FRENET

$\gamma : I \longrightarrow \mathbb{R}^n$ $\quad t \longmapsto \gamma(t)$ — curva $(n-1)$-regolare

$V_t : I \longrightarrow G_{n,k}(\mathbb{R})$ $\quad t \longmapsto \langle \gamma'(t), \gamma''(t), \dots, \gamma^{(k)}(t) \rangle$ (Gram-Schmidt) — base di $\mathbb{R}^n$ V_t

sotto GL $\quad E_t : I \longrightarrow O_n(\mathbb{R})$ $\quad t \longmapsto (e_1, \dots, e_n) = V_t \cdot T$ (ortonormalizzati Frenet equivalenti a γ)

$\frac{d}{dt} E_t = E_t \, K_t \qquad K_t = \begin{pmatrix} 0 & -k_1 & 0 & \dots & 0 \\ k_1 & 0 & -k_2 & & \vdots \\ 0 & k_2 & 0 & \ddots & 0 \\ \vdots & & \ddots & & -k_{n-1} \\ 0 & \dots & 0 & k_{n-1} & 0 \end{pmatrix}$

antisimmetrica, parte $E_t \in O_n(\mathbb{R})$
Solo $\mathrm{span}_{\mathbb{R}}(e_1, \dots, e_k)$ invariante per cambiamento GL_k a sinistra di V_t
$\in \mathbb{R}^{k \times k}$ in $GL(k)$ per cambiamento GL_k

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Esempi espliciti in dimensione 2 e 3:

dim 2:

Facciamo il disco:

sist. Frenet: $e_1 = t = \gamma'$ (poi divido per $\|\gamma'\|$)
$e_2 = n : \begin{cases} \langle e_1, e_2 \rangle = 0 \\ \det(e_1, e_2) = 1 \end{cases}$ ($\det > 0$ così n a sx)

$b = e_1 \wedge e_2 = e_1 e_2 = t \wedge n = \dfrac{\gamma' \wedge \gamma''}{\|\gamma'\|^2}$

eq. Frenet $(t, n, b)' = (t, n, b) \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$

$t' = kn$
$n' = -kt + \tau b$
$b' = -\tau n$

$k = t' \cdot n = \langle \gamma' \rangle' \cdot \dfrac{\gamma'' \wedge \gamma'}{\|\gamma'\|^3} = \dfrac{\det(\gamma', \gamma'')}{\|\gamma'\|^3}$, $\|\gamma'\|^3$

$\tau = b' \cdot n = b \cdot n' = \dfrac{\gamma' \wedge \gamma'''}{\|\gamma'\|^6} \cdot \left\langle \dfrac{\gamma' \wedge \gamma''}{\|\gamma'\|^3} \right\rangle' = \dfrac{\gamma' \wedge \gamma''' \cdot \|\gamma' \wedge \gamma''\|}{\|\gamma'\|^6}$

$n \cdot b = 0$
$n' \cdot b + n \cdot b' = 0$

$\left(\dfrac{\gamma' \wedge \gamma''}{\|\gamma'\|^3} \right)' \cdot \gamma'' \wedge \gamma'$

se γ curva in un piano: $\gamma : I \longrightarrow \mathbb{R}^2$

$\left(\dfrac{\gamma'}{\|\gamma'\|}, \dfrac{\gamma''}{\|\gamma''\|} \right)$ $\quad \dfrac{d}{dt} = \dfrac{d}{ds} \dfrac{ds}{dt}$

$\left(\dfrac{\gamma'}{\|\gamma'\|} \right)' = \dfrac{\gamma''}{\|\gamma'\|} + \gamma' \left(\dfrac{1}{\|\gamma'\|} \right)'$ $\quad \dfrac{ds}{dt} = \|\gamma'\|$

$\dfrac{d}{dt} \dfrac{1}{\|\gamma'\|} = \dfrac{1}{\|\gamma'\|}$

$\dfrac{d}{ds} = \dfrac{1}{\|\gamma'\|} \dfrac{d}{dt}$

$\dfrac{d}{ds} \dfrac{\gamma'}{\|\gamma'\|} = \dfrac{\gamma''}{\|\gamma'\|^2} - \dfrac{\gamma' \cdot \gamma''}{\|\gamma'\|^4} \gamma'$

$= \dfrac{\|\gamma'\|^2 \gamma'' - (\gamma' \cdot \gamma'') \gamma'}{\|\gamma'\|^4}$

$= \dfrac{\gamma' \wedge (\gamma'' \wedge \gamma')}{\|\gamma'\|^4}$

$\gamma'' = \dfrac{d}{ds} \left(\dfrac{\gamma'}{\|\gamma'\|} \right) \dfrac{ds}{dt} + \dfrac{\gamma''}{\|\gamma'\|^2} \dfrac{ds}{dt}$ $\quad \gamma' \perp \gamma'' \Rightarrow \gamma' \cdot \gamma'' = 0$

$\tau = \dfrac{\det(\gamma', \gamma'')}{\|\gamma'\|^3} \cdot \dfrac{\gamma' \wedge \gamma''' \cdot \|\gamma' \wedge \gamma''\|}{\|\gamma' \wedge \gamma''\|^2} \cdot \dfrac{1}{\|\gamma'\|^3}$

$\|\gamma''\| = \dfrac{\|\gamma' \wedge (\gamma' \wedge \gamma'')\|}{\|\gamma'\|^4} = \dfrac{\|\gamma'\| \, \|\gamma' \wedge \gamma''\|}{\|\gamma'\|^4} = \dfrac{\|\gamma' \wedge \gamma''\|}{\|\gamma'\|^3}$

$= \dfrac{\det(\gamma', \gamma'')}{\|\gamma'\|^3}$

$\gamma' \wedge \gamma'' \cdot \gamma''' = \det(\gamma', \gamma'', \gamma''')$

$= \det \begin{pmatrix} x' & x'' & x''' \\ y' & y'' & y''' \\ 0 & 0 & 0 \end{pmatrix} \cdot e_3 \cdot e_3$

$= \det \begin{pmatrix} x' & x'' \\ y' & y'' \end{pmatrix} \cdot e_3 \cdot e_3$

$\tau = \dfrac{\det(\gamma', \gamma'')}{\|\gamma'\|^6} \cdot \dfrac{\det(\gamma', \gamma'')}{\|\gamma'\|^3} = \dfrac{\det(\gamma', \gamma'')^2}{\|\gamma'\|^9}$

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dim 3:

Facciamo il disco:

$\gamma(s) = \begin{pmatrix} \cos(s) \\ \sin(s) \\ 0 \end{pmatrix}$

$\begin{cases} t = \gamma' = \begin{pmatrix} -\sin \\ \cos \\ 0 \end{pmatrix} \\ n = -\cos(s) = \begin{pmatrix} -\cos \\ -\sin \\ 0 \end{pmatrix} \end{cases}$

eq. Frenet $(t, n, b)' = (t, n, b) \begin{pmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$t' = kn$
$n' = -kt$
$b' = 0$

$k = t' \cdot n = \gamma'' \cdot n = \begin{pmatrix} -\cos \\ -\sin \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\cos \\ -\sin \\ 0 \end{pmatrix} =$
$= -\gamma' \cdot \gamma'' = \gamma' \wedge \gamma'' =$
$= \begin{vmatrix} -\sin & -\cos \\ \cos & -\sin \end{vmatrix} = -|\gamma' \gamma''| = 1$

Note: $\tau = 0$ se γ, γ'' sono $\begin{cases} \text{ugualmente } 0 \text{ o} \\ \text{uno opposto all'altro} \end{cases}$

$\tau = 0 \Rightarrow \gamma \subseteq \pi$

$\gamma(s) = \begin{pmatrix} \cos \\ \sin \\ 0 \end{pmatrix} \in \pi$

Nota: $k = \pi / \|\gamma\|^3$ Con segno il determinante dell'accrescimento $\gamma' \gamma''$

se γ curva in un piano: $\gamma : I \longrightarrow \mathbb{R}^2$

$t = \dfrac{\gamma'}{\|\gamma'\|} = \dfrac{1}{\sqrt{\gamma'^2_1 + \gamma'^2_2}} \begin{pmatrix} \gamma'_1 \\ \gamma'_2 \end{pmatrix}$

$n = -\cos(\frac{\pi}{2}) t = \dfrac{1}{\sqrt{\gamma'^2_1 + \gamma'^2_2}} \begin{pmatrix} -\gamma'_2 \\ \gamma'_1 \end{pmatrix}$ $\quad (t, n) = (t, \omega) \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$

$K = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$

$t' = \left(\dfrac{\gamma'_1}{\|\gamma'\|} \right)' = \dfrac{\gamma''_1}{\|\gamma'\|} + \left(\dfrac{1}{\|\gamma'\|} \right) \gamma' \equiv \omega \cdot n$

$\Rightarrow \quad \omega = t' \cdot n$
$= \left(\dfrac{\gamma'}{\|\gamma'\|} \right)' \cdot n$
$= \dfrac{\gamma''}{\|\gamma'\|} \cdot n + \gamma' \left(\dfrac{1}{\|\gamma'\|} \right) \cdot n$
$= \dfrac{\gamma''}{\|\gamma'\|} \cdot n = \dfrac{1}{\sqrt{\gamma'^2_1 + \gamma'^2_2}} \cdot \begin{pmatrix} \gamma''_1 \\ \gamma''_2 \end{pmatrix} \cdot \begin{pmatrix} -\gamma'_2 \\ \gamma'_1 \end{pmatrix} \dfrac{1}{\sqrt{\gamma'^2_1 + \gamma'^2_2}}$
$= \dfrac{1}{\|\gamma'\|^2} \cdot |\gamma' \gamma''|$

$\Rightarrow \quad k = \dfrac{1}{\|\gamma'\|^3} \cdot |\gamma' \gamma''| = \dfrac{1}{\sqrt{\gamma'^2_1 + \gamma'^2_2}^3} \begin{vmatrix} \gamma'_1 & \gamma''_1 \\ \gamma'_2 & \gamma''_2 \end{vmatrix}$.

Formule esplicite per curve nel cambiamento CARTESIANO e POLARE:

NEL PIANO:

CARTESIANO: $\gamma(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$ $\quad \gamma'(x) = \begin{pmatrix} 1 \\ f' \end{pmatrix}$ $\quad \gamma''(x) = \begin{pmatrix} 0 \\ f'' \end{pmatrix}$

$t = \dfrac{1}{\sqrt{1+f'^2}} \begin{pmatrix} 1 \\ f' \end{pmatrix}$ $\quad n = \dfrac{1}{\sqrt{1+f'^2}} \begin{pmatrix} -f' \\ 1 \end{pmatrix}$

versore di Gauss (uscente): $n = \dfrac{|\gamma' \gamma''|}{\|\gamma'\|^3} \begin{pmatrix} -f' \\ 1 \end{pmatrix} = \dfrac{f''}{\sqrt{1+f'^2}^3} \begin{pmatrix} -f' \\ 1 \end{pmatrix}$

POLARE: $\rho(\vartheta)$ $\quad \gamma(\vartheta) = \rho(\vartheta) \begin{pmatrix} \cos \vartheta \\ \sin \vartheta \end{pmatrix}$

$\gamma'(\vartheta) = \begin{pmatrix} \rho' \cos \vartheta - \rho \sin \vartheta \\ \rho' \sin \vartheta + \rho \cos \vartheta \end{pmatrix}$ $\quad \gamma'' = \begin{pmatrix} \rho'' \cos \vartheta - 2\rho' \sin \vartheta - \rho \cos \vartheta \\ \rho'' \sin \vartheta + 2\rho' \cos \vartheta - \rho \sin \vartheta \end{pmatrix}$

$\begin{pmatrix} \rho \cos \vartheta \\ \rho \sin \vartheta \end{pmatrix}' \begin{pmatrix} \rho' \cos \vartheta - \rho \sin \vartheta \\ \rho' \sin \vartheta + \rho \cos \vartheta \end{pmatrix}$

$\|\gamma'\|^2 = \rho'^2 + \rho^2$

$dt/ds = 0$

$k = \dfrac{|\gamma' \gamma''|}{\|\gamma'\|^3} = \dots = \dfrac{2\rho'^2 - \rho\rho'' + \rho^2}{\sqrt{\rho'^2 + \rho^2}^3}$

Nello spazio:

CARTESIANO: $\gamma(x) = \begin{pmatrix} x \\ g(x) \\ h(x) \end{pmatrix}$ $\quad \gamma' = \begin{pmatrix} 1 \\ f' \\ g' \end{pmatrix}$ $\quad \gamma'' = \begin{pmatrix} 0 \\ f'' \\ g'' \end{pmatrix}$

$\|\gamma'\| = \sqrt{1 + f'^2 + g'^2}$

$\gamma' \times \gamma'' = \begin{pmatrix} e_1 & 1 & 0 \\ e_2 & f' & f'' \\ e_3 & g' & g'' \end{pmatrix} = \begin{pmatrix} g'' f' - f' g' \\ g' \cdot 1 - 0 \\ 1 \cdot f'' - 0 \end{pmatrix}$

$\|\gamma' \times \gamma''\|^2 = (f'g'' - g'f')^2 + g'^2 + f''^2$

$\|\gamma' \times \gamma''\|^2 - (f'^2 + g'^2 + 1) = (1 + f'^2 + g'^2)(1 + f''^2 + g''^2) - (f'f'' + g'g'')^2$

$\gamma' \times \gamma'' \cdot \gamma''' = \begin{vmatrix} 1 & 0 & 0 \\ f' & f'' & f''' \\ g' & g'' & g''' \end{vmatrix}$

$k = \dfrac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \dots$

$\tau = \dfrac{\gamma' \wedge \gamma'' \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2} = \dots$

CILINDRICHE: $\gamma(\vartheta) = \begin{pmatrix} \rho(\vartheta) \cos \\ \rho(\vartheta) \sin \\ h(\vartheta) \end{pmatrix}$

SFERICHE: $\gamma(\vartheta) = \begin{pmatrix} \rho(\vartheta) \cos \vartheta \cos \\ \rho(\vartheta) \cos \vartheta \sin \\ \rho(\vartheta) \sin \vartheta \end{pmatrix}$

Approssimazioni locali alle funzioni nei Polinomi:

1. Formula notevole locale curve in $\mathbb{R}^3$, in poli:

$\gamma : (t, n, b) = (t, n, b) \begin{pmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$

cioè: $\begin{cases} t' = kn \\ n' = -kt + \tau b \\ b' = -\tau n \end{cases}$

Proviamo a scrivere γ alla Taylor usando le eq. di Frenet:

$\gamma' = t = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ nel rif. (t, n, b)

$\gamma'' = t' = kn = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix}$ nel rif. (t, n, b)

$\gamma''' = (kn)'$
$= k'n + kn' = k'n + k(-kt + \tau b)$
$= -k^2 t + k'n + k\tau b = \begin{pmatrix} -k^2 \\ k' \\ k\tau \end{pmatrix}$ nel rif. (t, n, b)

$\gamma^{(4)} = \dots$

Sviluppando:

$\gamma(s) = \gamma(0) + s\gamma'(0) + \frac{s^2}{2!}\gamma''(0) + \frac{s^3}{3!}\gamma'''(0) + \dots + \frac{s^k}{k!}\gamma^{(k)}(0) + \dots$

$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{s^2}{2} \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} + \frac{s^3}{3!} \begin{pmatrix} -k^2 t + k'n + k\tau b \\ \dots \\ \dots \end{pmatrix} + \dots$

$= \begin{pmatrix} s - \frac{k^2}{6} s^3 + \dots \\ \frac{k}{2} s^2 + \frac{k'}{6} s^3 + \dots \\ \frac{k\tau}{6} s^3 + \dots \end{pmatrix}$

nel piano (t, n) OSCULATORE: $\begin{pmatrix} s \\ \frac{k}{2} s^2 + \dots \end{pmatrix}$

Ottima approssimante curva parabola

$\begin{cases} x = s \\ y = \frac{k}{2} s^2 \end{cases}$ $\quad \boxed{y = \frac{k}{2} x^2}$

nel piano (n, b) NORMALE: $\begin{pmatrix} \frac{k}{2} s^2 + \dots \\ \frac{k\tau}{6} s^3 + \dots \end{pmatrix}$

$\begin{cases} y = \frac{k}{2} s^2 \\ z = \frac{k\tau}{6} s^3 \end{cases}$ $\quad \begin{cases} s^2 = \frac{2y}{k} \\ z = \frac{k\tau}{6} s^3 \end{cases}$ $\quad \boxed{z^2 = \frac{k^2 \tau^2}{18} y^3}$

nel piano (t, b) RETTIFICANTE: $\begin{pmatrix} s \\ \frac{k\tau}{6} s^3 + \dots \end{pmatrix}$

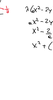
$\begin{cases} x = s \\ z = \frac{k\tau}{6} s^3 \end{cases}$ $\quad \boxed{z = \frac{k\tau}{6} x^3}$

Problema: generalizzare a $\mathbb{R}^n$

2. Piano osculatore di 3 non sui punti:
e cerchi osculatori a γ:

Nello sviluppo del Taylor dei 3 punti:
da prima approssimazione è retta (tangente)
da seconda è una parabola (osculatrice)

$\gamma(s) = \gamma(s_0) + \gamma'(s_0)(s - s_0) + \gamma''(s_0)\frac{(s-s_0)^2}{2!} + \dots = \gamma(s_0) + \gamma'(s_0)(s-s_0) + k(s_0)\frac{n(s_0)}{2}(s-s_0)^2$



Approssimazione di secondo ordine

Nel piano osculatore ma non parabola coi tre punti (ipercubica di equazione $y = ax^3 + bx^2 + cx + d$), abbiamo una sola cubica:

$\gamma(x^3 - 2x \cdot y) + y = 0$ circonferenza in $\begin{cases} x_{i+1} \\ y_{i+1} \end{cases}$

$ax^3 - 2x + y = 0$

$X^3 - \frac{2}{a} x + y = 0 \Rightarrow$

$X^3 + \left(Y - \frac{1}{2a} \right)^2 = \frac{1}{4a^2}$ $\quad$ centro $\begin{pmatrix} 0 \\ \frac{1}{2a} \end{pmatrix}$

raggio $\frac{1}{2a}$

Disegnare γ nel piano (t, b):

$\gamma(s_0) \quad n(s_0) \quad \kappa(s_0) = \pm < 0$
$\kappa(s_0) \quad \frac{\gamma(s_0) - n(s_0)}{2} = \pm < 0$
$|K(s_0)| \quad \frac{1}{\|\gamma(s_0)\|} > \pm$

$|K(s_0)| > \frac{1}{\|\gamma(s_0)\|}$ #

Nota: Data $\gamma : I \xrightarrow{C^2} \mathbb{R}^2$ liscia con curvatura $\neq 0$ ovunque, esiste sempre un'unica circonferenza $C \subseteq \mathbb{R}^2$ che approssima γ al secondo ordine, cioè $\gamma(s) = \frac{1}{\rho(s)} n(s)$

Data "forma del grafico di γ", trovare il cerchio osculatore... che curvatura non C_γ? che $\gamma \subseteq$ Palermo?

Mercoledì, 9 dicembre 2025

3. Proiezione dal cerchio:

$\gamma : I \longrightarrow \mathbb{R}^2$ curva regolare (generalizzata a $\mathbb{R}^n$)

$\exists \delta \in \mathbb{R} : \|\gamma(t)\| \neq 0$ definita massimo locale in t_0 $\quad \delta = \min$ distanza dalla curva γ

$\Rightarrow |K(t_0)| > \frac{1}{\|\gamma(t_0)\|} \quad \forall \kappa \in \mathbb{R} \quad |\kappa(t_0)| < \|\gamma(t_0)\|$

Dimostrazione (1): $\|\gamma(t_0)\|^2 = \gamma(t_0) \cdot \gamma(t_0)$ massimo locale in t_0:

$\begin{cases} \text{derivata prima nulla:} \\ \langle \gamma(t_0) \cdot \gamma(t_0) \rangle' = 2\,\gamma(t_0) \cdot \gamma'(t_0) = 0 \\ \gamma(t_0) \perp \gamma'(t_0) \end{cases}$

$\begin{cases} \text{derivata seconda negativa:} \\ \langle \gamma(t_0) \cdot \gamma(t_0) \rangle'' = \left(\gamma'(t_0) \cdot \gamma'(t_0) \right)' + \\ + \gamma(t_0) \cdot \gamma''(t_0) < 0 \end{cases}$

$\gamma'(t_0) \cdot \gamma'(t_0) + \gamma(t_0) \cdot \gamma''(t_0) < 0$

Supponiamo γ sia piana:

$\gamma(t_0) \quad n(t_0) \quad \kappa(t_0) = \pm < 0$
$\kappa(t_0) \quad \frac{\gamma(t_0) - n(t_0)}{2} = \pm < 0$
$|K(t_0)| \quad \frac{1}{\|\gamma(t_0)\|} > \pm$

$|K(t_0)| > \frac{1}{\|\gamma(t_0)\|}$ #

Dimostrazione (2):

Sviluppare $\gamma(s_0)$ su tre volte Taylor:

$\gamma(s) = \gamma(s_0) + \gamma'(s_0)(s - s_0) + \frac{\gamma''(s_0)}{2}(s - s_0)^2 + \frac{\gamma'''(s_0)}{3!}(s - s_0)^3 + \dots$

per calcolare $\gamma(s) \cdot \gamma(s) = 0$:

$= \gamma(s_0) \cdot \gamma(s_0) + 2\,\gamma(s_0) \cdot \gamma'(s_0)(s - s_0) + \left(\gamma'(s_0) \cdot \gamma'(s_0) + \gamma(s_0) \cdot \gamma''(s_0) \right) \frac{(s-s_0)^2}{2} + \dots$

e uniquare la circonferenza per massimo locale in t_0.

Problema di Frenet:

Esistono curve in $\mathbb{R}^n$ con $k_1, \dots, k_{n-1}$ di Frenet:

- t (tangente) è costante (rette?)
$t' = 0$
$kn = 0 \quad (n \neq 0) \Rightarrow k = 0$
(curvatura nulla caratterizza le rette: $\gamma(s) = p + sv \Rightarrow \gamma'(s) = v \Rightarrow k = 0$)

- n (normale) è costante (rette?)
$n' = 0$
$-kt + \tau b = 0 \Rightarrow$
t, b uscenti dal piano $\Rightarrow k = 0 \Rightarrow$ rette $\tau = 0$

- b (binormale) costante ($\gamma \subseteq$ piano)
$b' = 0$
$-\tau n = 0 \Rightarrow \tau = 0 \Rightarrow$ curva piana
(Teorema inverso: la curva sta dentro quel piano $\perp b$)

Passaggio per punto: γ curva data con:

- Data la γ di C^∞ passante per un punto P_0
$\forall k : \gamma(s) = \gamma(0) + \frac{s}{1!}\gamma'(0) + \frac{s^2}{2!}\gamma''(0) + \dots + \frac{s^k}{k!}\gamma^{(k)}(0) + \dots$
derivate $\gamma' = (\gamma \times \gamma')' = 0$
$\gamma' \times \gamma' + \gamma \times \gamma'' = 0$
$\gamma \times \gamma'' = 0$
$\gamma(\gamma \times \gamma') = 0 \Rightarrow \gamma \times \gamma' = 0$
$\begin{cases} f(s)\gamma = 0 \Rightarrow \gamma \times 0 \\ \gamma \times \gamma = 0 \end{cases}$ (sempre)

- Data la prima e la seconda derivata per un punto P_0
$\forall k$ $\quad \gamma(s) = \gamma(0) + s\gamma'(0) + \frac{s^2}{2}\gamma''(0) + \dots$
$\gamma' = \gamma \times \gamma' + \gamma \times \gamma'' = 0$
$\gamma \times \gamma' = \gamma \times (-k\gamma + \tau\gamma \times \gamma) = 0$
$(1 - \gamma k) \gamma \times \gamma' + \gamma \times \gamma \times \gamma \tau = 0$
$\begin{cases} 1 - \gamma k \neq 0 \\ \gamma \times \gamma' = 0 \end{cases} \Rightarrow \gamma \times \gamma' = 0$ (come sopra)
$\Rightarrow \gamma \times \gamma = 0$ (come sopra)
$\Rightarrow k = \frac{1}{\gamma}$ costante (cerchio)

- Data la binormale di γ passante per un P_0
$\forall k : \gamma(s) = \gamma(0) + s\gamma(0) + \frac{s^2}{2}\gamma'(0) + \dots$
$\gamma = \gamma(0) + \gamma(0) \times \gamma(0) + \frac{s^2}{2}\gamma' \times \gamma(0) + \dots$
$\gamma \times \gamma(0) = 0 + \gamma(0) \times \gamma(0) + \frac{s^2}{2}\gamma' \times \gamma(0) + \dots$
$\gamma \times \gamma(0) = \frac{s^2}{2}\gamma' \times \gamma(0) + \dots$
$\gamma \times \gamma(0) = \frac{s^2}{2}(\gamma \times \gamma(0))'