\mathrm{Im}(\bar{\alpha}\beta)$

retta per α, β : $\quad \bar{\gamma} z + \gamma \bar{z} + c = 0$

$\gamma = -i(\beta - \alpha)$, $\quad c = -2\,\mathrm{Im}(\bar{\alpha}\beta)$

$\bar{\gamma} = i(\alpha - \beta)$, $\quad -c = 2\,\mathrm{Im}(\bar{\alpha}\beta)$

Se $\alpha, \beta \in S^1$ allora $\bar{\alpha} = \frac{1}{\alpha}$, $\bar{\beta} = \frac{1}{\beta}$

si deve trovare una formula più semplice, quale,

$z + \alpha\beta\bar{z} = \alpha + \beta$.

(che non è del tipo "canonico")

retta : $\quad \bar{\alpha} z + \alpha \bar{z} + c = 0$

significati geometrici di α, c ?

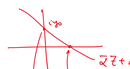
se $c \neq 0$ $\qquad$ $(\alpha \neq 0)$ divido per $|\alpha| \in \mathbb{R}$

$\bar{\alpha} z + \alpha \bar{z} + 1 = 0$ $\qquad \boxed{\bar{\alpha} z + \alpha \bar{z} + c = 0 \text{ con } |\alpha| = 1}$

α rappresenta una NORMALE della retta $\bar{\alpha} z + \alpha \bar{z} = 0$

e $i\alpha$ è punto della retta

ma c ?



$\bar{\alpha} z + \alpha \bar{z} + c = 0$ con $|\alpha| = 1$, $c > 0$

$\bar{\alpha} x_0 + \alpha x_0 + c = 0 \Rightarrow x_0 = -\frac{c}{\bar{\alpha} + \alpha} = -\frac{c}{2\,\mathrm{Re}(\alpha)}$

$\bar{\alpha} i y_0 + \alpha(-i y_0) + c = 0 \Rightarrow y_0 = -\frac{c}{i(\bar{\alpha} - \alpha)} = -\frac{c}{2\,\mathrm{Im}(\alpha)}$

$x_0 y_0 = \frac{c^2}{4\,\mathrm{Re}(\alpha)\,\mathrm{Im}(\alpha)}$

$x_0^2 + y_0^2 = \frac{c^2}{4\,\mathrm{Re}(\alpha)^2} + \frac{c^2}{4\,\mathrm{Im}(\alpha)^2} = \frac{c^2}{4\,\mathrm{Re}(\alpha)^2\,\mathrm{Im}(\alpha)^2}$

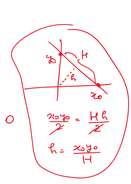
$\frac{|x_0 y_0|}{\sqrt{x_0^2 + y_0^2}} = \frac{c^2/4}{c/2} = \frac{c}{2}$

dunque $c = 2\,\frac{|x_0 y_0|}{\sqrt{x_0^2 + y_0^2}} =$

$= 2$ distanza tra r e 0

oppure si fa "spinta α" usando $\frac{1}{x_0^2} + \frac{1}{y_0^2} = \frac{4}{c^2}$

$\frac{x_0^2 + y_0^2}{x_0^2 y_0^2}$ $\qquad c^2 = 4\,\frac{x_0^2 y_0^2}{x_0^2 + y_0^2} \Rightarrow$



$\frac{x_0 y_0}{r} = \frac{H\rho}{\cancel{r}}$

$\rho = \frac{x_0 y_0}{H}$

14/11/2023

Sistema: $\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ \lambda x_1 + \mu x_2 + \lambda x_3 + \mu x_4 = 0 \\ (1-\lambda)x_1 + (1-\mu)x_2 + x_3 + x_4 = 1 + \mu \end{cases}$

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ \lambda & \mu & \lambda & \mu & 0 \\ 1-\lambda & 1-\mu & 1 & 1 & 1+\mu \end{array}\right) \xrightarrow[\text{III}-(1-\lambda)\text{I}]{\text{II}-\lambda\text{I}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & \mu-\lambda & 0 & \mu-\lambda & -\lambda \\ 0 & 2-\mu & \lambda & \lambda & \mu+\lambda \end{array}\right)$

$(1-\mu)-(1-\lambda)$ $\quad 1-(1-\lambda)$ $\quad 1-(1-\lambda)$ $\quad (1+\lambda)-(1-\lambda)$

$\lambda = \mu$ $\qquad \xrightarrow{\text{III} \leftrightarrow \text{II}} \qquad \lambda \neq \mu$

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & \mu-\lambda & 0 & \mu-\lambda & -\lambda \\ 0 & 0 & \lambda & \lambda & \mu \end{array}\right)$

$\mu - \lambda \neq 0 \Rightarrow rg(inc) = 3$, dim sol $= 4 - 3 = 1$

$\lambda = 0 \ (\Rightarrow \mu \neq 0) \Rightarrow rg(inc) = 3$, dim sol $= 4 - 3$

$\mu - \lambda = 0$ $\quad \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & \lambda & \lambda & \lambda \end{array}\right)$

$\lambda = \mu \neq 0$ (incompatibile) $rg(inc) = 2$ $rg(cogn) = 3$ $\quad \emptyset$

$\lambda = \mu = 0$ $\quad \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & & & & 0 \end{array}\right)$ $rg(inc) = 1$, $rg(cogn\,sol) = 1$, dim sol $= 4 - 1 = 3$

$\begin{cases} x_1 = 1 - x_2 - x_3 - x_4 = 1 - \alpha - \beta - \gamma \\ x_2 = \alpha \\ x_3 = \beta \\ x_4 = \gamma \end{cases}$

sol : $\begin{pmatrix} 1-\alpha-\beta-\gamma \\ \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

sol $= \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{sol. part.}} + \underbrace{\left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle}_{\text{sol. omog.}}$

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right)$

$\begin{cases} x_1 = 1 - x_2 - x_3 - x_4 = 1 - \alpha \\ x_2 = -\frac{\lambda}{\mu - \lambda} - \frac{(\mu-\lambda)}{\mu-\lambda} x_4 \\ x_3 = \frac{\mu}{\lambda} - \frac{\mu}{\lambda} x_4 \\ x_4 = \alpha \end{cases}$ $\qquad \begin{cases} x_1 = 1 - \left(-\frac{\lambda}{\mu-\lambda}\right) - \left(\frac{\mu}{\lambda} - \frac{\alpha}{\lambda}\right) - \alpha = \left(1 + \frac{\lambda}{\mu-\lambda} - \frac{\mu}{\lambda}\right) + \alpha\left(\frac{\mu}{\lambda}\right) \\ x_2 = -\frac{\lambda}{\mu-\lambda} - \alpha \\ x_3 = \frac{\mu}{\lambda} - \alpha\frac{\mu}{\lambda} \\ x_4 = \alpha \end{cases}$

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & \mu & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu & \mu \end{array}\right)$ $\quad \begin{cases} x_1 = 1 - x_2 - x_3 - x_4 = 1 - \beta \\ x_2 = -\frac{\mu x_4}{\mu} = -1 \\ x_3 = \beta \\ x_4 = 1 \end{cases}$ $\quad x_3$, $\mu \neq 0$

$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$

$\begin{cases} x_1 = e + \alpha c' \\ x_2 = d + \alpha d' \\ x_3 = c + \alpha c' \\ x_4 = \alpha \end{cases}$ $\qquad \begin{pmatrix} e + \alpha c' \\ d + \alpha d' \\ c + \alpha c' \\ \alpha \end{pmatrix} = \begin{pmatrix} e \\ d \\ c \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} c' \\ d' \\ c' \\ 1 \end{pmatrix}$ $\quad \alpha \in \mathbb{R}$

soluzioni $= \underbrace{\begin{pmatrix} e \\ d \\ c \\ 0 \end{pmatrix}}_{\text{sol. part.}} + \underbrace{\left\langle \begin{pmatrix} c' \\ d' \\ c' \\ 1 \end{pmatrix} \right\rangle}_{\text{sol. omog.}}$

$\begin{pmatrix} 1-\beta \\ -1 \\ \beta \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

sol $= \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

Tecnica : $\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_3 - x_4 + x_5 = 0 \end{cases}$

$U \leq \mathbb{R}^5$

$\dim_{\mathbb{R}} U = 5 - 2$ $\quad rg\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right)$

per scrivere una base

$\begin{pmatrix} x_1 = -\alpha - 2\beta \\ x_2 = \alpha \\ x_3 = \beta - \gamma \\ x_4 = \beta \\ x_5 = \gamma \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$\alpha = 1$, $\beta = 0$, $\gamma = 0$ $\qquad \alpha = 0$, $\beta = 1$, $\gamma = 0$

oppure : $\underbrace{\begin{pmatrix} * \\ * \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} * \\ 0 \\ * \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{sono vettori indipendenti perché si vede } x_1, x_3 \ldots}$

U equazioni $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$

generatori $\begin{pmatrix} -1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Calcolo $= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$a x_1 + b x_2 + c x_3 + d x_4 + e x_5 = 0$

$\begin{pmatrix} a & b & c & d & e \\ & & & & \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \end{pmatrix}$

⊛ $\mathbb{R}[x]_{\leq 5}$ $\quad$ sp. vett. dim 6 $\quad = \left\{ \sum_{i=0}^{5} a_i x^i \mid a_i \in \mathbb{R} \right\} \cong \left\{ \begin{pmatrix} a_0 \\ \vdots \\ a_5 \end{pmatrix} \mid a_i \in \mathbb{R} \right\} = \mathbb{R}^6$

$V_1 = \{ P(x) : P(x) = x^5 P(1/x) \}$

$V_{-1} = \{ P(x) : P(x) = -x^5 P(1/x) \}$

dim, base per $V_1, V_{-1}, V_1 \cap V_{-1}, V_1 + V_{-1}$?

V_1 : $\quad P(x) = \sum a_i x^i$

$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$x^5 P(1/x) = a_0 x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$

sono = sse $\begin{cases} a_0 = a_5 \\ a_1 = a_4 \\ a_2 = a_3 \end{cases}$ $\quad \begin{cases} a_0 - a_5 = 0 \\ a_1 - a_4 = 0 \\ a_2 - a_3 = 0 \end{cases}$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$

eq. cart. – dist rg 3

$\Rightarrow \dim V_1 = 6 - 3 = 3$

base : $\quad P(x) = a + b(x + x^4) + c(x^2 + x^3) + b x^4 + a x^5$

$= a(1 + x^5) + b(x + x^4) + c(x^2 + x^3)$

$V_1 = \langle 1 + x^5, x + x^4, x^2 + x^3 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

V_{-1} : $\quad P(x) = \sum a_i x^i$

uguali a $-x^5 P(1/x)$

cioè $\begin{cases} a_0 = -a_5 \\ a_1 = -a_4 \\ a_2 = -a_3 \end{cases}$ $\quad \begin{cases} a_0 + a_5 = 0 \end{cases}$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

ovvero $\langle 1 - x^5, x - x^4, x^2 - x^3 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$\dim(V_{-1}) = 3$

Calcolo l'intersezione $V_1 \cap V_{-1}$

metto insieme le equazioni cartesiane:

$\begin{cases} V_1 \left(\begin{cases} a_0 - a_5 = 0 \\ a_1 - a_4 = 0 \\ a_2 - a_3 = 0 \end{cases} \right. \\ V_{-1} \left(a_0 + a_5 = 0 \right. \end{cases}$ $\quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{IV}-\text{I}, \ \text{V}-\text{II}, \ \text{VI}-\text{III}]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}$

$rg = 6$

$\dim(V_1 \cap V_{-1}) = 6 - 6 = 0$

$\dim \mathbb{R}[x]_{\leq 5}$

$\boxed{V_1 \cap V_{-1} = 0}$

$\Rightarrow \dim(V_1 + V_{-1}) = \dim(V_1) + \dim(V_{-1}) - \dim(V_1 \cap V_{-1})$

$= 3 + 3 - 0 = 6$

$\Rightarrow V_1 + V_{-1} = \mathbb{R}[x]_{\leq 5}$

Quindi V_1 e V_{-1} sono $V_1 \oplus V_{-1} = \mathbb{R}[x]_{\leq 5}$

Conclusione ogni polinomio $P(x) \in \mathbb{R}[x]_{\leq 5}$ si scrive in modo unico

$P(x) = \underbrace{A(x)}_{\in V_1} + \underbrace{B(x)}_{\in V_{-1}}$

perché ogni $\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \underbrace{\alpha_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{V_1} + \underbrace{\alpha_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{V_{-1}}$

$\begin{cases} \alpha_0 + \alpha_3 = a_0 \\ \alpha_0 - \alpha_3 = a_5 \\ \vdots \end{cases} \Rightarrow \begin{cases} \alpha_0 = \frac{a_0 + a_5}{2} \\ \alpha_3 = \frac{a_0 - a_5}{2} \end{cases}$