Ripasso:

(4) $\varphi: V \to W$ $\quad v^1 = (v_1, \dots, v_n)$ base V, $\quad w^1 = (w_1, \dots, w_m)$ base W

$\varphi(v_j) \in \mathbb{K}^m_{w^1}$ (vale vettore i.c.)

- $\varphi(v_j) = \sum_i a_{ij}(\varphi)\, w_i$
- $M_{w^1}(\varphi) = \begin{pmatrix} a_{11}(\varphi) & \cdots & a_{1n}(\varphi) \\ \vdots & & \vdots \\ a_{m1}(\varphi) & \cdots & a_{mn}(\varphi) \end{pmatrix} \in \mathbb{K}^{m \times n}$ $\quad \forall \varphi \in \mathcal{L}(V,W)$

(vale anche $\mathbb{K}^{m \times n} \cong \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)$)

$\forall$ determinata da $\varphi(v_1) \dots \varphi(v_n)$, quindi da $M_{w^1}(\varphi)$

Inoltre

$(\varphi(v_1) \dots \varphi(v_n)) = (w_1 \dots w_m) \begin{pmatrix} | & & | \\ a_1(\varphi) & \cdots & a_n(\varphi) \\ | & & | \end{pmatrix}$

$a_j(\varphi)$ = coordinate $(\varphi(v_j))$ su w^1 (vettore colonna $a_j(\varphi)$ rappresenta $\varphi(v_j)$ $\forall j$)

$\varphi: V \to W$, $v^1 = (v_1, \dots, v_n)$, $w^1 = (w_1, \dots, w_m)$

$\varphi(v) = \sum_j x_j v_j \Rightarrow \varphi(v) = \sum_j x_j \varphi(v_j) = \sum_j x_j \sum_i a_{ij}(\varphi) w_i = \sum_i \left(\sum_j a_{ij}(\varphi) x_j \right) w_i$

$\varphi(v) = (w_1 \dots w_m) \begin{pmatrix} a_{11}(\varphi) x_1 + \dots + a_{1n}(\varphi) x_n \\ \vdots \\ a_{m1}(\varphi) x_1 + \dots + a_{mn}(\varphi) x_n \end{pmatrix} = (w_1 \dots w_m) M_{w^1}(\varphi) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

(5) $\varphi_{v^1 w^1}: \mathrm{Hom}_{\mathbb{K}}(V,W) \to M_{m \times n}(\mathbb{K})$

$\varphi \longmapsto M_{w^1}(\varphi)$

è un spazio vettoriale

$\varphi_2 = \varphi_{v^2 w^2}: \mathrm{End}_{\mathbb{K}}(V) \to M_{n \times n}(\mathbb{K})$

è un ed $\mathbb{K}$-algebra.

Problema:

- Per matrici quadrate $A, B \in M_n(\mathbb{K})$ del tipo $AB \neq BA$.

Domanda: quali sono le matrici che COMUTANO con TUTTE? (cioè A t.c. $AX = XA$ $\forall X \in M_n(\mathbb{K})$)

(Si chiama "CENTRO DELL'ALGEBRA" per dettagli da, di cui stanno —).

- Quali sono le matrici che commutano con TUTTE le MATRICI DIAGONALI $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$?

(5) $\varphi_{v^1 w^1}: \mathrm{Hom}_{\mathbb{K}}(V,W) \to M_{m \times n}(\mathbb{K})$

$\varphi \longmapsto M_{w^1}(\varphi)$

è un ed $\mathbb{K}$-spazio vettoriale.

$\varphi_2 = \varphi_{v^2 v^2}: \mathrm{End}_{\mathbb{K}}(V) \to M_{n \times n}(\mathbb{K})$

è un ed $\mathbb{K}$-algebra.

(E')

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_V"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\mathrm{id}_W"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\varphi"] \\ V \arrow[r, "\varphi"] & V \end{tikzcd}$$

$a_{id_V}(\varphi) = a_{id_W}(\mathrm{id}_V \circ \varphi \circ \mathrm{id}_V) =$

$= a_{\varphi}(\mathrm{id}_V) \cdot a_{\varphi}(\varphi) \cdot a_{\varphi}(\mathrm{id}_V)$

$B = C A H$ $\qquad B = H^{-1} A H$

Esempio: Scrivere alcune matrici di una funzione su $\mathbb{K}^3$.

$\mathbb{K}^3 = U \oplus V$ $\quad U = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rangle$ dim 2

$V = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle$ dim 1

$\pi_V^U: \mathbb{K}^3 \to \mathbb{K}^3$

$x \mapsto x_U \mapsto \pi_V^U(x) = x_V$

$U \quad V$

domanda: $a_{\mathcal{E}}(\pi_V^U)$? $\quad \mathcal{E} = (e_1, e_2, e_3)$ base canonica $\mathbb{K}^3$

$a_{\mathcal{E}}(\pi_V^U)$? $\quad \mathcal{E} = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

da calcolare $\mathbb{K}^3$ tra le basi?

$\pi(e_1) = 0$, $\pi(e_2) = 0$, $\pi(e_3) = e_3$ $\Rightarrow a_{\mathcal{E}}(\pi_V^U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *

$v = x_1 e_1 + x_2 e_2 + x_3 e_3 \Rightarrow \pi(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= (0, 0, x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \pi(v) = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$

Utilizza per calcolare $a_{\mathcal{E}}(\pi_V^U)$ dove:

- U calcolare $\pi(e_1)$, $\pi(e_2)$, $\pi(e_3)$ $\in \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\mathbb{K}$ t.c.
- ogni proiezione sul vettore $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in V$

e calcolare $\pi_V^U(x)$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\in V} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, dobbiamo trovare x_1, x_2, x_3 (funzioni x_1, x_2, x_3)

$x = y + z$

$\pi(x) = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} + \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \\ x_3 \end{pmatrix}$

$\begin{cases} 2x_1 = -x_1 - x_2 + x_3 \\ 2x_2 = -x_1 - x_2 + x_3 \\ 2x_3 = x_1 + x_2 + x_3 \end{cases}$

$\pi(x) = x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{x_1 + x_2 + x_3}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$

$= \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + x_2 - x_3 \\ x_1 + x_2 - x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$a_{\mathcal{E}}(\pi_V^U)$

Un altri Esempio:

$\pi_V^U(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\pi_V^U(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\pi_V^U(e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

nota $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Soluzione la soluzione fra

$a_{\mathcal{E}}(\pi_V^U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad a_{\mathcal{E}}(\pi_V^U) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\mathbb{K}^3 \xrightarrow{\pi_V^U} \mathbb{K}^3$

$\mathbb{K}^3 \xrightarrow{\pi_V^U} \mathbb{K}^3$

$\pi_V^U(e_1) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

e_1, e_2, e_3

Scrivere $\pi_V^U(e_1)$ in termini della base e_1, e_2, e_3

$\pi_V^U(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{moltiplicazione}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0$ $\quad 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0$

(1)

- allora abbiamo che:

$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$?

$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

28/3/2022

Ripasso: V spazio vettoriale su $\mathbb{K}$, $v_1, \dots, v_n$ base V $\quad (n = \dim_{\mathbb{K}} V)$

$\varphi_v: V \to \mathbb{K}^n = V_n(\mathbb{K})$

$v \mapsto (x_1, \dots, x_n)$ $\quad v(v) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Se $w_1, \dots, w_n$ altra base su V allora:

$\varphi_w: V \to \mathbb{K}^n = V_n(\mathbb{K})$

$v \mapsto y_1 e_1 + \dots + y_n e_n = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\begin{tikzcd} V \arrow[r, "\varphi_v"] \arrow[dr, "\varphi_w"'] & \mathbb{K}^n \arrow[d, "\mathrm{id}"] \\ & \mathbb{K}^n \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, mapsto] \arrow[dr, mapsto] & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ & \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \end{tikzcd}$$

$v \mapsto v \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad \varphi_v(v) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$v \mapsto v \mapsto \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $\quad \varphi_w(v) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\varphi_w(v) = \mathrm{id}(\varphi_v(v))$

$\varphi_w = \mathrm{id} \circ \varphi_v$

$\varphi_w = \mathrm{id} \circ \varphi_v$ $\quad \mathrm{id}: \mathbb{K}^n \to \mathbb{K}^n$

$\mathrm{id}(e_1) = e_1$

$\mathrm{id}(e_2) = e_2$

$\mathrm{id}(e_n) = e_n$

$\varphi_w(v) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \varphi_v(v)$

$\varphi_w(v) = \mathrm{id}(\varphi_v(v))