

Finite Groups that are the union of at most 25 proper subgroups

Martino Garonzi,
Dipartimento di Matematica Pura ed Applicata,
Via Trieste 63, 35121 Padova, Italy.
E-mail address: mgaronzi@studenti.math.unipd.it

Abstract

For a finite group G let $\sigma(G)$ (the “sum” of G) be the least number of proper subgroups of G whose set-theoretical union is equal to G , and $\sigma(G) = \infty$ if G is cyclic. We say that a group G is σ -elementary if for every non-trivial normal subgroup N of G we have $\sigma(G) < \sigma(G/N)$. In this article we produce the list of all the σ -elementary groups of sum up to 25. We also show that $\sigma(\text{Aut}(PSL(2, 8))) = 29$.

1 Introduction

Let G be a finite group. If G is not cyclic we can consider the “covers” of G , the families of proper subgroups of G whose set-theoretical union is equal to G , and we can define $\sigma(G)$ (the “sum” of G , a concept introduced by Cohn in 1994: see [6]) to be the least cardinality of a cover of G , i.e. the cardinality of a minimal cover of G . If G is cyclic then the sum is not well defined because no proper subgroup contains any generator of G ; in this case we define $\sigma(G) = \infty$, with the convention that $n < \infty$ for every integer n . An easy result is that if $N \trianglelefteq G$ then $\sigma(G) \leq \sigma(G/N)$, because every cover of G/N corresponds to a cover of G .

Definition 1 (σ -elementary groups). *G is said to be σ -elementary if for every non trivial normal subgroup N of G we have $\sigma(G) < \sigma(G/N)$. G is said to be n -elementary if G is σ -elementary and $\sigma(G) = n$.*

It is an easy exercise to show that $\sigma(G) \neq 2$ for every group G . In this article we find all the n -elementary groups for every $3 \leq n \leq 25$. We will produce an explicit tabular of them. In particular we obtain that:

Theorem 1. *A finite σ -elementary non-abelian group G with $\sigma(G) \leq 25$ is either of affine type or almost-simple with socle of prime index.*

In 1926 Scorza proved that $\sigma(G) = 3$ if and only if $C_2 \times C_2$ is an epimorphic image of G (cf. [4]): in our terminology $C_2 \times C_2$ is the unique 3-elementary group. Cohn in 1994 found all the σ -elementary groups with sum up to 6 (cf.

[6]). Tomkinson in 1997 found the sum of the solvable groups: he found that if G is a finite solvable non-cyclic group then $\sigma(G) = q + 1$ where q is the least order of a chief factor of G with more than a complement (cf. [8]). It turned out that it is interesting to ask whether a sum can be not of the form $q + 1$ with q a prime power. Tomkinson found the answer for the first such integer, 7: he showed that $\sigma(G) \neq 7$ for every finite group G , i.e. that 7 is not a sum (cf. [8]). Some time later the next three integer of this kind, 11, 13 and 15, were solved: in 1999, R.A. Bryce, V. Fedri and L. Serena found that $\sigma(PSL(3, 2)) = 15$ (cf. [9]); in 2007 Abdollahi, Ashraf, Shaker showed that $\sigma(\text{Sym}(6)) = 13$ (cf. [1]); in 2008 Lucchini and Detomi showed that 11 is not a sum (cf. [2]). The integers between 16 and 25 which are not of the form $q + 1$ with q a prime power are 16, 19, 21, 22, 23, 25. In this work we find in particular that:

Theorem 2. *The integers between 16 and 25 which are not sums are 19, 21, 22, 25.*

We also obtain the following:

Theorem 3. $\sigma(\text{Aut}(PSL(2, 8))) = 29$.

In sections 2 and 3 we will recall some known results on the structure of the σ -elementary groups. With the help of these results we will be able to prove that if $n \leq 25$ then an n -elementary group has a primitive permutation representation of degree smaller than n . Therefore a crucial part of our proof is the study of $\sigma(G)$ for G a primitive permutation group of low degree: all the needed results in this direction will be collected in section 4.

2 Preliminary results

Let G be a finite group. We recall the definition of the primitive monolithic group X associated to a non-Frattini minimal normal subgroup N of G . We consider two cases:

- N is abelian. Since N is non-Frattini there exists a complement H of N in G . Then we define $X := N \rtimes H/C_H(N)$.
- N is non-abelian. In this case we define $X := G/C_G(N)$.

In any case X is a primitive monolithic group with socle isomorphic with N . Except for the result about the center (which is proved in [6]), the following results can be found in [2]:

Theorem 4. *Let G be a finite, non-abelian and σ -elementary group.*

1. $\Phi(G) = Z(G) = 1$;
2. G has at most one abelian minimal normal subgroup;
3. let $\text{soc}(G) = G_1 \times \dots \times G_n$ be the socle of G , where G_1, \dots, G_n are the minimal normal subgroups of G . Then G is a subdirect product of the primitive monolithic groups X_i associated to the G_i 's. In particular every X_i is an epimorphic image of G .

Proposition 1. *Let G be a non-abelian σ -elementary group, N a minimal normal subgroup of G and X the primitive monolithic group associated to N . Then:*

- *If $X = N$ then $G = X = N$.*
- *If $|X/N|$ is a prime then $G = X$.*

Proposition 2. *Let H be a group, and let V be a H -module. Define $G := V \rtimes H$ and suppose that $C_V(H) = 0$ (this is the case if V is a non-central minimal normal subgroup of G). Then:*

1. *If $H^1(H, V) \neq 0$ then $\sigma(G) = \sigma(H)$;*
2. *if $\sigma(H) \geq 2|V|$ then $H^1(H, V) = 0$.*

3 About solvable σ -elementary groups

In this paragraph we will give and prove a known result about solvable σ -elementary groups, which is a consequence of the next result of Tomkinson (cf. [8]):

Theorem 5 (Tomkinson). *Let G be a solvable non-cyclic group. Then $\sigma(G) = |S/K| + 1$ where $|S/K|$ is the least order of a chief factor of G with more than a complement.*

Let G be a non-abelian solvable σ -elementary group. By theorem 4 we know that G is monolithic (let V be its socle), and by theorem 5 we have $\sigma(G) = |S/K| + 1$ for a chief factor S/K of G with multiple complements, and whose order is minimal among the orders of the chief factors with this property. Suppose $K \neq 1$, i.e. $V \leq K$. If G/V is not cyclic then

$$|S/K| + 1 = \sigma(G) < \sigma(G/V) = 1 + |S_0/K_0|,$$

where S_0/K_0 is a smallest chief factor of G/V with more than a complement. The inequality $|S/K| < |S_0/K_0|$ gives then a contradiction, and $K = 1$. This means that $\sigma(G) = |V| + 1$. Since G/V acts faithfully and irreducibly on V , by the result of Gaschütz (see [10]) the chief factors of G/V have order $< |V|$, and this implies $|V| + 1 = \sigma(G) \leq \sigma(G/V) < |V| + 1$, contradiction. We deduce that G/V is cyclic, and $\sigma(G) = |V| + 1$ by lemma 2.1 in [8].

Summarizing:

Theorem 6. *Let G be a finite solvable non-abelian group. Then G is σ -elementary if and only if it is monolithic and $G/\text{soc}(G)$ is cyclic. In this case $\sigma(G) = |\text{soc}(G)| + 1$.*

4 About the sum of some primitive groups

We study in this section the sum of primitive groups of degree up to 24. This will simplify the study of n -elementary groups. We begin with a known lemma.

Lemma 1. *If H is a maximal subgroup of a group G and $\sigma(H) > \sigma(G)$ then H appears in every minimal cover of G . In particular if H is maximal and non-normal then $\sigma(H) < [G : H]$ implies $\sigma(G) \geq \sigma(H)$.*

Proof. Let $\{H_1, \dots, H_n\}$ be a minimal cover of G , where $n = \sigma(G)$. Then $H = (H \cap H_1) \cup \dots \cup (H \cap H_n)$ is a union of less than $\sigma(H)$ subgroups equal to H , so at least one of them must be unproper (by definition of $\sigma(H)$): $H \cap H_i = H$ for some $i \in \{1, \dots, n\}$, i.e. $H = H_i$ since H is maximal. Every conjugate of H is a maximal subgroup of G isomorphic to H , thus if H is not normal then in every minimal cover of G there are all the $[G : H]$ conjugates of H . \square

4.1 Primitive groups with non-abelian socle

In this paragraph we will prove the bounds on the sums of the primitive groups of degree up to 24 and non-abelian socle which are stated in the following tabular.

deg	σ
5	$\sigma(\text{Alt}(5)) = 10; \sigma(\text{Sym}(5)) = 16.$
6	$\sigma(\text{Alt}(5)) = 10; \sigma(\text{Sym}(6)) = 13; \sigma(\text{Sym}(5)) = 16.$
7	$\sigma(\text{SL}(3, 2)) = 15; \sigma(\text{Alt}(7)) = 31; \sigma(\text{Sym}(7)) = 64.$
8	$\sigma(\text{PSL}(2, 7)) = \sigma(\text{PGL}(2, 7)) = 29;$ $\sigma(\text{Alt}(8)) \geq 64; \sigma(\text{Sym}(8)) \geq 29.$
9	$\sigma(\text{Aut}(\text{PSL}(2, 8))) = 29; \sigma(\text{PSL}(2, 8)) = 36;$ $\sigma(\text{Alt}(9)) \geq 80; \sigma(\text{Sym}(9)) \geq 172.$
10	$\sigma(\text{Alt}(5)) = 10; \sigma(\text{Sym}(6)) = 13; \sigma(\text{Alt}(6)) = 16;$ $\sigma(\text{PGL}(2, 9)) = 46; \sigma(M_{10}), \sigma(\text{Alt}(10)), \sigma(\text{Sym}(10)) \geq 45, \sigma(\text{P}\Gamma\text{L}(2, 9)) = 3.$
11	$\sigma(M_{11}) = 23; \sigma(\text{PSL}(2, 11)) = 67; \sigma(\text{Alt}(11)), \sigma(\text{Sym}(11)) \geq 512.$
12	$\sigma(M_{11}) = 23; \sigma(\text{PSL}(2, 11)) = \sigma(\text{PGL}(2, 11)) = 67;$ $\sigma(M_{12}), \sigma(\text{Alt}(12)), \sigma(\text{Sym}(12)) \geq 67.$
13	$\sigma(\text{PSL}(3, 3)), \sigma(\text{Alt}(13)), \sigma(\text{Sym}(13)) \geq 144.$
14	$\sigma(\text{PSL}(2, 13)), \sigma(\text{PGL}(2, 13)), \sigma(\text{Alt}(14)), \sigma(\text{Sym}(14)) \geq 92.$
15	$\sigma(\text{Sym}(6)) = 13; \sigma(\text{Alt}(6)) = 16; \sigma(\text{Alt}(7)) = 31;$ $\sigma(\text{PSL}(4, 2)), \sigma(\text{Alt}(15)), \sigma(\text{Sym}(15)) \geq 64.$
16	$\sigma(\text{Alt}(16)) \geq 2^{14}; \sigma(\text{Sym}(16)) > 6435.$
17	$\sigma(\text{PSL}(2, 16)), \sigma(\text{PSL}(2, 16) : 2,$ $\sigma(\text{P}\Gamma\text{L}(2, 16)), \sigma(\text{Alt}(17)), \sigma(\text{Sym}(17)) \geq 68.$
18	$\sigma(\text{PSL}(2, 17)), \sigma(\text{PGL}(2, 17)), \sigma(\text{Alt}(18)), \sigma(\text{Sym}(18)) \geq 2^{16}.$
19	$\sigma(\text{Alt}(19)), \sigma(\text{Sym}(19)) \geq 2^{17}$
20	$\sigma(\text{PSL}(2, 19)), \sigma(\text{PGL}(2, 19)), \sigma(\text{Alt}(20)), \sigma(\text{Sym}(20)) \geq 191.$
21	$\sigma(\text{SL}(3, 2)) = 15; \sigma(\text{Alt}(7)) = 31; \sigma(\text{P}\Gamma\text{L}(3, 4)) = 3;$ $\sigma(\text{PGL}(2, 7)), \sigma(\text{Sym}(7)), \sigma(\text{PSL}(3, 4)), \sigma(\text{P}\Sigma\text{L}(3, 4)),$ $\sigma(\text{PGL}(3, 4)), \sigma(\text{Alt}(21)), \sigma(\text{Sym}(21)) \geq 64.$

deg	σ
22	$\sigma(M_{22}), \sigma(M_{22} : 2), \sigma(\text{Alt}(22)), \sigma(\text{Sym}(22)) \geq 67.$
23	$\sigma(M_{23}), \sigma(\text{Alt}(23)), \sigma(\text{Sym}(23)) \geq 64.$
24	$\sigma(M_{24}), \sigma(\text{PSL}(2, 23)), \sigma(\text{PGL}(2, 23)),$ $\sigma(\text{Alt}(24)), \sigma(\text{Sym}(24)) \geq 277.$

The bounds about $\sigma(\text{Sym}(n))$ when n is odd or $n \geq 14$, $\sigma(\text{Alt}(n))$ when $n \neq 7$ and $\sigma(M_{11})$ are proved in the work of Maroti [3]. $\sigma(\text{Sym}(5))$ and $\sigma(\text{Alt}(5))$ are computed by Cohn in [6], $\sigma(\text{Sym}(6))$ is found by Abdollahi, Ashraf and Shaker in [1], the bound on $\sigma(M_{10})$ is proved by Lucchini and Detomi in [2], the sums of the groups of the form $\text{PGL}(2, q)$ and $\text{PSL}(2, q)$ are found by Bryce, Fedri and Serena in [9], $\sigma(\text{Alt}(7))$ is found by Kappe in [7]. The only groups that we are left to study to prove what is stated in the tabular are the following:

- $\text{Sym}(8)$. It admits $\text{PGL}(2, 7)$ as a maximal subgroup of index 120 and sum 29, so $\sigma(\text{Sym}(8)) \geq 29$ by lemma 1.
- Let $G := \text{Aut}(\text{PSL}(2, 8)) = \text{P}\Gamma\text{L}(2, 8)$. We are going to show that $\sigma(G) = 29$ and that there exists only one minimal cover of G .
 G is an almost simple group of order $1512 = 2^3 \cdot 3^3 \cdot 7$. $\text{PSL}(2, 8)$, its non-trivial proper normal subgroup, is a maximal subgroup of sum 36, thus if $\sigma(G) < 36$ then $\text{PSL}(2, 8)$ appears in every minimal cover.
Now, since $\text{soc}(G)$ together with the normalizers of the 3-Sylow subgroups of $\text{PSL}(2, 8)$ form a cover of G consisting of 29 subgroups, we have $\sigma(G) \leq 29$ so $\text{soc}(G)$ appears in every minimal cover of G . The only maximal subgroups of G which contain elements of order 9 are $\text{soc}(G) = \text{PSL}(2, 8)$ and the normalizers of the 3-Sylow subgroups of $\text{PSL}(2, 8)$. It follows that if P is a 3-Sylow subgroup of $\text{soc}(G) = \text{PSL}(2, 8)$ then $N_G(P)$ is the only maximal subgroup of G which contains the elements of order 9 in $N_G(P) - P$. So the 28 normalizers of the 3-Sylow subgroups of $\text{soc}(G)$ appear in every minimal cover. So $\sigma(G) = 29$.
- $\text{Sym}(10)$. Its maximal subgroups (cf. [5]) are:
 - $\text{Sym}(9)$;
 - $\text{Sym}(8) \times C_2$;
 - $\text{Sym}(7) \times \text{Sym}(3)$;
 - $(\text{Sym}(5) \times \text{Sym}(5)) : 2$;
 - $\text{Sym}(6) \times \text{Sym}(4)$;
 - $2^5 : \text{Sym}(5)$;
 - $\text{Alt}(6) : 2^2$.

$\text{Sym}(8)$ and $\text{Sym}(9)$ do not have elements of order 21. Thus $\text{Sym}(8) \times C_2$ has no elements of order 21, and the only maximal subgroups of $\text{Sym}(10)$ which contain elements of order 21 are of the kind $\text{Sym}(7) \times \text{Sym}(3)$. Now $\text{Sym}(10)$ has $6! \cdot \binom{10}{7} \cdot 2! = 172800$ elements of order 21, and $\text{Sym}(7) \times$

$\text{Sym}(3)$ has $6! \cdot 2! = 1440$ such elements, so in order to cover the elements of order 21 we need at least $172800/1440 = 120$ proper subgroups. Therefore $\sigma(\text{Sym}(10)) \geq 120$.

- M_{12} . It admits $PSL(2, 11)$ as a maximal non normal subgroup of sum 67 and index 144, so $\sigma(M_{12}) \geq 67$ by lemma 1.
- $\text{Sym}(12)$. It admits $PGL(2, 11)$ as a maximal non normal subgroup of sum 67 and index $9!$, so $\sigma(\text{Sym}(12)) \geq \sigma(PGL(2, 11)) = 67$ by lemma 1.
- $PSL(3, 3)$. It has 1728 elements of order 13, and its maximal subgroups which contain elements of order 13 are of the kind $C_{13} \rtimes C_3$, and such subgroups are 144. Since the $C_{13} \rtimes C_3$ have 12 elements of order 13, in order to cover the elements of order 13 we need at least $1728/12 = 144$ proper subgroups. In particular $\sigma(G) \geq 144$.
- $G := PSL(2, 16) : 2$ admits $PSL(2, 16)$ as a maximal and normal subgroup. The maximal subgroups of G are:
 - 17 occurs of $((2^4 \cdot 5) \cdot 3) \cdot 2$;
 - 120 occurs of 17.4;
 - 68 occurs of $C_2 \times \text{Alt}(5)$;
 - 1 occur of $PSL(2, 16)$;
 - 136 occurs of $\text{Sym}(3) \times D_{10}$.

The only maximal subgroups which contain elements of order 10 are $C_2 \times \text{Alt}(5)$ (which has 24 elements of order 10) and $\text{Sym}(3) \times D_{10}$ (which contains 12 elements of order 10), and since G contains 1632 elements of order 10, we need at least $1632/24 = 68$ proper subgroups to cover the elements of order 10. In particular $\sigma(G) \geq 68$.

- $G := P\Gamma L(2, 16) = PSL(2, 16) \rtimes C_4$. The maximal subgroups of G are:
 - 17 occurs of $((2^4 \cdot 5) \cdot 3) \cdot 4$;
 - 136 occurs of $(5 \cdot 4) \times \text{Sym}(3)$;
 - 68 occurs of $\text{Alt}(5) \cdot 4$;
 - 120 occurs of 17.8;
 - 1 occur of $PSL(2, 16) \rtimes C_2$.

The only maximal subgroups which contain elements of order 12 are $(5 \cdot 4) \times \text{Sym}(3)$ (which contains 20 elements of order 12) and $\text{Alt}(5) \cdot 4$ (which contains 40 elements of order 12), so to cover the elements of order 12 (which are 2720) we need at least $2720/40 = 68$ proper subgroups, so that $\sigma(G) \geq 68$.

- $PSL(3, 4) = M_{21}$. The only maximal subgroup of $PSL(3, 4)$ which contains elements of order 7 is $PSL(2, 7)$, which contains 48 elements of order 7. Since $PSL(3, 4)$ has 5760 elements of order 7, the sum of $PSL(3, 4)$ is at least $5760/48 = 120$.
- $P\Sigma L(3, 4) = PSL(3, 4) : 2$. The only maximal subgroup of $P\Sigma L(3, 4)$ which contains elements of order 14 is $PSL(2, 7) \times C_2$, which contains 48 elements of order 14. Since $P\Sigma L(3, 4)$ has 5760 elements of order 14, the sum of $P\Sigma L(3, 4)$ is at least 120.
- $PGL(3, 4)$. The only maximal subgroup of $PGL(3, 4) = PSL(3, 4) : 3$ which contains elements of order 21 is $(7 : 3) \times 3$, which contains 12 elements of order 21. Since $PGL(3, 4)$ contains 11520 elements of order 21, the sum of $PGL(3, 4)$ is at least $11520/12 = 960$.
- $P\Gamma L(3, 4)$. It is a 3-sum group.
- M_{22} . It contains 80640 elements of order 11, and the only maximal subgroup of M_{22} which contains elements of order 11 is $PSL(2, 11)$, which contains 120 such elements. Thus the sum of M_{22} is at least $80640/120 = 672$.
- $M_{22} : 2$. It admits $PGL(2, 11)$ as a maximal and non normal subgroup of index 672 and sum 67, so $\sigma(M_{22} : 2) \geq 67$ by lemma 1.
- M_{23} . It admits $\text{Alt}(8)$ as a maximal non normal subgroup of index 506 and sum ≥ 64 , so $\sigma(M_{23}) \geq 64$ by lemma 1.
- M_{24} . It admits $PSL(2, 23)$ as a maximal non normal subgroup of sum 277 and index 40320, so $\sigma(M_{24}) \geq 277$ by lemma 1.

4.2 Primitive groups with abelian socle

The sums or some bounds on the sums of the primitive groups of degree up to 24 and abelian socle are summarized in the following tabular. The solvable primitive groups which are not σ -elementary and the cyclic groups are omitted. If p is a prime, the subgroups of $AGL(1, p)$ which contain the socle \mathbb{F}_p are p -primitive and σ -elementary of sum $p + 1$ (this follows from section 3); they are omitted.

deg	σ
4	$\sigma(\text{Alt}(4)) = 5$.
8	$\sigma(AGL(1, 8)) = 9; \sigma(ASL(3, 2)) = 15$.
9	$\sigma(3^2 : 4) = \sigma(AGL(1, 9)) = 10$.
16	$\sigma(AGL(4, 2)) \geq 31, \sigma(A\Gamma L(2, 4)) \leq 4, \sigma(ASL(2, 4) : 2) \leq 16,$ $\sigma(AGL(2, 4)) \leq 10, \sigma(ASL(2, 4)) \leq 10, \sigma(2^4 : \text{Sym}(6)) \leq 13,$ $\sigma(2^4 : \text{Alt}(6)) \leq 16, \sigma(2^4 : \text{Sym}(5)) \leq 16, \sigma(2^4 : \text{Alt}(5)) \leq 10,$ $\sigma(2^4 : \text{Alt}(7)) = 31, \sigma(2^4 : 5) = \sigma(AGL(1, 16)) = 17$.

To show this recall theorem 6: if a non-cyclic solvable and primitive monolithic group G is such that $G/\text{soc}(G)$ is cyclic then G is σ -elementary and

$\sigma(G) = |\text{soc}(G)| + 1$. This solves all the cases except for $ASL(3, 2)$ and the 16-primitive groups. We have $\sigma(ASL(3, 2)) = 15$ by proposition 2 because $H^1(SL(3, 2), \mathbb{F}_2^3) = C_2$ (in fact in [11] it is shown that $H^1(GL(d, 2), \mathbb{F}_2^d) = 0$ if $d \neq 3$ and $H^1(SL(3, 2), \mathbb{F}_2^3) = C_2$) and $\sigma(SL(3, 2)) = 15$ (by [9]). The 16-primitive groups with abelian socle are of the kind $V \rtimes H$ with $H \leq GL(V)$ and $V = C_2^4$. In almost all the cases the bounds given in the tabular are of the kind “ $\sigma(V \rtimes H) \leq \sigma(H)$ ” using for $\sigma(H)$ the information collected in the previous section. Only two cases require more attention:

- $G := AGL(4, 2) = \mathbb{F}_2^4 \rtimes GL(4, 2)$. We want to show that $\sigma(G) \geq 31$. We observe that $GL(4, 2) \cong \text{Alt}(8)$ and G is monolithic. Let $V := \mathbb{F}_2^4$ and $H := GL(4, 2)$, so that $G = V \rtimes H$. Since $\sigma(\text{Alt}(8)) \geq 69$, if (as we can suppose) $\sigma(G) < 69$ then every complement of V must appear in every minimal cover $\{M_1, \dots, M_n\}$ of G , where $n = \sigma(G)$. We have $H^1(H, V) = 0$ by the result in [11], so that V has exactly 16 complements in G , let them be M_1, \dots, M_{16} . If $g \in GL(4, 2)$ stabilizes a non-zero vector $v \in V$ then the function $\mathbb{F}_2^4 \rightarrow \mathbb{F}_2^4$ which sends x to $x^g - x$ is not injective (having v in its kernel), so it is not surjective: there exists $w \in V$ such that $x^g - x \neq w$ for every $x \in V$. In this case $wg \in G$ does not belong to any complement of V , because the complements of V are conjugate to H , and $wg \in H^x$ means $g \in H$ and $w = x - x^g$. Thus wg must lie in at least one M_i with $i \geq 17$, so g must belong to it, because the maximal subgroups which do not complement V must contain it (V is the only minimal normal subgroup of G). It turns out that $M_{17}/V, \dots, M_n/V$ must cover all the point stabilizers of $GL(4, 2)$. If v is a non-zero vector then the point stabilizer of v in $GL(4, 2)$ is isomorphic to $ASL(3, 2)$, which is a 15-sum group. So either every point stabilizer is one of the M_i/V with $i \geq 17$ or the M_i/V with $i \geq 17$ are at least 15. In any case the M_i/V with $i \geq 17$ are at least 15, so $\sigma(G) \geq 15 + 16 = 31$.
- $G := \mathbb{F}_2^4 \rtimes \text{Alt}(7)$. Let $V := \mathbb{F}_2^4$ and $H := \text{Alt}(7)$. We want to show that $\sigma(G) \geq 31$. Suppose by contradiction that $\sigma(G) \leq 30$, so that all the complements of V appear in every minimal cover of G (because $\sigma(\text{Alt}(7)) = 31$). Since $\sigma(G) \leq \sigma(H) = 31$ we have $H^1(H, V) = 0$ (otherwise we would have at least 32 complements of V). Let M_1, \dots, M_{16} be the 16 complements of V in G . Since H acts faithfully on 16 elements, we have an injection $\text{Alt}(7) \rightarrow \text{Sym}(16)$, and a 7-cycle h of $\text{Alt}(7)$ must fix a non-zero vector in this action (the image of a 7-cycle in $\text{Sym}(16)$ is either a 7-cycle or a product of two disjoint 7-cycles), let v be this vector. Then $vh \in G$ does not belong to any complement of H because it has order 14, and H has no elements of order 14. Thus vh lies in a M_i with $i \geq 17$, and every such M_i contains V (because it does not complement it), so $M_{17}/V, \dots, M_n/V$ contain together all the 7-cycles. But to cover the 7-cycles in $\text{Alt}(7)$ we need at least 15 subgroups, because the 7-cycles in $\text{Alt}(7)$ are 6! and the only maximal subgroups of $\text{Alt}(7)$ which contain 7-cycles are the $SL(3, 2)$, and they contain 48 7-cycles. We

deduce that $\sigma(G) \geq 16 + 15 = 31$, contradiction. In particular since $\sigma(G) \leq \sigma(\text{Alt}(7)) = 31$, G is a non- σ -elementary 31-sum group.

5 The n -elementary groups with $n \leq 25$

If X is a primitive monolithic group and $N := \text{soc}(X)$, we denote by $l_X(N)$ the minimal index of a proper supplement of N in X . We recall that X is $l_X(N)$ -primitive. In particular if N is abelian then the supplements of N are in fact complements, so in this case $l_X(N) = |N|$ and X is $|N|$ -primitive.

The following lemma summarizes some known results which can be found in [2].

Lemma 2. *If G is an abelian non-cyclic σ -elementary group then $G \cong C_p \times C_p$ and $\sigma(G) = p + 1$, for some prime p . If G is non-abelian, G_1, \dots, G_n are its minimal normal subgroups and X_1, \dots, X_n are the associated primitive monolithic groups then:*

1. $\sigma(G) \geq \sum_{i=1}^n l_{X_i}(G_i)$.
2. If G_1 is abelian then $\frac{1}{2}\sigma(G) < |G_1| \leq \sigma(G) - 1$.

With the following lemma we reduce our study to monolithic groups.

Lemma 3. *The non-abelian σ -elementary groups of sum ≤ 33 are primitive and monolithic.*

Proof. Let G be a non-abelian σ -elementary group. Let G_1, \dots, G_n be the minimal normal subgroups of G , and let X_1, \dots, X_n be the primitive monolithic groups associated to G_1, \dots, G_n respectively. We have then $\sigma(G) \geq \sum_{i=1}^n l_{X_i}(G_i)$, and $l_{X_i}(G_i) = |G_i|$ if G_i is abelian. We may suppose by theorem 4 (2) that G_2, \dots, G_n are not abelian.

Consider a non-abelian minimal normal subgroup G_i . In the following discussion we will apply proposition 1.

- If $t := l_{X_i}(G_i) \leq 16$ and $t \neq 6, 10$ then since X_i is t -primitive we immediately see that X_i/G_i has order ≤ 3 , so either $X_i = G_i$ or X_i/G_i is a prime, and this implies $G = X_i$ and $n = 1$.
- If $l_{X_i}(G_i) = 6$ then X_i is 6-primitive so $G_i \in \{\text{Alt}(5), \text{Alt}(6)\}$. If $G_i = \text{Alt}(5)$ then $G \in \{\text{Alt}(5), \text{Sym}(5)\}$ and $n = 1$, otherwise if $G_i = \text{Alt}(6)$ then X_i is a subgroup of $\text{Aut}(\text{Alt}(6))$ containing $\text{Alt}(6)$, so either X_i/G_i has order ≤ 2 , and if this is the case $G = X_i$ and $n = 1$, or $X_i = \text{Aut}(\text{Alt}(6))$ and X_i is a 3-sum group since it admits $C_2 \times C_2$ as an epimorphic image. Since X_i is a quotient of G , this implies that $\sigma(G) = 3$. This contradicts $\sigma(G) \geq l_{X_i}(G_i)$.
- If $l_{X_i}(G_i) = 10$ then

$$X_i \in \{\text{Alt}(5), \text{Sym}(5), \text{PSL}(2, 9), \text{PGL}(2, 9),$$

$\text{Sym}(6), M_{10}, PGL(2, 9), \text{Alt}(10), \text{Sym}(10)\}$.

Now since $l_{X_i}(G_i) \leq \sigma(G)$, and since X_i is a quotient of G we cannot have $X_i = PGL(2, 9)$ because $\sigma(PGL(2, 9)) = 3$; it follows that $X_i/G_i \in \{1, C_2\}$ so $G = X_i$ and $n = 1$.

We deduce that if G_i is non-abelian then either $n = 1 = i$ or $l_{X_i}(G_i) \geq 17$. In particular if G_1 is non-abelian and $\sigma(G) \leq 33$ then $n = 1$ since $33 \geq \sigma(G) \geq \sum_{i=1}^n l_{X_i}(G_i) \geq 17n$. Let us suppose that G_1 is abelian. If $n \geq 2$ then $l_{X_i}(G_i) \geq 17$ for $i \geq 2$, so $|G_1| \leq \sigma(G) - 17$ and $\sigma(G) - 17 \geq |G_1| > \frac{1}{2}\sigma(G)$, thus $\sigma(G) > 34$. We deduce that if G is σ -elementary, non-abelian and non monolithic then $\sigma(G) \geq 34$. \square

Now the study of n -primitive groups with $3 \leq n \leq 25$ is easy. Since abelian non-cyclic σ -elementary groups are $(p+1)$ -elementary and isomorphic to $C_p \times C_p$ for some prime p , we will look for non-abelian σ -elementary groups. Since every non-abelian n -elementary group with sum $n \leq 25$ is primitive of degree $\leq n - 1$, it suffices to look at the tabulars of section 4 to obtain the following tabular, in which there are for every sum n all the n -elementary groups.

sum	groups
3	$C_2 \times C_2$
4	$C_3 \times C_3, \text{Sym}(3)$
5	$\text{Alt}(4)$
6	$C_5 \times C_5, D_{10}, AGL(1, 5)$
7	\emptyset
8	$C_7 \times C_7, D_{14}, 7 : 3, AGL(1, 7)$
9	$AGL(1, 8)$
10	$3^2 : 4, AGL(1, 9), \text{Alt}(5)$
11	\emptyset
12	$C_{11} \times C_{11}, 11 : 5, D_{22}, AGL(1, 11)$
13	$\text{Sym}(6)$
14	$C_{13} \times C_{13}, D_{26}, 13 : 3, 13 : 4, 13 : 6, AGL(1, 13)$
15	$SL(3, 2)$
16	$\text{Sym}(5), \text{Alt}(6)$
17	$2^4 : 5, AGL(1, 16)$
18	$C_{17} \times C_{17}, D_{34}, 17 : 4, 17 : 8, AGL(1, 17)$
19	\emptyset
20	$C_{19} \times C_{19}, AGL(1, 19), D_{38}, 19 : 3, 19 : 6, 19 : 9$
21	\emptyset
22	\emptyset
23	M_{11}
24	$C_{23} \times C_{23}, D_{46}, 23 : 11, AGL(1, 23)$
25	\emptyset

References

- [1] Abdollahi, A.; Ashraf, F.; Shaker, S. M. “The symmetric group of degree six can be covered by 13 and no fewer proper subgroups”. *Bull. Malays. Math. Sci. Soc.* (2) 30 (2007), no. 1, 57-58.
- [2] A. Lucchini, E. Detomi, “On the Structure of Primitive n -Sum Groups”, *CUBO A Mathematical Journal* Vol.10 n. 03 (195-210), 2008.
- [3] A. Maroti, “Covering the symmetric groups with proper subgroups”, *J. Combin. Theory Ser. A*, 110(1) (2005), 97-111.
- [4] G. Scorza, “I gruppi che possono pensarsi come somme di tre loro sottogruppi”, *Boll. Un Mat. Ital.* (1926) 216-218.
- [5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, “Atlas of finite groups”, Clarendon Press Oxford.
- [6] J.H.E. Cohn, “On n -sum groups”, *Math. Scand.*, 75(1) (1994), 44-58.
- [7] L. C. Kappe, J. L. Redden, “On the Covering Number of Small Alternating Groups” (preprint).
- [8] M.J. Tomkinson, “Groups as the union of proper subgroups”, *Math. Scand.*, 81(2) (1997), 191-198.
- [9] R.A. Bryce, V. Fedri, L. Serena, “Subgroup coverings of some linear groups”, *Bull. Austral. Math. Soc.*, 60(1999), 227-238.
- [10] W. Gaschütz, “Existenz und Konjugiertsein von Untergruppen, die in endlichen auflösbaren Gruppen durch gewisse Indexschränken definiert sind”, *Journal of Algebra* 53 (1978), 1-20.
- [11] W. Jones, B. Parshall, “On the 1-cohomology of finite groups of Lie type”, proceeding of the conference on Finite Groups (Univ. Utah, Park City, Utah 1975) 313-328 Academic Press, New York 1976.

Martino Garonzi, Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, 35121 Padova, Italy. E-mail address: mgaronzi@studenti.math.unipd.it