ELICHE CIRCOLARI in $\mathbb{R}^3$:

sono curve di equazione parametrica

$$\gamma(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix} \qquad a, b \in \mathbb{R}_{>0}$$

"scala a chiocciola"

$$\gamma' = \begin{pmatrix} -a\sin t \\ a\cos t \\ b \end{pmatrix} \qquad \gamma'' = \begin{pmatrix} -a\cos t \\ -a\sin t \\ 0 \end{pmatrix} \qquad \gamma''' = \begin{pmatrix} a\sin t \\ -a\cos t \\ 0 \end{pmatrix}$$

$$\|\gamma'\| = \sqrt{a^2+b^2}$$

$$s(t) = \sqrt{a^2+b^2}\, t$$

$$\vec{T}(t) = \begin{pmatrix} a\cos\frac{s}{\sqrt{a^2+b^2}} \\ a\sin\frac{s}{\sqrt{a^2+b^2}} \\ b\frac{s}{\sqrt{a^2+b^2}} \end{pmatrix}$$

$$\gamma' \times \gamma'' = \begin{pmatrix} ab\sin t \\ -ab\cos t \\ a^2 \end{pmatrix} \qquad \|\gamma' \times \gamma''\| = \sqrt{(\gamma'\times\gamma'')\cdot(\gamma'\times\gamma'')} = a\sqrt{a^2+b^2}$$

$$\kappa(t) = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2} \quad \text{(costante!)}$$

$$\tau(t) = \frac{|\gamma' \gamma'' \gamma'''|}{\|\gamma' \times \gamma''\|^2} = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2} \quad \text{(costante!)}$$

dunque κ e τ sono costanti determinate dai parametri a, b:

$$\begin{cases} \kappa = \dfrac{a}{a^2+b^2} \\ \tau = \dfrac{b}{a^2+b^2} \end{cases}$$

e viceversa: a e b sono determinati da κ e τ:

$$\kappa^2 + \tau^2 = \frac{a^2}{(a^2+b^2)^2} \Rightarrow \begin{cases} a = \kappa(a^2+b^2) = \dfrac{\kappa}{\kappa^2+\tau^2} \\ b = \tau(a^2+b^2) = \dfrac{\tau}{\kappa^2+\tau^2} \end{cases}$$

dunque possiamo invertire il TFC e dire:
qui c'è una curva con κ e τ costanti (>0)
e a meno di isometrie in $\mathbb{R}^3$ una sola!
calcolando a, b come sopra.

Problema: trovare *direttamente* la curva curva in $\mathbb{R}^3$ con $\kappa, \tau \in$ costanti $\neq 0$ usando l'elica circolare:

κ, τ costanti

$$\Rightarrow \begin{cases} n' = -\kappa t' + \tau b \\ \downarrow \\ -\kappa' t n - \tau^2 n \\ \downarrow \\ -(\kappa^2+\tau^2) n \end{cases}$$

$$\kappa(s) = \sigma \sin\left(\tfrac{s}{\sqrt{\sigma^2+\beta^2}}\right) + \omega \sin\left(\tfrac{s}{\sqrt{\sigma^2+\beta^2}}\right)$$

$$\downarrow \text{ con } \sigma, \omega \text{ valori opportuni}$$

$$\begin{pmatrix} \cos\left(\frac{s}{\sqrt{\sigma^2+\beta^2}}\right) \\ \sin\left(\frac{s}{\sqrt{\sigma^2+\beta^2}}\right) \\ 0 \end{pmatrix}$$

$$t(s) = \kappa n = \kappa \begin{pmatrix} \cos\left(\frac{s}{\sqrt{\sigma^2+\beta^2}}\right) \\ \sin\left(\frac{s}{\sqrt{\sigma^2+\beta^2}}\right) \\ 0 \end{pmatrix}$$

$$t(s) = \begin{pmatrix} * \cos(\quad) \\ * \sin(\quad) \\ * \end{pmatrix} \longleftarrow \text{antiderivo}$$

$$\gamma(s) = \int t(s) ds = \ldots \text{ ELICA CIRCOLARE!}$$

Inoltre $\dfrac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix} = \delta(t)$ è elica circolare unitaria

$\Rightarrow \gamma(s)$

problema: calcolare C_γ = centro dei cerchi di γ

$$C_\gamma = \gamma + \frac{1}{\kappa} n_\gamma$$

$$\vdots \ldots$$

$$= \begin{pmatrix} -b^2/a \\ -b^2/a \\ bt \end{pmatrix}$$

dunque elica circolare $\quad r_{C_\gamma} = \dfrac{-b^2/a}{\left(\frac{b}{a^2+b^2}\right)^2 a^2} = \dfrac{-b^2/a}{b^2(a^2+b^2)^2/a^2} = \dfrac{-a}{a^2+b^2} = -\kappa$

$$\tau_{C_\gamma} = \frac{b}{\left(\frac{b}{a^2+b^2}\right)^2 a^2} = \frac{b}{b^2(a^2+b^2)^2 a^2} = \frac{a^2}{b(a^2+b^2)} = \frac{\kappa^2}{\tau}$$

in $C_\gamma = \gamma$ finché $\begin{cases} r_{C_\gamma} = \kappa \\ \tau_{C_\gamma} = \tau \end{cases}$.

ELICHE GENERALI:

Per curve γ in $\mathbb{R}^3$ con $\kappa, \tau \neq 0$ sono equivalenti e sceglienti tra loro:

② τ/κ costante

③ le tangenti formano angolo fisso con un vettore $\vec{v}$ FISSATO.

④ Le normali sono parallele a un fissato piano ($\perp \vec{v}$ del punto ③)

⑤ le binormali formano angolo fisso con un vettore $\vec{v}$ FISSATO

⑥ sono del tipo: $\begin{pmatrix} a\int \cos\theta(t) dt \\ a\int \cos\theta(t) dt \\ bt \end{pmatrix} = \gamma(t)$

a meno di isometrie in $\mathbb{R}^3$.

Cose facili: $t \cdot v =$ costante $\Leftrightarrow (t \cdot v)' = 0$ $\qquad b \cdot v =$ costante

② $\Leftrightarrow t' \cdot v = 0$ ⑤

$\Leftrightarrow t \cdot v = 0$

$\Leftrightarrow n \perp v$ ④

(dunque ③ = ⑤ = ⑥ equivalenti)

⑤ $\Rightarrow$ ④

$n \perp v \Rightarrow \exists\, c \in \langle t, b \rangle$

$v = t \cdot \cos\vartheta + b \cdot \sin\vartheta \quad$ con ϑ costante

$0 = v' = t' \cdot \cos\vartheta + b' \sin\vartheta$

$= \kappa n \cdot \cos\vartheta - \tau n \sin\vartheta$

$= (\kappa \cos\vartheta - \tau \sin\vartheta) n$

dunque $\kappa \cos\vartheta - \tau \sin\vartheta = 0$

$\Rightarrow \tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta} = \frac{\tau}{\kappa} = \text{costante!}$ ④

③ $\Leftrightarrow$ ⑤ ④

⑥ $\Rightarrow$ ④

④ $\Rightarrow$ ⑥ direttamente.

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④ $\Rightarrow$ ⑦ $\quad \frac{\tau}{\kappa} =$ costante $= \frac{b}{a}$ con ϑ angolo fissato

possiamo $\sigma := t \cdot \cos\vartheta + b \sin\vartheta$ (a priori dipende da t, b, ma vedremo che non è così...)

allora $\sigma' = t' \cdot \cos\vartheta + b' \sin\vartheta$

$= \kappa n \cdot \cos\vartheta - \tau n \sin\vartheta$

$= (\kappa \cos\vartheta - \tau \sin\vartheta) n$

$= \left(\frac{\kappa}{\tau} \tau \cos\vartheta - \tau \sin\vartheta\right) n = 0$

dunque σ costante

e quindi t forma angolo fisso con t, b, perché $\sigma \cdot t = t \cdot t \cos\vartheta + b \cdot t \sin\vartheta =$ costante costante.

$\sigma \cdot b = t \cdot b \cos\vartheta =$ costante.

dunque ⑤ ④

⑦ $\Rightarrow$ ⑥ $\quad \gamma' = \begin{pmatrix} a\cos\theta(t) \\ a\sin\theta(t) \\ b \end{pmatrix}$ $\qquad \gamma'' = \begin{pmatrix} -a\sin\theta(t)\,\theta'(t) \\ a\cos\theta(t)\,\theta'(t) \\ 0 \end{pmatrix}$ $\qquad \gamma''' = \begin{pmatrix} -a\cos\theta(t)\,\theta'(t)^2 - a\sin\theta(t)\,\theta''(t) \\ -a\sin\theta(t)\,\theta'(t)^2 + a\cos\theta(t)\,\theta''(t) \\ 0 \end{pmatrix}$

$\|\gamma'\| = \sqrt{a^2+b^2}$

$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} = \frac{\sqrt{a^2+b^2}\; a\,|\theta'(t)|}{(\sqrt{a^2+b^2})^3} = \frac{a\,|\theta'(t)|}{a^2+b^2}$ $\qquad \left| \begin{array}{l} k!=t!= \\ \cdot \frac{a^2}{(a^2+b^2)^2}\theta'^2 + \frac{b^2}{(a^2+b^2)^2}\theta'^2 \\ = \frac{a^2+b^2}{(a^2+b^2)^2}\theta'^2 = \frac{\theta'^2}{a^2+b^2} \end{array} \right.$

$\tau = \frac{|\gamma' \gamma'' \gamma'''|}{\|\gamma' \times \gamma''\|^2} = \frac{-b\cdot a^2\cdot|\theta'(t)|^3}{(a^2+b^2)\cdot a^2\cdot\theta'(t)^2} = -\frac{b}{a^2+b^2}\theta'(t)$ costante

$\frac{\tau}{\kappa} = -\frac{b}{a}\ (=$ costante, dunque ⑥

viceversa: ② $\Rightarrow$ ⑦

data γ con $\frac{\tau}{\kappa}$ costante

cerchiamo a, b, θ: $\quad \kappa^2 + \tau^2 = \frac{a^2}{(a^2+b^2)^2}$

$$\begin{cases} a = \kappa\,\frac{a^2+b^2}{a^2} = \frac{\kappa}{\kappa^2+\tau^2}\,\theta' \\ b = \frac{\tau}{\kappa^2+\tau^2}\,\theta' \\ a^2+b^2 = 1 \end{cases}$$

questo determina a, b, θ, allora possiamo $\gamma(t)$ usando $\kappa(t), \tau(t)$ con $\frac{\tau(t)}{\kappa(t)}$ costante

Possiamo poi le isometrie finali. Come si vuole e viceversa dal teorema sopra.

Osservazione:

$$\begin{cases} \kappa^2 = \frac{a^2}{(a^2+b^2)^2}\,\theta'^2 \\ \tau^2 = \frac{b^2}{(a^2+b^2)^2}\,\theta'^2 \end{cases} \Rightarrow \begin{cases} a^2 = \\ b^2 = \end{cases}$$

Posso usare fatti di intermedio anche ②③ ⇒ ⑤

consegno: $\gamma = \begin{pmatrix} \sigma \\ \beta \end{pmatrix}'$

con $\sigma \cdot t =$ costante ②

$\sigma \cdot v = 0$ ③

$$\Rightarrow \gamma' = t = \begin{pmatrix} a\cos\theta(t) \\ a\sin\theta(t) \\ b \end{pmatrix} \quad \text{con } a^2+b^2=1$$

$\sigma \cdot t_3 = \begin{pmatrix} \theta \\ \beta \end{pmatrix}' \cdot t_3 =$ costante

$\gamma = \int \gamma' ds$ dunque ⑤.

Nota: se τ/κ costante, κ costante allora κ, τ sono costanti: elica circolare.

CURVE SFERICHE: unitaria, regolare, $\kappa, \tau \neq 0$

$\subset \mathbb{R}^3$

$\gamma \subset$ sfera $\Leftrightarrow \left(\frac{\kappa'}{\tau\kappa^2}\right)' =$ costante $= R^2$ $\qquad \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\tau\kappa^2}\right)^2 =$ costante $= R^2$ $\qquad \left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\right)^2 \left(\frac{1}{\kappa}\right)^2 = R^2$

$$\begin{cases} \rho = \frac{1}{\kappa} \\ \delta = \frac{1}{\tau\kappa^2} \end{cases} \Rightarrow \begin{cases} \rho' = -\frac{\kappa'}{\kappa^2} \\ \delta\rho' + \rho = R \end{cases}$$

$\Leftrightarrow \frac{\rho'}{\rho} = -(\delta\rho)' \Leftrightarrow \frac{\rho'}{\rho} + (\delta\rho)' = 0$

$\Leftrightarrow \rho^2 + (\rho\delta\rho')^2 =$ costante $= R^2$

$\Leftarrow (\rho^2 + (\rho\delta\rho')^2)' = 0$

$2\rho\rho' + 2(\rho\delta\rho)(\rho\delta\rho') = 0$

$\rho'(\rho + \delta \cdot (\rho\delta\rho')) = 0$

$\rho + \delta \cdot (\rho\delta\rho') = 0$

$\frac{\rho'}{\rho} + (\rho\delta)' = 0$

Vediamo che $\gamma \subset$ sfera ha le proprietà cercate:

$\gamma \subset \partial D(C, R)$ $\quad$ con C punto $\in \mathbb{R}^3$, R reale >0

$(\gamma - C)\cdot(\gamma - C) = R^2$ $\quad$ (cioè $\|\gamma - C\| = R$)

$\Rightarrow (\gamma - C)\cdot\gamma' = 0$ $\qquad (\gamma - C)\cdot t = 0$

$\Rightarrow \gamma'\cdot\gamma' + (\gamma - C)\cdot\gamma'' = 0$

$t\cdot t + (\gamma - C)\cdot(\kappa n) = 0$

$1 + \kappa(\gamma - C)\cdot n = 0$ $\qquad (\gamma - C)\cdot n = -\frac{1}{\kappa}$

$\Rightarrow 0 + n'\cdot(\gamma - C) + \kappa\,\gamma\cdot t + \kappa(\gamma - C)\cdot n' = 0$

$\kappa^2(-\frac{1}{\kappa}) + \kappa\cdot(\gamma - C)\cdot(-\tau\kappa t + \tau b) = 0$

$-\frac{\kappa'}{\kappa} + \kappa\tau\,(\gamma - C)\cdot b = 0$ $\qquad (\gamma - C)\cdot b = \frac{\kappa'}{\kappa^2\tau}$

$\Leftrightarrow \gamma\cdot b + (\gamma - C)\cdot b' = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$

$(\gamma - C)\cdot(-\tau n) = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$

$-\tau\,(\gamma - C)\cdot n = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$

$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$

$\rightarrow$ t, n, b ort. determinate

$$(\gamma - C) = ((\gamma - C)\cdot t)\,t + ((\gamma - C)\cdot n)\,n + ((\gamma - C)\cdot b)\,b$$

$$= \left(-\tfrac{1}{\kappa}\right) n + \left(\tfrac{\kappa'}{\tau\kappa^2}\right) b$$

$$\|\gamma - C\|^2 = R^2$$

$$\left(\tfrac{1}{\kappa}\right)^2 + \left(\tfrac{\kappa'}{\tau\kappa^2}\right)^2$$

Viceversa, se $\gamma \subset$ curva con proprietà notevole che ha $\kappa \neq 0$, vogliamo vedere che è $\subset$ sfera:

poniamo

$$C = \gamma + \frac{1}{\kappa} n - \frac{\kappa'}{\tau\kappa^2} b \qquad \text{e poi } \partial \text{ un punto che dipende dal parametro } s$$

Facciamo la derivata:

$$C' = \gamma' + \left(\tfrac{1}{\kappa} n\right)' - \left(\tfrac{\kappa'}{\tau\kappa^2} \cdot b\right)'$$

$$= t + \left(\tfrac{1}{\kappa}\right)' n + \tfrac{1}{\kappa} n' - \left(\tfrac{\kappa'}{\tau\kappa^2}\right)' b - \tfrac{\kappa'}{\tau\kappa^2} b'$$

$$= \kappa\, t + \left(\tfrac{1}{\kappa}\right)' n + \tfrac{1}{\kappa}(-\kappa t + \tau b) - \left(\tfrac{\kappa'}{\tau\kappa^2}\right)' b - \tfrac{\kappa'}{\tau\kappa^2}(-\tau n)$$

$$= 0 \text{ finale!}$$

poiché $C \in$ punto costante $\forall s$, γ sfera, con dipendenza liscia,

$$\|\gamma - C\|^2 = \left\|\tfrac{1}{\kappa} n + \tfrac{\kappa'}{\tau\kappa^2} b\right\|^2 = \ldots = R^2 \text{ costante}$$

per ipotesi

dunque $\gamma \subset \partial D(C, R)$,

cioè $\gamma \subset$ curva sferica.

Vediamo casi particolari di curve sferiche

UNITARIE sulla sfera unitaria $\subset \partial D(0,1)$

poniamo $\quad \gamma := \gamma \times \gamma' \cdot \gamma'' = |\gamma\,\gamma'\gamma''|$

$\gamma' := |\gamma'\times\gamma''|\,t = |\gamma\,\gamma'\gamma''|\,t$

allora $\quad \kappa = \sqrt{1+\tau^2}$

$\tau = \gamma'/\kappa^2$

dunque $\kappa \in \tau$ sono determinati completamente da γ, ce prendi determina completamente la curva γ,

e γ è $\subset$ sfera se $\gamma = \text{costante}$ ($\exists^1 \neq 0 \Rightarrow \tau = 0$, γ piana)

CERCHIO massimo se $\gamma = 0$ ($\kappa = 1, \tau = 0$, γ cerchio massimo)

Domanda: una curva con dei $\kappa \in \tau$ è necessariamente sferica (nonostante sia la funzione γ? Posto verificare se è soddisfatta la condizione del teorema $\left(\frac{\kappa'}{\tau\kappa^2}\right)' = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$

se $\kappa = \sqrt{1+\tau^2}$

$\tau = \gamma'/\kappa^2$

allora $\quad \left(\frac{\kappa'}{\tau\kappa^2}\right) = \frac{\gamma'}{(1+\tau^2)\tau\kappa^2}$

$$\left(\frac{\kappa'}{\tau\kappa^2}\right) = \left(\frac{\gamma'}{(1+\tau^2)\tau\kappa^2}\right)'$$

$$= \left(\frac{\gamma'}{(1+\tau^2)\tau}\right)'$$

$$= \frac{\gamma''(1+\tau^2) - \gamma'\tau'}{(1+\tau^2)^2\tau}$$

$$= \frac{\gamma''(1+\tau^2) - \gamma'\gamma'/\tau}{(1+\tau^2)^2\tau}$$

$$= \frac{\gamma''}{(1+\tau^2)\tau\kappa^2}$$

Perché la costante è uniforme per ipotesi.

Per calcolare $\kappa \in \tau$ conviene usare un riferimento diverso da quello di Frenet:

usiamo $\quad \gamma, \gamma', \gamma\gamma'$ come base ortonormale mobile.

$\gamma \subset S^2 \Rightarrow \|\gamma\| = 1$

$\Rightarrow \gamma' \perp \gamma$

γ unitaria $\Rightarrow \|\gamma'\| = 1$

$\gamma' \times \gamma$ unitaria (unitaria)

$$\begin{cases} (\gamma\cdot\gamma = 1 \quad (\gamma \in \mathbb{R}^3) \\ 2\,\gamma\cdot\gamma' = 0 \quad (\gamma \times \gamma') \\ 2\,\gamma\cdot\gamma' + \gamma\cdot\gamma'' = 0 \Rightarrow \gamma\cdot\gamma'' = -1 \\ \gamma'\cdot\gamma' = 1 \quad (\gamma \text{ unitaria}) \\ \gamma'\cdot\gamma'' = 0 \\ \gamma'\cdot\gamma''' + \gamma''\cdot\gamma'' = 0 \Rightarrow \gamma'\cdot\gamma''' = 0 \end{cases}$$

$$\gamma'' = \begin{vmatrix} \gamma & \gamma' & \gamma\gamma' \\ \gamma'' & \gamma'' & \gamma''\gamma' \\ 0 & 0 & \gamma''\gamma' \end{vmatrix} = (\gamma\gamma'\cdot\gamma'')\,\gamma\gamma' = -\gamma + J(\gamma\gamma')$$

$$\gamma''' = -\gamma' + J(\gamma\gamma'') = -\gamma' + J(-\gamma + J(\gamma\gamma')) = -\gamma' - \gamma = -\begin{pmatrix} \gamma \\ \gamma' \end{pmatrix}$$

$\kappa = \|\gamma''\| = \sqrt{1+\tau^2}$

$\tau = -b\cdot n'$

$$\begin{cases} t = \gamma' \\ n = \gamma'' = -\gamma + J(\gamma\gamma') \\ b = t \times n = \frac{1}{\sqrt{1+\tau^2}}\,\gamma - \frac{\tau}{\sqrt{1+\tau^2}}\,(\gamma\gamma') = \frac{1}{\kappa}\left(\frac{\gamma}{\gamma'}\right) \end{cases}$$

$$\tau = -b\cdot n' = \left(\frac{\gamma}{\gamma'}\right)' \cdot \left(\frac{\gamma}{J\gamma}\right) \cdot \frac{1}{1+\tau^2}$$

$$= \left(\frac{\gamma'}{-\gamma}\right) \cdot \left(\frac{J\gamma}{\gamma\gamma'}\right) \cdot \frac{1}{1+\tau^2}$$

$$= \left(\frac{\gamma'}{-\gamma}\right) \cdot \left(\frac{J\gamma + (\gamma\gamma')'}{\gamma\gamma'}\right) \cdot \frac{1}{1+\tau^2}$$

$$= \left(\frac{\gamma'}{-\gamma}\right) \cdot \left(\frac{J\gamma + (-\gamma + J(\gamma\gamma'))}{\gamma\gamma'}\right) \cdot \frac{1}{1+\tau^2}$$

$$= \frac{|\gamma\,\gamma'\,\gamma''|}{1+\tau^2} = \frac{\gamma\gamma'}{1+\tau^2}$$

$\gamma'' \cdot \gamma = 0$

$\gamma'' \cdot \gamma' = 1$

$\gamma'' \cdot (\gamma\gamma') = |\gamma\,\gamma'\gamma''| = \gamma'$

Abbastanza: curiosità

$$\left(\tfrac{1}{\tau}\right)' + \left(\tfrac{1}{\tau}\right)'\left(\tfrac{1}{\tau}\right)^2 = 1$$

$$\left(\tfrac{1}{\tau}\right)' + \left(\tfrac{\tau\gamma''}{\gamma'}\right)\left(\tfrac{1}{\tau\gamma'}\right)^2 = 1$$

$$\left(\tfrac{1}{\tau}\right)' + \left(\tfrac{-\gamma'(-\tau+\gamma\gamma')}{\gamma'}\right)\left(\tfrac{1}{\tau\gamma'}\right)^2 = 1 - \frac{1}{1+\tau^2}$$

$$\left(\tfrac{1}{\tau}\right)' + \left(\tfrac{\gamma'\tau^2}{\gamma'}\right)\left(\tfrac{1}{(1+\tau^2)\tau^2}\right) = \frac{\tau^2}{1+\tau^2}$$

ok sembra sempre vero!

Elica circolare come sfera curva circolare:
γ curva unitaria, sulla sfera unitaria.

Studiamo:

$S: \quad S = \gamma'$ ($\subset$ curva in S^2, non unitaria!)

$S' = \gamma'' = -\gamma + J(\gamma\gamma')$ $\qquad \|S\| = \sqrt{1+\tau^2}$

$\gamma''' \cdot S'' = -\gamma' + J'(\gamma\gamma') + J(\gamma\gamma')' = -S + J'(\gamma\gamma') + J(\gamma' S)$

$= -S + J'(\gamma\gamma') + J\,\gamma'(-\gamma + J(\gamma\gamma'))$

$= -(\gamma + J)S' + J(\gamma\gamma')$

$S \times S' = \begin{pmatrix} \gamma\gamma' \\ S \\ S' \end{pmatrix} = \begin{pmatrix} \gamma\gamma' \\ \gamma' \\ -\gamma + J(\gamma\gamma') \end{pmatrix} = \gamma(\gamma\gamma')\,\gamma - J'\gamma' + (\gamma\gamma')\,(\gamma\gamma')$

$\gamma^{(4)} \cdot S''' = \begin{pmatrix} \gamma'' \\ \gamma''' \\ \gamma^{(4)} \end{pmatrix} = \begin{pmatrix} \gamma'' \\ \gamma''' \\ \gamma''' + \ldots \end{pmatrix} = \gamma(\gamma\gamma')\,\gamma\gamma' - J'\gamma'\gamma'$

$= -(1+\tau^2)\,\gamma'' - 2J'\gamma' + J''(\gamma\gamma') + J'(\gamma\gamma')$

$= -(1+\tau^2)(-\gamma + J(\gamma\gamma')) - 2J'\gamma' + J''(\gamma\gamma') - J'$

$= (+\tau^2)\,\gamma - 3J'\gamma' + (J'' - (1+\tau^2))\,(\gamma\gamma')$

$\gamma_g = \dfrac{\|S' \times S''\|}{\|S'\|^3} = \dfrac{\sqrt{\gamma'^4 + (1+\tau^2)^2}}{\sqrt{1+\tau^2}^3} = \sqrt{1 + \frac{\gamma'^2}{(1+\tau^2)^3}}$

$\tau_g = \dfrac{|S', S'', S'''|}{\|S' \times S''\|^2} = \dfrac{\begin{vmatrix} \gamma\gamma' & \gamma' & -\gamma + J(\gamma\gamma') \\ \gamma' & -\gamma + J(\gamma\gamma') & \gamma(\gamma\gamma')\,\gamma - J'\gamma' + (\gamma\gamma')\,(\gamma\gamma') \\ -\gamma + J(\gamma\gamma') & \gamma(\gamma\gamma')\,\gamma - J'\gamma' + (\gamma\gamma')\,(\gamma\gamma') & \gamma''' \cdot S'' \end{vmatrix}}{\gamma'^4 + (1+\tau^2)^2} = \dfrac{\gamma'^2(1+\tau^2) - 3\gamma'^2}{\gamma'^4 + (1+\tau^2)^2}$

$\lambda = \int \gamma$ (curva unitaria, non sferica!)

$\lambda' = \gamma$ $\qquad \|\lambda'\| = \|\gamma\| = 1$ $\qquad \gamma \subset S^2$, γ è unitaria

$\lambda'' = \gamma' \quad \Rightarrow \kappa_\lambda = \|\gamma'\| = \tau$

$\lambda''' = \gamma'' \qquad \tau_\lambda = \frac{|\gamma\,\gamma'\,\gamma''|}{\|\gamma'\|^2} = \frac{\|\gamma\,\gamma'\,\gamma''\|}{\|\gamma'\|^2} = 1$

$\kappa_\lambda = \tau$

$\tau_\lambda = 1$

se questo valesse si chiuderebbe cerchio dei concetti costanti e...

$\mu = \int \gamma\,\gamma'\,\gamma''$ $\qquad \|\gamma\,\gamma'\,\gamma''\| = 1$ (curva unitaria)

$\mu' = \gamma\,\gamma'\,\gamma'' \qquad \|\mu'\| = 1$

$\mu'' = (\gamma\gamma')' \cdot \gamma'\gamma'' + \gamma\gamma' \cdot (-\gamma + J(\gamma\gamma')) = -\gamma'$

$\mu''' = (\gamma\gamma'')' \cdot (-\gamma) - \gamma'^2 = -\gamma' \cdot \gamma - \gamma'^2 = -\gamma'^2$

$\kappa_\mu = \|\gamma'\|$

$\tau_\mu = 1$

$\tau_\mu = \frac{|\mu'\,\mu''\,\mu'''|}{\|\mu'\times\mu''\|^2} = \frac{\begin{vmatrix} \gamma\gamma' & \gamma & -\gamma + J(\gamma\gamma') \\ \gamma & -\gamma + J(\gamma\gamma') & -\gamma' \\ -\gamma + J(\gamma\gamma') & -\gamma' & -\gamma'^2 \end{vmatrix}}{\gamma'^4} = \frac{\gamma'^2}{\gamma'^4} = \frac{1}{\gamma'^2} < 1$

qui torniamo comunque a...

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martedì 16 dicembre 2025