G1-234-sppA

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① $U=\left\langle \begin{pmatrix}1\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\right\rangle$ $V=\left\langle \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\rangle$ $W=\left\langle \begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right\rangle = \begin{cases} x_1=x_3\\ x_2=x_4\end{cases}$

$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{Q}^4$

$$X-\alpha\begin{pmatrix}1\\1\\1\\0\end{pmatrix}-\beta\begin{pmatrix}0\\1\\1\\1\end{pmatrix}=\begin{pmatrix}x_1-\alpha\\x_2-\alpha-\beta\\x_3-\alpha-\beta\\x_4-\beta\end{pmatrix}\in W \text{ sse } \begin{cases}x_1-\cancel{\alpha}=x_3-\cancel{\alpha}-\beta\\ x_2-\alpha-\cancel{\beta}=x_4-\cancel{\beta}\end{cases}$$

$$\text{sse } \begin{cases}\alpha=x_2-x_4\\ \beta=x_3-x_1\end{cases}$$

dunque
$$\pi_U^W(X)=(x_2-x_4)\begin{pmatrix}1\\1\\1\\0\end{pmatrix}+(x_3-x_1)\begin{pmatrix}0\\1\\1\\1\end{pmatrix}=\begin{pmatrix}x_2-x_4\\x_2-x_4+x_3-x_1\\x_2-x_4+x_3-x_1\\x_3-x_1\end{pmatrix}=\underbrace{\begin{pmatrix}0&1&0&-1\\-1&1&1&-1\\-1&1&1&-1\\-1&0&1&0\end{pmatrix}}_{\alpha_\mathcal{E}(\pi_U^W)}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

e allora $\alpha_\mathcal{E}(\sigma_U^W)=2\alpha_\mathcal{E}(\pi_U^W)-\mathbb{1}_4=\begin{pmatrix}-1&2&0&-2\\-2&1&2&-2\\-2&2&1&-2\\-2&0&2&-1\end{pmatrix}$

(perché $\sigma_U^W=\pi_U^W-\pi_W^U=\pi_U^W-(id-\pi_U^W)=2\pi_U^W-id$)

Stesso ragionamento per V e W, oppure scriviamo

$$X=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\alpha\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\gamma\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\delta\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

cioè $\begin{cases}\gamma=x_1\\ \beta+\delta=x_2\\ \alpha+\gamma=x_3\\ \delta=x_4\end{cases}\Rightarrow\begin{cases}\alpha=x_3-x_1\\ \beta=x_2-x_4\\ \gamma=x_1\\ \delta=x_4\end{cases}$

e allora
$$\sigma_W^V(X)=\alpha\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\\0\end{pmatrix}-\gamma\begin{pmatrix}1\\0\\1\\0\end{pmatrix}-\delta\begin{pmatrix}0\\1\\0\\1\end{pmatrix}=$$
$$=(x_3-x_1)\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+(x_2-x_4)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}-x_1\begin{pmatrix}1\\0\\1\\0\end{pmatrix}-x_4\begin{pmatrix}0\\1\\0\\1\end{pmatrix}=$$
$$=\begin{pmatrix}-x_1\\x_2-2x_4\\x_3-2x_1\\-x_4\end{pmatrix}=\underbrace{\begin{pmatrix}-1&0&0&0\\0&1&0&-2\\-2&0&1&0\\0&0&0&-1\end{pmatrix}}_{\alpha_\mathcal{E}(\sigma_W^V)}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

per rispondere alle altre domande conviene usare una base migliore:

$U=\langle v_1+v_3, v_2+v_4\rangle$ $U=\langle v_1,v_2\rangle$ $W=\langle v_3,v_4\rangle$

e allora le matrici sono

$$A'=\left(\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 2&0&-1&0\\0&2&0&-1\end{array}\right)\qquad B'=\left(\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 0&0&-1&0\\0&0&0&-1\end{array}\right)$$

$\left.\begin{array}{l}v_1+v_3\mapsto v_1+v_3\\ v_3\mapsto -v_3\end{array}\right\}\Rightarrow v_1\mapsto v_1+2v_3$

$\left.\begin{array}{l}v_2+v_4\mapsto v_2+v_4\\ v_4\mapsto -v_4\end{array}\right\}\Rightarrow v_2\mapsto v_2+2v_4$

da cui:
$$A'B'=\left(\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 2&0&-1&0\\0&2&0&-1\end{array}\right)\qquad B'A'=\left(\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline -2&0&-1&0\\0&-2&0&-1\end{array}\right)$$

che sono diverse tra di loro, ma sono simili perché

$$\underset{H^{-1}}{A}(AB)\underset{H}{A}=BA$$
(perché $A^2=\mathbb{1}$)

(in generale, se A è invertibile, allora AB simile a BA perché $A^{-1}(AB)A=BA$ ...)

(naturalmente, $AB=BA$ sse $A'B'=B'A'$ perché $A'=H^{-1}AH$, $B'=H^{-1}BH$ ...)

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② $A=\begin{pmatrix}1&0&1&0\\0&1&0&1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}$

$$p_A(x)=\begin{vmatrix}x-1&0&-1&0\\0&x-1&0&-1\\1&0&x-1&0\\0&1&0&x-1\end{vmatrix}=\begin{vmatrix}x-1&-1&0&0\\1&x-1&0&0\\0&0&x-1&-1\\0&0&1&x-1\end{vmatrix}=\begin{vmatrix}x-1&-1\\1&x-1\end{vmatrix}^2=((x-1)^2+1)^2$$

dunque ha autovalori (complessi non reali) di molteplicità 2: $x-1=\pm i$, $x=1\pm i$

vediamo gli autospazi:

$A-(1+i)=\underbrace{\begin{pmatrix}-i&0&1&0\\0&-i&0&1\\-1&0&-i&0\\0&-1&0&-i\end{pmatrix}}_{rg=2}$ ha nucleo $\left\langle\begin{pmatrix}1\\0\\i\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\i\end{pmatrix}\right\rangle$ di dim 2

$A-(1-i)=$ coniugato nucleo $\left\langle\begin{pmatrix}1\\0\\-i\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\-i\end{pmatrix}\right\rangle$

Quindi $m_A(x)=(x-1)^2+1=(x-1+i)(x-1-i)$ (diagonalizzabile su $\mathbb{C}$ per il primo criterio)

e $A\underset{\mathbb{C}}{\sim}\begin{pmatrix}1+i&&&\\&1+i&&\\&&1-i&\\&&&1-i\end{pmatrix}=H^{-1}AH$ con $H=\begin{pmatrix}1&0&1&0\\0&1&0&1\\i&0&-i&0\\0&i&0&-i\end{pmatrix}$

Invece su $\mathbb{Q}$ non è diagonalizzabile, e nemmeno triangolarizzabile (e nemmeno su $\mathbb{R}$) perché non ha gli autovalori nel campo.

Tuttavia
$$A\underset{\mathbb{C}}{\sim}\left(\begin{array}{cc|cc}1+i&&&\\&1-i&&\\\hline&&1+i&\\&&&1-i\end{array}\right)$$

dunque $A\underset{\mathbb{Q}}{\sim}\left(\begin{array}{cc|cc}1&1&0&0\\-1&1&0&0\\\hline0&0&1&1\\0&0&-1&1\end{array}\right)$

usando la base $Re\begin{pmatrix}1\\0\\i\\0\end{pmatrix}, Im\begin{pmatrix}1\\0\\i\\0\end{pmatrix}, Re\begin{pmatrix}0\\1\\0\\i\end{pmatrix}, Im\begin{pmatrix}0\\1\\0\\i\end{pmatrix}$
$=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

cioè con $R=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$ otteniamo $R^{-1}AR=\left(\begin{array}{cc|cc}1&1&0&0\\-1&1&0&0\\\hline0&0&1&1\\0&0&-1&1\end{array}\right)$ //

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③ $N=\begin{pmatrix}0&1&&\\&0&1&\\&&0&1\\&&&0\end{pmatrix}$ $N^2=\begin{pmatrix}0&0&1&0\\&0&0&1\\&&0&0\\&&&0\end{pmatrix}\sim\left(\begin{array}{cc|cc}0&1&&\\&0&&\\\hline&&0&1\\&&&0\end{array}\right)$ $N^3=\begin{pmatrix}0&0&0&1\\&0&0&0\\&&0&0\\&&&0\end{pmatrix}\sim\left(\begin{array}{cc|cc}0&1&&\\&0&&\\\hline&&0&\\&&&0\end{array}\right)$

pol. car. x^4
pol. min. x^4
dim. nuclei gen.: 1,2,3,4
Young: colonna di 4 caselle

N^2: x^4, x^2, 2,4 — 2 blocchi J
$\ker N^2\subsetneq\ker(N^2)^2$
$\langle e_1,e_2\rangle$ $\langle e_1,e_2,e_3,e_4\rangle$
base Jord.: e_1, e_3, e_2, e_4 ($e_1=N^2e_3$, $e_2=N^2e_4$)

N^3: x^4, x^2, 3,4 — 3 blocchi J
$\ker N^3\subsetneq\ker(N^3)^2$
$\langle e_1,e_2,e_3\rangle$ $\langle e_1,e_2,e_3,e_4\rangle$
base Jord.: $e_1\ e_4\ e_2\ e_3$ ($e_1=N^2e_4$)

$\mathcal{C}(N)=\{X\in M_4(K): NX=XN\}\leq M_n(K)$

sottospazio: $X,Y\in\mathcal{C}(N)$ allora $\alpha X+Y\in\mathcal{C}(N)$
$0_4\in\mathcal{C}(N)$

$X=\begin{pmatrix}x_{11}&x_{12}&x_{13}&x_{14}\\x_{21}&x_{22}&x_{23}&x_{24}\\x_{31}&x_{32}&x_{33}&x_{34}\\x_{41}&x_{42}&x_{43}&x_{44}\end{pmatrix}$:

$NX=\begin{pmatrix}x_{21}&x_{22}&x_{23}&x_{24}\\x_{31}&x_{32}&x_{33}&x_{34}\\x_{41}&x_{42}&x_{43}&x_{44}\\0&0&0&0\end{pmatrix}$ molt a sx per N "sposta in su" le righe di X

$XN=\begin{pmatrix}0&x_{11}&x_{12}&x_{13}\\0&x_{21}&x_{22}&x_{23}\\0&x_{31}&x_{32}&x_{33}\\0&x_{41}&x_{42}&x_{43}\end{pmatrix}$ molt a dx per N "sposta a dx" le colonne di X

affinché siano uguali: $X=\begin{pmatrix}x_{11}&x_{12}&x_{13}&x_{14}\\0&x_{11}&x_{12}&x_{13}\\0&0&x_{11}&x_{12}\\0&0&0&x_{11}\end{pmatrix}$, cioè X "triangolare superiore con diagonali costanti" (x_{14} qualsiasi)

$=x_{11}\mathbb{1}+x_{12}N+x_{13}N^2+x_{14}N^3$
$\in K[N]$

base $\mathbb{1},N,N^2,N^3$, $\dim_K\mathcal{C}(N)=4$ e $K[N]=\mathcal{C}(N)$
(perché $\leq$ e stessa dim.: $\dim_K K[N]=\dim_K K[x]/\underbrace{(x^4)}_{\text{pol. minimo}}=4$

$\mathcal{C}(N^2)=\{X\in M_4(K): N^2X=XN^2\}\leq M_4(K)$

stessa tecnica:

$N^2X=\begin{pmatrix}x_{31}&x_{32}&x_{33}&x_{34}\\x_{41}&x_{42}&x_{43}&x_{44}\\0&0&0&0\\0&0&0&0\end{pmatrix}$

$XN^2=\begin{pmatrix}0&0&x_{11}&x_{12}\\0&0&x_{21}&x_{22}\\0&0&x_{31}&x_{32}\\0&0&x_{41}&x_{42}\end{pmatrix}$

dunque $X=\begin{pmatrix}x_{11}&x_{12}&x_{13}&x_{14}\\x_{41}&x_{22}&x_{23}&x_{24}\\0&0&x_{11}&x_{12}\\0&0&x_{43}&x_{22}\end{pmatrix}$

$$=x_{11}\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&0\end{pmatrix}+x_{12}\begin{pmatrix}0&1&0&0\\&0&0&0\\&&0&1\\&&&0\end{pmatrix}+x_{13}\begin{pmatrix}0&0&1&0\\&0&0&0\\&&0&0\\&&&0\end{pmatrix}+x_{14}\begin{pmatrix}0&0&0&1\\&0&0&0\\&&0&0\\&&&0\end{pmatrix}$$
$$x_{21}\begin{pmatrix}0&0&&\\1&0&0&\\&&0&0\\&&1&0\end{pmatrix}+x_{22}\begin{pmatrix}0&&&\\&1&&\\&&0&\\&&&1\end{pmatrix}+x_{23}\begin{pmatrix}0&0&0&0\\&0&1&0\\&&0&0\\&&&0\end{pmatrix}+x_{24}\begin{pmatrix}0&0&0&0\\&0&0&1\\&&0&0\\&&&0\end{pmatrix}$$

e $\dim_K\mathcal{C}(N^2)=8$

mentre $\dim_K K[N^2]=\dim_K K[x]/(x^2)=2$ con base $\mathbb{1},N^2$ (perché $(N^2)^2=0$)

quindi $K[N^2]\subsetneq\mathcal{C}(N)$.

Note: per studiare $\mathcal{C}(N^2)$ si poteva usare la forma di Jordan di N^2, che è $\begin{pmatrix}N_2&0\\0&N_2\end{pmatrix}=J$

perché se $H^{-1}N^2H=J$ abbiamo $H^{-1}\mathcal{C}(N^2)H=\mathcal{C}(J)$
e $\mathcal{C}(N^2)=H\,\mathcal{C}(J)H^{-1}$

e se $JY=YJ$ ragioniamo a blocchi: $Y=\begin{pmatrix}Y_1&Y_2\\Y_3&Y_4\end{pmatrix}$

$$\underset{\begin{pmatrix}N_2Y_1&N_2Y_2\\N_2Y_3&N_2Y_4\end{pmatrix}}{\begin{pmatrix}N_2&0\\0&N_2\end{pmatrix}\begin{pmatrix}Y_1&Y_2\\Y_3&Y_4\end{pmatrix}}=\underset{\begin{pmatrix}Y_1N_2&Y_2N_2\\Y_3N_2&Y_4N_2\end{pmatrix}}{\begin{pmatrix}Y_1&Y_2\\Y_3&Y_4\end{pmatrix}\begin{pmatrix}N_2&0\\0&N_2\end{pmatrix}}$$

da cui ogni Y_i commuta con N_2,

quindi $Y=\left(\begin{array}{cc|cc}a&b&c&d\\0&a&0&c\\\hline e&f&g&h\\0&e&0&g\end{array}\right)$

dipende da $2\cdot4=8$ parametri...

Naturalmente abbiamo $K[N^2]\leq K[N]$ (matrici polinomiali in N^2 sono polinomiali in N)
e $\mathcal{C}(N)\leq\mathcal{C}(N^2)$ (matrici che commutano con N commutano con N^2)

abbiamo visto $K[N]=\mathcal{C}(N)$ (dim 4), quindi $\underset{\dim 2}{K[N^2]}\subsetneq\underbrace{K[N]=\mathcal{C}(N)}_{\dim 4}\subsetneq\underset{\dim 8}{\mathcal{C}(N^2)}$

Note: si poteva studiare anche $\mathcal{C}(N^3)$ e $K[N^3]$:

$$\begin{tikzcd}[column sep=small]
& 12 \arrow[dl, no head] \arrow[dr, no head] & \\
\mathcal{C}(N^2)\ 8 & & \mathcal{C}(N^3)\ 10 \\
& 6 \arrow[ul, no head] \arrow[ur, no head] \arrow[d, no head] & \\
& K[N]=\mathcal{C}(N)=\langle\mathbb{1},N,N^2,N^3\rangle\ 4 \arrow[d, no head, "3"'] & \\
& \bullet \arrow[dl, no head] \arrow[dr, no head] & \\
K[N^2]=\langle\mathbb{1},N^2\rangle\ 2 & & 2\ K[N^3]=\langle\mathbb{1},N^3\rangle \\
& \langle\mathbb{1}\rangle \arrow[ul, no head] \arrow[ur, no head] &
\end{tikzcd}$$