COVERING CERTAIN MONOLITHIC GROUPS WITH PROPER SUBGROUPS

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Abstract. Given a finite non-cyclic group G, call $\sigma(G)$ the least number of proper subgroups of G needed to cover G. In this paper we give lower and upper bounds for $\sigma(G)$ for G a group with a unique minimal normal subgroup N isomorphic to A_n^m where $n \geq 5$ and G/N is cyclic. We also show that $\sigma(A_5 \wr C_2) = 57$.

1. Introduction

Given a finite non-cyclic group G, call $\sigma(G)$ the least number of proper subgroups of G needed to cover G set-theoretically. This notion has been introduced the first time by Cohn in 1994 in [5]. We usually call "cover" of G a family of proper subgroups of G which covers G, and "minimal cover" of G a cover of G consisting of exactly $\sigma(G)$ elements. If G is cyclic then $\sigma(G)$ is not well defined because no proper subgroup contains any generator of G; in this case we define $\sigma(G) = \infty$, with the convention that $n < \infty$ for every integer n. In [15] Tomkinson showed that if G is a finite solvable group then $\sigma(G) = q + 1$, where q is the least order of a chief factor of G with more than one complement. The behavior of the function σ has been intensively studied for the almost simple groups. The alternating and symmetric groups have been considered by Maróti in [12]. In [2] Britnell, Evseev, Guralnick, Holmes and Maróti studied the linear groups $GL(n, q)$, $PGL(n, q)$, $SL(n, q)$, $PSL(n, q)$. In [10] Lucido studied the Suzuki groups. In [9] Lucchini and Maróti found an asymptotic formula for the function which assigns to the positive integer x the number of positive integers n at most x with the property that $\sigma(S) = n$ for some non-abelian simple group S.

If N is a normal subgroup of a finite group G then $\sigma(G) \leq \sigma(G/N)$, since every cover of G/N can be lifted to a cover of G. We say that G is "σ-primitive" if $\sigma(G) < \sigma(G/N)$ for every non-trivial normal subgroup N of G. Since every finite group has a σ-primitive epimorphic image with the same σ, the structure of the σ-primitive groups is of big interest. It was studied by Lucchini and Detomi in [8]. They proved for instance that every σ-primitive group is a subdirect product of monolithic groups (i.e. groups with only one minimal normal subgroup). This and other partial results lead us to believe that the monolithic groups have a crucial role in this story. In the same paper Lucchini and Detomi conjectured that every non-abelian σ-primitive group is monolithic. This motivates us in the study of the function σ for the monolithic σ-primitive groups.

Let us consider a monolithic σ-primitive group G. If $\mathrm{soc}(G)$ is abelian then it is easy to prove that $\mathrm{soc}(G)$ is complemented in G and $\sigma(G) = c + 1$, where c is the number of complements of $\mathrm{soc}(G)$ in G. Let now n, m be positive integers

Date: 27th of August, 2011.

with $n \geq 5$. Suppose that $\mathrm{soc}(G) = A_n^m$ and that $G/\mathrm{soc}(G)$ is cyclic. Write $\mathrm{soc}(G) = T_1 \times \cdots \times T_m = T^m$, with $T = A_n$, and define $X := N_G(T_1)/C_G(T_1)$. Then either $X \cong A_n$ ("even case") or $X \cong S_n$ ("odd case"). In the even case $G \cong A_n \wr C_m$ (cfr. [1], Definition 1.1.8 and Remark 1.1.40.13). These groups have been studied in [13] obtaining lower and upper bounds for $\sigma(G)$ and its exact value in the case $n \equiv 2 \mod (4)$.

Consider now the odd case. Let $\gamma \in G$ be such that $\gamma\, \mathrm{soc}(G)$ generates $G/\mathrm{soc}(G)$, so that $G = \langle T^m, \gamma \rangle$. Since

$$T^m < G \leq \mathrm{Aut}(T^m) \cong T \wr \mathrm{Sym}(m),$$

every element of G has the form $(x_1, \ldots, x_m)\gamma^k$ with $x_1, \ldots, x_m \in T$ and k an integer. Moreover γ itself is of the form $(y_1, \ldots, y_m)\delta$ with $y_1, \ldots, y_m \in \mathrm{Aut}(A_n)$, and $\delta \in \mathrm{Sym}(m)$ is an m-cycle since G acts transitively on the m factors of T^m. γ can be chosen in such a way that each y_i is either 1 or equal to $\tau := (12) \in S_n - A_n$. Since we are in the odd case the number of indices $i \in \{1, \ldots, m\}$ such that $y_i = \tau$ is odd. It is easy to show that γ is conjugate to $(1, \ldots, 1, \tau)\delta$ in G. Therefore we may choose γ to be $(1, \ldots, 1, \tau)\delta$ and clearly it is not restrictive to choose $\delta := (1 \cdots m)$. It turns out that G is the semidirect product

$$A_n^m \rtimes \langle \gamma \rangle.$$

Let us fix some notation. Let $C := C_G(T_1)$. Let U be a maximal subgroup of G supplementing the socle N of G. U is called "of product type" if $U = N_G(M \times M^{a_2} \times \cdots \times M^{a_m})$ with M a maximal $N_U(T_1)$-invariant subgroup of T_1 (cfr. [1], Remark 1.1.40.20) and $a_2, \ldots, a_m \in \mathrm{Aut}(A_n)$. In this case $M = N_U(T_1) \cap T_1$ and $N_U(T_1)C/C$ is a maximal subgroup of $N_G(T_1)/C \cong S_n$ (cfr. [1], Remark 1.1.40.21) whose intersection with T_1C/C is $MC/C \cong M$, so that M is of the form $K \cap A_n$ with K maximal in S_n. U is said to be of "diagonal type" if $U = N_G(\Delta)$ where $\Delta = \Delta_1 \times \cdots \times \Delta_{m/q}$, where q is a prime divisor of m and $\Delta_i = \{(x, x^{\alpha_{i_1}}, \ldots, x^{\alpha_{i_q}}) \mid x \in A_n\}$, where $\alpha_{i_k} \in \mathrm{Aut}(A_n)$ for $k = 1, \ldots, q$. In this case we also say that U is of "diagonal type q". It turns out that every maximal subgroup of G supplementing the socle is either of product type or of diagonal type.

In this paper we establish the following result, generalizing the results in [12] about $\sigma(S_n)$ (which corresponds to the case $m = 1$). The arguments we use involve the same covers of S_n considered in [12], and this is why the results have similar flavour: in particular, we obtain an exact formula for $\sigma(G)$ when n is odd with some exceptions, and an asymptotic formula when n is even.

Theorem 1. *Let m, n be positive integers, and let $G := A_n \rtimes C_{2m}$ as above. Let $\omega(x)$ denote the number of prime factors of the positive integer x. The following holds.*

(1) *Suppose that $n \geq 7$ is odd and $m \neq 1$ if $n = 9$. Then*

$$\sigma(G) = \omega(2m) + \sum_{i=1}^{(n-1)/2} \binom{n}{i}^m.$$

(2) *If $n = 5$ then*

$$10^m \leq \sigma(G) \leq \omega(2m) + 5^m + 10^m.$$

If $n = 5$ and every prime divisor of m is either 2 or 3 then

$$\sigma(G) = \omega(2m) + 5^m + 10^m.$$

(3) *Suppose that $n \geq 8$ is even. Then*

$$\left(\frac{1}{2}\binom{n}{n/2}\right)^m \leq \sigma(G) \leq \omega(2m) + \left(\frac{1}{2}\binom{n}{n/2}\right)^m + \sum_{i=1}^{[n/3]} \binom{n}{i}^m.$$

In particular $\sigma(G) \sim \left(\frac{1}{2}\binom{n}{n/2}\right)^m$ as $n \to \infty$.

(4) *If $n = 6$ then*

$$\sigma(G) = \omega(2m) + 2 \cdot 6^m.$$

Here the upper bound for $\sigma(G)$ is always given by the cardinality of a cover consisting of the $\omega(2m)$ maximal subgroups of G containing its socle and suitable maximal subgroups of product type, $N_G(M \times M^{a_2} \times \cdots \times M^{a_m})$, where the $N_{S_n}(M)$'s cover $S_n - A_n$.

We also compute $\sigma(A_5 \wr C_2)$ (corresponding to the even case when $(n, m) = (5, 2)$), which is not computed in [13]. Similarly as above and as in the results in [13], a minimal cover of $A_5 \wr C_2$ consists of the maximal subgroups containing the socle and a family of subgroups of product type corresponding to a cover of A_5 (consisting of the normalizers of the Sylow 5-subgroups and four point stabilizers).

Theorem 2. *$\sigma(A_5 \wr C_2) = 1 + 4 \cdot 5 + 6 \cdot 6 = 57$.*

Compare this result with the corresponding odd case: $\sigma(A_5^2 \rtimes C_4) = 1 + 5 \cdot 5 + 10 \cdot 10 = 126$. Note that $A_5 \wr C_2$ is the easiest example of a non-almost-simple monolithic group with non-abelian socle.

2. Preliminary lemmas

In the present section we collect some technical lemmas which will be useful in the next section.

Let n be a positive integer and let $c_1, \dots, c_k \in \{1, \dots, n\}$ be such that $c_1 + ... + c_k = n$. A "$(c_1, \dots, c_k)$-cycle" will be an element of S_n which can be written as the product of k pairwise disjoint cycles of length $c_1, \dots, c_k$. An "intransitive subgroup of S_n (resp. A_n) of type $(c_1, \dots, c_k)$" will be the biggest subgroup of S_n (resp. A_n) acting on $\{1, \dots, n\}$ with k given orbits of size $c_1, \dots, c_k$. It is clearly isomorphic to $S_{c_1} \times \cdots \times S_{c_k}$ (resp. $(S_{c_1} \times \cdots \times S_{c_k}) \cap A_n$).

Proposition 1 (Stirling's formula)**.** *For all positive integers n we have*

$$\sqrt{2\pi n}(n/e)^n e^{1/(12n+1)} < n! < \sqrt{2\pi n}(n/e)^n e^{1/(12n)}.$$

The following lemma is shown in the proof of lemma 2.1 in [11].

Lemma 1. *For a positive integer n at least 8 we have*

$$((n/a)!)^a a! \geq ((n/b)!)^b b!$$

whenever a and b are divisors of n with $a \leq b$.

Lemma 2. *Let $n \neq 9, 15$ be an odd positive integer, and let $a \geq 3$ be a proper divisor of n. Then*

$$\left(\frac{n-1}{2}\right)! \left(\frac{n-3}{2}\right)! \geq (n/a)!^a \cdot a!.$$

Proof. Proceed by inspection for $21 \leq n \leq 299$, using lemma 1. Assume $n \geq 300$. Let us use Stirling's formula. We are reduced to prove that

$$\sqrt{\pi(n-1)}((n-1)/2e)^{(n-1)/2}\sqrt{\pi(n-3)}((n-3)/2e)^{(n-3)/2} \geq$$
$$\geq 2\sqrt{2\pi n/a}^{a}(n/ae)^n\sqrt{2\pi a}(a/e)^a.$$

Using the inequalities $\pi \geq \sqrt{2\pi}$ and $n-3 \geq a$ we are reduced to prove that

$$(n-1)^{1/2}(n-1)^{(n-1)/2}(n-3)^{(n-3)/2} \geq 2/(2e)^2\sqrt{2\pi n/a}^{a}(2n/a)^n(a/e)^a,$$

and using $n-1 \geq n-3$ we obtain:

$$(n-1)^{1/2}(n-3)^{n-2} \geq (2/(4e^2))(2\pi n/a)^{a/2}(2n/a)^n(a/e)^a.$$

Using the inequality $3 \leq a \leq \sqrt{n}$ we obtain:

$$(n-1)^{1/2}(n-3)^{n-2} \geq (2/4e^2)(2\pi n/3)^{\sqrt{n}/2}(2n/3)^n(\sqrt{n}/e)^{\sqrt{n}}.$$

Take logarithms and divide by n, obtaining

$$(1/2n)\log(n-1) + ((n-2)/n)\log(n-3) \geq (1/n)\log(2/4e^2) + (1/2\sqrt{n})\log(2\pi/3)+$$
$$+(1/2\sqrt{n})\log(n) + \log(2n/3) + (1/\sqrt{n})\log(\sqrt{n}/e).$$

Since $\sqrt{n-1} \geq 2/4e^2$ and $(1/2\sqrt{n})\log(2\pi/3) \leq 1/\sqrt{n}$ we are reduced to show that

$$\log(n-3) \geq (2/n)\log(n-3) + (1/\sqrt{n})\log(n) + \log(2n/3).$$

Since $n \geq 300$ we have that $(2/n)\log(n-3)+(1/\sqrt{n})\log(n) < 0.37$, hence it suffices to show that $\log(n-3) \geq 0.37 + \log(2n/3)$, i.e. $n-3 \geq (2/3)e^{0.37} \cdot n$. This is true since $(2/3)e^{0.37} < 0.97$. $\square$

Corollary 1. *Let $n \geq 11$ be an odd integer. Then the order of an intransitive maximal subgroup of S_n (resp. A_n) is bigger than the order of any transitive maximal subgroup of S_n (resp. A_n) different from A_n.*

Proof. The imprimitive case follows from the lemma noticing that $((n+1)/2)!((n-1)/2)! \geq ((n-1)/2)!((n-3)/2)!$, and if $n = 15$ then $((n+1)/2)!((n-1)/2)! \geq (n/a)!^a a!$ for $a \in \{3,5\}$. By [11] the order of a primitive maximal subgroup of A_n or S_n is at most 2.6^n and $((n+1)/2)!((n-1)/2)! \geq 2.6^n$. $\square$

Lemma 3. *Let n, a, b be positive integers, with $a > b$.*

(1) *Suppose n is odd. Let K be an intransitive maximal subgroup of A_n. If $(n^2-1)^a \geq 4^a e^{2(a-b)} n^{2b}$, then $|K|^{a/b} \geq |A_n|$.*
(2) *Suppose n is even. Let K be a maximal imprimitive subgroup of A_n of the form $(S_{n/2} \wr S_2) \cap A_n$. If $n^a \geq 2^a e^{a-b} n^b$, then $|K|^{a/b} \geq |A_n|$.*

Proof. We prove only (1), since the proof of (2) is similar. Suppose n is odd. Since the smallest intransitive maximal subgroups of A_n are the ones of type $((n-1)/2, (n+1)/2)$, what we have to prove is the following inequality:

$$(1/2)^{a/b}((n-1)/2)!^{a/b}((n+1)/2)!^{a/b} \geq n!/2.$$

Since $e^{\frac{a/b}{6(n-1)+1}+\frac{a/b}{6(n+1)+1}} \geq e^{1/12n}$ for every positive integer n, using Stirling's formula we see that it is sufficient to show that

$$(1/2)^{a/b}((n-1)/2e)^{a(n-1)/2b}\sqrt{(\pi(n-1))^{a/b}}((n+1)/2e)^{a(n+1)/2b}\sqrt{(\pi(n+1))^{a/b}} \geq$$
$$\geq (1/2)(n/e)^n\sqrt{2\pi n}.$$

Re-write this as follows:

$$((n^2-1)/4e^2)^{a(n-1)/2b}(\pi/2)^{a/b}(n^2-1)^{a/2b}((n+1)/2e)^{a/b} \geq$$
$$\geq (1/2)(n/e)^n\sqrt{2\pi n}.$$

In other words:

$$((n^2-1)/4e^2)^{an/2b}(\pi(n+1)/2)^{a/b} \geq (1/2)\sqrt{2\pi n}(n/e)^n.$$

Since $\pi(n+1)/2 \geq (1/2)\sqrt{2\pi n}$ we are reduced to prove that

$$((n^2-1)/4e^2)^{an/2b} \geq (n/e)^n,$$

i.e.

$$(n^2-1)^a \geq (n/e)^{2b}(4e^2)^a = 4^a e^{2(a-b)} n^{2b}.$$

□

Lemma 4. *Let n be an odd positive integer at least 5, let a be a $(2, n-2)$-cycle in S_n, and let b be a $(n-1)$-cycle in S_n. No primitive maximal subgroup of S_n contains a, no imprimitive maximal subgroup of S_n contains b, and no intransitive maximal subgroup of S_n contains both a and b.*

Proof. The second and the third statement are clear. If a primitive subgroup of S_n contains a then it contains the transposition a^{n-2}, thus it contains A_n by the Jordan theory (cfr. for example [4], Theorem 6.15 and Exercise 6.6). □

In the rest of this section we will use the notations which we fixed in the introduction.

Lemma 5. *Let $1 \leq k < 2m$ be an integer coprime to $2m$. In the following let the subscripts be identified with their reductions modulo m, and let $b_1 := 1$, b_2, …, $b_m \in S_n$, x_1, …, $x_m \in A_n$. Let M be a subgroup of A_n. The following holds.*

(1) *Suppose $k < m$. For $d \in \{1, \ldots, m\}$ define τ_d to be τ if $d > m-k$, and 1 if $d \leq m-k$. Then the element $(x_1, \ldots, x_m)\gamma^k \in G$ belongs to $N_G(M \times M^{b_2} \times \cdots \times M^{b_m})$ if and only if*

$$\eta_d := b_d x_d \tau_d b_{d+k}^{-1} \in N_{S_n}(M), \qquad \forall d = 1, \ldots, m.$$

Moreover in this case

$$\eta := \eta_1\eta_{1+k}\eta_{1+2k}\cdots\eta_{1+(m-1)k} =$$
$$= x_1\tau_1 x_{1+k}\tau_{1+k}\cdots x_{1+(m-1)k}\tau_{1+(m-1)k} \in N_{S_n}(M) - A_n.$$

(2) *Suppose $k > m$. For $d \in \{1, \ldots, m\}$ define τ_d to be τ if $d \leq 2m-k$, and 1 if $d > 2m-k$. The element $(x_1, \ldots, x_m)\gamma^k \in G$ belongs to $N_G(M \times M^{b_2} \times \cdots \times M^{b_m})$ if and only if*

$$\eta_d := b_d x_d \tau_d b_{d+k-m}^{-1} \in N_{S_n}(M), \qquad \forall d = 1, \ldots, m.$$

Moreover in this case

$$\eta := \eta_1\eta_{1+k-m}\eta_{1+2(k-m)}\cdots\eta_{1+(m-1)(k-m)} =$$
$$= x_1\tau_1 x_{1+k-m}\tau_{1+k-m}\cdots x_{1+(m-1)(k-m)}\tau_{1+(m-1)(k-m)} \in N_{S_n}(M) - A_n.$$

(3) *If $N_{S_n}(M)$ contains η (which depends only on $x_1, \ldots, x_m$), then there exist $a_2, \ldots, a_m \in A_n$ such that*

$$(x_1, \ldots, x_m)\gamma^k \in N_G(M \times M^{a_2} \times \cdots \times M^{a_m}).$$

Proof. Assume first that $k < m$. The element

$$(x_1, \dots, x_m)\gamma^k = (x_1, \dots, x_{m-k}, x_{m-k+1}\tau, \dots, x_m\tau)\delta^k$$

belongs to $N_G(M \times M^{b_2} \times \cdots \times M^{b_m})$ if and only if

$$(M^{x_1} \times M^{b_2x_2} \times \cdots \times M^{b_{m-k}x_{m-k}} \times M^{b_{m-k+1}x_{m-k+1}\tau} \times \cdots \times M^{b_mx_m\tau})^{\delta^k} =$$

$$= M \times M^{b_2} \times \cdots \times M^{b_m},$$

if and only if

$$M^{b_{m-k+1}x_{m-k+1}\tau} \times \cdots \times M^{b_mx_m\tau} \times M^{x_1} \times M^{b_2x_2} \times \cdots \times M^{b_{m-k}x_{m-k}} =$$

$$= M \times M^{b_2} \times \cdots \times M^{b_m}.$$

In other words:

$$b_{m-k+1}x_{m-k+1}\tau, b_{m-k+2}x_{m-k+2}\tau b_2^{-1}, \dots, b_mx_m\tau b_k^{-1},$$

$$x_1b_{k+1}^{-1}, b_2x_2b_{k+2}^{-1}, \dots, b_{m-k}x_{m-k}b_m^{-1} \in N_{S_n}(M).$$

For $d \in \{1, \dots, m\}$ define τ_d to be τ if $d > m-k$, and 1 if $d \leq m-k$. The conditions we have are the following:

$$\eta_d := b_dx_d\tau_db_{d+k}^{-1} \in N_{S_n}(M), \qquad d = 1, \dots, m.$$

Observe that since k and m are coprime,

$$\{\tau_1, \tau_{1+k}, \tau_{1+2k}, \dots, \tau_{1+(m-1)k}\} = \{\tau_1, \dots, \tau_m\}.$$

Now

$$\eta := \eta_1\eta_{1+k}\eta_{1+2k} \cdots \eta_{1+(m-1)k} =$$

$$= x_1\tau_1x_{1+k}\tau_{1+k} \cdots x_{1+(m-1)k}\tau_{1+(m-1)k} \in N_{S_n}(M)$$

is an odd element of S_n since $\eta \equiv \tau^k \mod (A_n)$ and k is odd (being coprime to $2m$).

Point (2) follows easily from point (1) by noticing that $((x_1, \dots, x_m)\gamma^k)^{-1} = (x_1^{-1}, \dots, x_m^{-1})^{\gamma^k}\gamma^{2m-k}$.

Let us prove point (3). Suppose that the normalizer of M in S_n contains η. Assume that $k < m$ (the case $k > m$ is similar). For fixed elements $b_2, \dots, b_m \in S_n$ define $\eta_d := b_dx_d\tau_db_{d+k}^{-1}$, for $d = 1, \dots, m$, and now choose $b_2, \dots, b_m$ in such a way that η_{1+k}, η_{1+2k},..., $\eta_{1+(m-1)k} \in N_{S_n}(M)$. Let η_1 be the element of S_n such that $\eta_1\eta_{1+k} \cdots \eta_{1+(m-1)k} = \eta$. Then since $\eta \in N_{S_n}(M)$, also $\eta_1 \in N_{S_n}(M)$. Now, a suitable power of $(x_1, \dots, x_m)\gamma^k$ is of the form $(y_1, \dots, y_m)\gamma$, with $y_1, \dots, y_m \in A_n$. Since the element $(y_1, \dots, y_m)\gamma \in G$ belongs to $N_G(M \times M^{b_2} \times \cdots \times M^{b_m})$ we have

$$b_my_m\tau, y_1b_2^{-1}, b_2y_2b_3^{-1}, \dots, b_{m-1}y_{m-1}b_m^{-1} \in N_{S_n}(M).$$

We may choose $a_2 := y_1$, $a_3 := y_1y_2$,..., $a_m := y_1y_2 \cdots y_{m-1}$. In this way we get $M^{b_i} = M^{a_i}$ and $a_i \in A_n$, for $i = 2, \dots, m$. □

From the proof of this proposition it easily follows that:

Corollary 2. *If $M \leq A_n$, $b_2, \dots, b_m \in \mathrm{Aut}(A_n)$ and $N_G(M \times M^{b_2} \times \cdots \times M^{b_m})$ contains an element of the form $(x_1, \dots, x_m)\gamma$ with $x_1, \dots, x_m \in A_n$ then there exist $a_2, \dots, a_m \in A_n$ such that $M^{b_i} = M^{a_i}$ for $i = 2, \dots, m$.*

Lemma 6. *Let r be a divisor of m, and let $x_1,\dots,x_m \in A_n$, $a_1 := 1, a_2,\dots,a_m \in S_n$. Let M be a subgroup of A_n. The element $(x_1,\dots,x_m)\gamma^r \in G$ belongs to $N_G(M \times M^{a_2} \times \cdots \times M^{a_m})$ if and only if the following conditions are satisfied:*

$$a_{m-r+i}x_{m-r+i}\tau a_i^{-1} \in N_{S_n}(M) \qquad \forall i = 1,\dots,r;$$
$$a_i x_i a_{r+i}^{-1} \in N_{S_n}(M) \qquad \forall i = 1,\dots,m-r.$$

In particular

$$x_i x_{i+r} x_{i+2r} \cdots x_{i+m-r}\tau \in N_{S_n}(M)^{a_i} \qquad \forall i = 1,\dots,r.$$

Now assume that m is odd. Then the element $(x_1,\dots,x_m)\gamma^2 \in G$ belongs to $N_G(M \times M^{a_2} \times \cdots \times M^{a_m})$ if and only if the following conditions are satisfied:

$$a_{m-1}x_{m-1}\tau, a_m x_m \tau a_2^{-1},$$
$$x_1 a_3^{-1}, a_2 x_2 a_4^{-1},\dots,a_{m-2}x_{m-2}a_m^{-1} \in N_{S_n}(M).$$

In particular

$$x_1 x_3 \cdots x_m \tau x_2 x_4 \cdots x_{m-1}\tau \in N_{S_n}(M).$$

Proof. The element $(x_1,\dots,x_m)\gamma^r = (x_1,\dots,x_{m-r},x_{m-r+1}\tau,\dots,x_m\tau)\delta^r$ normalizes $M \times M^{a_2} \times \cdots \times M^{a_m}$ if and only if

$$(M \times M^{a_2} \times \cdots \times M^{a_m})^{(x_1,\dots,x_{m-r},x_{m-r+1}\tau,\dots,x_m\tau)\delta^r} = M \times M^{a_2} \times \cdots \times M^{a_m},$$

in other words

$$M^{a_{m-r+1}x_{m-r+1}\tau} \times \cdots \times M^{a_m x_m \tau} \times M^{x_1} \times M^{a_2 x_2} \times \cdots \times M^{a_{m-r}x_{m-r}} =$$
$$= M \times M^{a_2} \times \cdots \times M^{a_m},$$

and this leads to what is stated.

Now assume m is odd. The element

$$(x_1,\dots,x_m)\gamma^2 = (x_1,\dots,x_{m-2},x_{m-1}\tau,x_m\tau)\delta^2$$

normalizes $M \times M^{a_2} \times \cdots \times M^{a_m}$ if and only if

$$(M \times M^{a_2} \times \cdots \times M^{a_m})^{(x_1,\dots,x_{m-2},x_{m-1}\tau,x_m\tau)\delta^2} = M \times M^{a_2} \times \cdots \times M^{a_m},$$

in other words

$$M^{a_{m-1}x_{m-1}\tau} \times M^{a_m x_m \tau} \times M^{x_1} \times M^{a_2 x_2} \times \cdots \times M^{a_{m-2}x_{m-2}} =$$
$$= M \times M^{a_2} \times \cdots \times M^{a_m},$$

and this leads to what is stated. □

Lemma 7. *Let r be a divisor of m. The element $(x_1,\dots,x_m)\gamma$ normalizes*

$$\Delta := \{(y_1,\dots,y_{m/r},y_1^{b_{21}},\dots,y_{m/r}^{b_{2,m/r}},\dots,y_1^{b_{r,1}},\dots,y_{m/r}^{b_{r,m/r}}) \mid y_1,\dots,y_{m/r} \in A_n\}$$

if and only if (here $b_{1i} = 1$ for all $i = 1,\dots,m/r$)

$$b_{r,m/r}x_m\tau b_{i1} = b_{i-1,m/r}x_{(i-1)m/r} \qquad \forall i = 2,\dots,r$$

and

$$x_j b_{i,j+1} = b_{i,j}x_{(i-1)(m/r)+j} \qquad \forall i = 2,\dots,r,\ j = 1,\dots,m/r-1.$$

In particular

$$x_1 \cdots x_m \tau = [x_1 \cdots x_{m/r-1}(b_{r,m/r}x_m\tau)]^r.$$

For $b \in S_n$ let $l_r(b)$ be the number of elements $s \in S_n$ such that $s^r = b$. Then

$$|\{(x_1,\dots,x_m)\gamma \in N_G(\Delta) \mid x_1 \cdots x_m\tau = b\}| = l_r(b) \cdot |A_n|^{m/r-1}.$$

In particular this number is 0 *if* $b \in A_n$ *or if* r *is even.*

Proof. It is a direct computation. The element $(x_1, \dots, x_m)\gamma$ belongs to $N_G(\Delta)$ if and only if for every $y_1, \dots, y_{m/r} \in A_n$ the element

$$(y_{m/r}^{b_{r,m/r}x_m\tau}, y_1^{x_1}, \dots, y_{m/r}^{x_{m/r}}, y_1^{b_{21}x_{m/r+1}}, \dots, y_{m/r}^{b_{2,m/r}x_{2m/r}}, \dots,$$
$$y_1^{b_{r,1}x_{(r-1)m/r+1}}, \dots, y_{m/r-1}^{b_{r,m/r-1}x_{m-1}})$$

belongs to Δ, and this leads to the stated conditions.

Using these conditions we see that for every $1 \leq i \leq r-1$,

$$x_1 \cdots x_{m/r-1} b_{r,m/r} x_m \tau =$$
$$= b_{i,1} x_{(i-1)m/r+1} x_{(i-1)m/r+2} \cdots x_{(i-1)m/r+m/r-1} x_{im/r} b_{i+1,1}^{-1},$$

and

$$x_1 \cdots x_{m/r-1} b_{r,m/r} x_m \tau = b_{r,1} x_{(r-1)m/r+1} \cdots x_{m-1} x_m \tau.$$

It follows that

$$(x_1 \cdots x_{m/r-1} b_{r,m/r} x_m \tau)^r = x_1 \cdots x_m \tau.$$

The last two statements follow easily from the first two. □

3. Proof of Theorem 1

In this section we prove Theorem 1 for $m \geq 2$ (the case $m = 1$ is proved in [12]).

The next definition was introduced in [12].

Definition 1 (Definite unbeatability)**.** *Let* X *be a finite group. Let* $\mathcal{H}$ *be a set of proper subgroups of* X*, and let* $\Pi \subseteq X$*. Suppose that the following four conditions hold on* $\mathcal{H}$ *and* Π*.*

(1) $\Pi \cap H \neq \emptyset$ *for every* $H \in \mathcal{H}$*;*
(2) $\Pi \subseteq \bigcup_{H \in \mathcal{H}} H$*;*
(3) $\Pi \cap H_1 \cap H_2 = \emptyset$ *for every distinct pair of subgroups* H_1 *and* H_2 *of* $\mathcal{H}$*;*
(4) $|\Pi \cap K| \leq |\Pi \cap H|$ *for every* $H \in \mathcal{H}$ *and* $K < X$ *with* $K \notin \mathcal{H}$*.*

Then $\mathcal{H}$ *is said to be definitely unbeatable on* Π*.*

For $\Pi \subseteq X$ let $\sigma_X(\Pi)$ be the least cardinality of a family of proper subgroups of X whose union contains Π. The next lemma is straightforward so we state it without proof.

Lemma 8. *If* $\mathcal{H}$ *is definitely unbeatable on* Π *then* $\sigma_X(\Pi) = |\mathcal{H}|$*.*

It follows that if $\mathcal{H}$ is definitely unbeatable on Π then $|\mathcal{H}| = \sigma_X(\Pi) \leq \sigma(X)$.

Let us fix the notations.

Notations 1. *Let* n, m *be positive integers, with* $m \geq 2$ *and* $n \geq 5$*. Let* A, B *be two fixed subsets of* $S_n - A_n$*, and let* C *be a fixed subset of* A_n*. For a prime divisor* r *of* m *define* Ω_r *to be the set*

$$\{(x_1, \dots, x_m)\gamma^r \mid x_1 x_{1+r} x_{1+2r} \cdots x_{1+m-r}\tau \in A,\ x_2 x_{2+r} x_{2+2r} \cdots x_{2+m-r}\tau \in B\}.$$

If m *is odd let*

$$\Omega_2 := \{(x_1, \dots, x_m)\gamma^2 \mid x_1 x_3 \cdots x_m \tau x_2 x_4 \cdots x_{m-1}\tau \in C\}.$$

For a prime divisor r *of* $2m$ *let* H_r *be the pre-image of* $\langle \gamma^r \rangle$ *via the projection* $G \to \langle \gamma \rangle$*. Let* Π *be a fixed subset of* $S_n - A_n$*, and let*

$$\Omega_1 := \{(x_1, \dots, x_m)\gamma \mid x_1 \cdots x_m \tau \in \Pi\}.$$

Assume that $n \geq 5$ is odd. Let $K_1, \dots, K_t$ be the intransitive maximal subgroups of A_n. Let Σ be the subset of S_n consisting of the $(k, n-k)$-cycles where $1 \leq k \leq n-1$, and let Π be a fixed subset of Σ. Call $I := \{i \in \{1, \dots, t\} \mid N_{S_n}(K_i) \cap \Pi \neq \emptyset\}$. Let

$$\mathcal{L} := \{N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m}) \mid i \in I,\ a_2, \dots, a_m \in A_n\}.$$

Let A be the set of the $(2, n-2)$-cycles of S_n, let B be the set of the $(n-1)$-cycles of S_n, for m odd let C be:

- the set of the n-cycles of S_n if either $n \geq 7$, or $n = 5$ and $m \notin \{5, 7\}$;
- a subset of S_5 consisting of 12 5-cycles, two in each Sylow 5-subgroup, if $n = 5$ and $m \in \{5, 7\}$.

If m is even or $(n, m) = (5, 3)$ let $C = \emptyset$. We have $|A| = |A_n|/(n-2)$, $|B| = 2|A_n|/(n-1)$, $|C| = 2|A_n|/n$ if $n \geq 7$ or $n = 5$, $m \notin \{3, 5, 7\}$, $|C| = 12$ if $n = 5$, $m \in \{5, 7\}$, and $|\Omega_r| = \frac{2}{(n-1)(n-2)}|A_n|^m$ if $r \neq 2$ or m is even, while if $r = 2$ and m is odd then $|\Omega_r| = (2/n)|A_n|^m$. Suppose we are in one of the following cases:

1. $n = 5$ and $\Pi = \{(2354), (4521), (4132), (1253), (4531), (3245), (1352), (2314), (4125), (3541)\}$;
2. $n \geq 7$ and $\Pi = \Sigma$.

Let $r_1, \dots, r_{\omega(2m)}$ be the distinct prime factors of $2m$. By Lemma 5 and Corollary 2 the family $\mathcal{H} := \mathcal{L} \cup \{H_{r_1}, \dots, H_{r_{\omega(2m)}}\}$ covers G if $n \neq 5$. In fact the odd elements of S_n are covered by the intransitive maximal subgroups of S_n.

Proposition 2. *With the notations and assumptions above, we have:*

1. *If $(n, m) \neq (5, 3)$, $\mathcal{H}$ is definitely unbeatable on $\Omega := \Omega_1 \cup \Omega_{r_1} \cup \cdots \cup \Omega_{r_{\omega(2m)}}$.*
2. *$\mathcal{L}$ is definitely unbeatable on Ω_1.*

Proof. We will verify the four conditions of Definition 1 for both $\mathcal{H}$ and $\mathcal{L}$. Lemmas 4 and 6 imply that if H is a maximal subgroup of G of product type and r is a prime divisor of m then $H \cap \Omega_r = \emptyset$; in particular $H \cap \Omega = H \cap (\Omega_1 \cup \Omega_2)$. If $H \in \mathcal{L}$ then $H \cap \Omega_2 = \emptyset$. Moreover $\Omega_r \subset H_r$ for every prime divisor r of $2m$ and $H_r \cap H_s \cap \Omega = \emptyset$ for every two distinct prime divisors r, s of $2m$. All this implies that the first three conditions of Definition 1 hold for $\mathcal{H}$ if they hold for $\mathcal{L}$. We will check them now.

Recall first that if K is a subgroup of A_n and $x_1, \dots, x_m, a_2, \dots, a_m \in A_n$ then $(x_1, \dots, x_m)\gamma \in N_G(K \times K^{a_2} \times \cdots \times K^{a_m})$ if and only if

$$a_m x_m \tau \in N_{S_n}(K),\ x_1 \in K a_2,\ x_2 \in a_2^{-1} K a_3,\ \cdots,\ x_{m-1} \in a_{m-1}^{-1} K a_m.$$

1. We show that $\Omega_1 \cap H \neq \emptyset$ for every $H = N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m}) \in \mathcal{L}$. Choose the element $(x_1, \dots, x_m)\gamma$ in the intersection in this way: $x_1 = a_2$, $x_2 = a_2^{-1}a_3, \dots,$ $x_{m-1} = a_{m-1}^{-1}a_m$ and x_m such that $x_1 \cdots x_m \tau \in \Pi \cap N_{S_n}(K_i)$.
2. We show that $\Omega_1 \subseteq \bigcup_{H \in \mathcal{L}} H$. Given $(x_1, \dots, x_m)\gamma \in \Omega_1$ choose $i \in I$ such that $x_1 \cdots x_m \tau \in N_{S_n}(K_i)$ and $a_2 = x_1$, $a_3 = x_1 x_2, \dots,$ $a_m = x_1 x_2 \cdots x_{m-1}$. Then choose $H := N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m})$.
3. We show that $\Omega_1 \cap N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m}) \cap N_G(K_j \times K_j^{b_2} \times \cdots \times K_j^{b_m}) = \emptyset$ for $N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m}) \neq N_G(K_j \times K_j^{b_2} \times \cdots \times K_j^{b_m})$ belonging to $\mathcal{L}$. If $(x_1, \dots, x_m)\gamma$ belongs to the stated intersection then $x_1 \cdots x_m \tau \in N_{S_n}(K_i) \cap N_{S_n}(K_j) \cap \Pi$ with $i \neq j$ (which is impossible) or $i = j$ and

$$x_k \in a_k^{-1} K_i a_{k+1} \cap b_k^{-1} K_i b_{k+1}$$

for $k = 1, \dots, m$, where $a_1 := 1$. This easily implies that $K_i^{a_k} = K_i^{b_k}$ for $k = 2, \dots, m$, contradiction.

We now prove that $|H \cap \Omega| \geq |H' \cap \Omega|$ for every $H \in \mathcal{H}$, H' maximal subgroup of G with $H' \not\in \mathcal{H}$. Note that this indeed proves condition (4) of Definition 1 for both $\mathcal{H}$ and $\mathcal{L}$ since for every prime divisor r of $2m$ and every $H \in \mathcal{L}$ we have $H_r \cap \Omega_1 = \emptyset$ and $H \cap \Omega_r = \emptyset$.

First we prove that if $K \not\in \{K_i \mid i \in I\}$ is a subgroup of A_n of the form $R \cap A_n$ where R is a maximal subgroup of S_n (cfr. section 1) then

$$|\Omega \cap N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m})| \geq |\Omega \cap N_G(K \times K^{b_2} \times \cdots \times K^{b_m})|.$$

Notice that since the right hand side of this inequality is zero if K is intransitive (this can happen if $n = 5$), we may assume that K is transitive. As we have already noticed this inequality re-writes as

$$|\Omega_1 \cap N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m})| \geq |(\Omega_1 \cup \Omega_2) \cap N_G(K \times K^{b_2} \times \cdots \times K^{b_m})|.$$

The size of $\Omega_1 \cap N_G(L \times L^{a_2} \times \cdots \times L^{a_m})$ in general (for a subgroup L of A_n and some $a_2, \dots, a_m \in S_n$) is $|L|^{m-1} \cdot |N_{S_n}(L) \cap \Pi|$, and if m is odd the size of $\Omega_2 \cap N_G(L \times L^{a_2} \times \cdots \times L^{a_m})$ is $|L|^{m-1} \cdot |L \cap C|$. Therefore we have to show that

$$|K_i|^{m-1} \cdot |N_{S_n}(K_i) \cap \Pi| \geq |K|^{m-1} \cdot |N_{S_n}(K) \cap (\Pi \cup C)|. \quad (*)$$

- Suppose $n = 5$. The transitive maximal subgroups of A_5 have order 10. Moreover the only intransitive maximal subgroups of A_5 whose normalizers in S_5 intersect Π are the five point stabilizers. If m is even or $m = 3$ then $C = \emptyset$ and $|N_{S_5}(K_i) \cap \Pi| = |N_{S_5}(K) \cap \Pi| = 2$ for every $i \in I$, thus $(*)$ is true. If $m \not\in \{5, 7\}$ is odd then $|N_{S_n}(K) \cap (\Pi \cup C)| = 6$ and $(*)$ becomes $12^{m-1} \cdot 2 \geq 10^{m-1} \cdot 6$, which is true for $m \geq 8$. If $m \in \{5, 7\}$ then $|N_{S_n}(K) \cap (\Pi \cup C)| = 4$ and $(*)$ becomes $12^{m-1} \cdot 2 \geq 10^{m-1} \cdot 4$, which is true.
- Suppose $n = 7$. The left hand side is at least $72^{m-1} \cdot 12$. Since the transitive maximal subgroups of S_7 different from A_7 have size 42 and contain 20 elements of $\Pi \cup C$, it suffices to show that $72^{m-1} \cdot 12 \geq 21^{m-1} \cdot 20$, i.e. $(72/21)^m \geq 40/7$, which is true for $m \geq 2$.
- Suppose $n = 9$. The smallest maximal intransitive subgroup of A_9 is the one of type $(4, 5)$, it has size 1440 and the size of the intersection of its normalizer in S_9 with Π is the smallest possible, $3! \cdot 4! = 144$. Thus the left hand side of $(*)$ is at least $1440^{m-1} \cdot 144$. The right hand side is at most $\max(216^{m-1} \cdot 72, 648^{m-1} \cdot 432)$ (note that the maximal subgroups of A_9 isomorphic to $\mathrm{Aut}(PSL(2, 8))$ are not of the form $R \cap A_9$ with R maximal in S_9: cfr. section 1). Therefore it suffices to show that $1440^{m-1} \cdot 144 \geq 648^{m-1} \cdot 432$, and this is true for $m \geq 3$. If $m = 2$ then $C = \emptyset$ and it suffices to show that $1440 \cdot 144 \geq 648 \cdot 288$ (recall that the imprimitive maximal subgroups of S_9 contain 144 9-cycles and 288 $(6, 3)$-cycles), which is true.
- Suppose $n \geq 11$. Then $|K_i| \geq |K|$ by Corollary 1, and the inequality $|N_{S_n}(K_i) \cap \Pi| \geq |N_{S_n}(K) \cap (\Pi \cup C)|$ is proved in claim 3.2 of [12].

Now we prove that if $N_G(\Delta)$ is a maximal subgroup of G of diagonal type (its existence implies that m is not a power of 2 by Lemma 7) and $i \in \{1, \dots, t\}$, $a_2, \dots, a_m \in A_n$ then

$$|\Omega \cap N_G(K_i \times K_i^{a_2} \times \cdots \times K_i^{a_m})| \geq |\Omega \cap N_G(\Delta)|.$$

The right hand side is at most $|N_G(\Delta)| \leq 2m|A_n|^{m/p}$, where p is the smallest prime divisor of m, hence we are reduced to prove that $|K_i|^{m-1} \cdot |N_{S_n}(K_i) \cap \Pi| \geq 2m|A_n|^{m/p}$. Since if K_i is of type $(k, n-k)$ then $|N_{S_n}(K_i) \cap \Pi| = (k-1)!(n-k-1)!$, we obtain $(2/(k(n-k)))|K_i|^m \geq 2m|A_n|^{m/p}$. Since $k(n-k) \leq ((n-1)/2)((n+1)/2)$, it suffices to show that

$$\frac{8}{n^2-1}|K_i|^m \geq 2m \cdot |A_n|^{m/p}. \tag{1}$$

Note that if s is a divisor of m and L_s denotes the set of elements of G of the form $(x_1, \ldots, x_m)\gamma^s$ then $|N_G(\Delta) \cap L_s| = |\Delta|$. Therefore by Lemma 7 if $N_G(\Delta)$ is of diagonal type 2 then it suffices to show that

$$\frac{8}{n^2-1}|K_i|^m \geq \omega(m) \cdot |A_n|^{m/2}. \tag{2}$$

- If $n = 5$ then $|K_i| = \frac{1}{2}4! = 12$, and if $n = 9$ then $|K_i| \geq \frac{1}{2}4!5! = 1440$; in both these cases (1) is true for $p \geq 3$ and (2) is true for $m \geq 6$. If $n = 11$ then $|K_i| \geq \frac{1}{2}5!6! = 43200$ and (1) is true for $m \geq 2$. If $n = 13$ then $|K_i| \geq \frac{1}{2}6!7! = 1814400$ and (1) is true for $m \geq 2$.
- Suppose $n = 7$. Then $|K_i| \geq 72$, thus it suffices to show (1): $72^m \geq 12m \cdot 2520^{m/2}$, i.e. $(72/\sqrt{2520})^m/m \geq 12$. This is true for $m \geq 15$. If $p \geq 3$ it suffices to show that $(72/\sqrt[3]{2520})^m/m \geq 12$, which is true for $m \geq 3$. Thus we are done if p is odd. If $m \in \{10, 12, 14\}$ then $\omega(m) = 2$ and using (2) we are reduced to show that $(72/\sqrt{2520})^m \geq 12$, which is true.

  We are left with the case $m = 6$. It is easy to see that in general if H is a maximal subgroup of G of diagonal type 2 and r is a prime divisor of m then $|\Omega_r \cap H| \leq |A_n|^{m/2-1} \cdot \min(|A|, |B|)$ (just use the definition of Ω_r). In our case $\min(|A|, |B|) = |A| = 504$, and $72^6 \geq \omega(6) \cdot 6 \cdot 2520^2 \cdot 504$.
- Suppose $n \geq 15$. Then $|K_i|^{3/2} \geq |A_n|$ by Lemma 3, so using (1) we are reduced to prove that $(8/(n^2-1))|A_n|^{\frac{2}{3}m} \geq 2m|A_n|^{m/2}$, i.e. $|A_n|^{m/6} \geq (m/4)(n^2-1)$. This is clearly true for every m since $n \geq 15$.

Now we prove that if $(n, m) \neq (5, 3)$ then $|H_r \cap \Omega| \geq |H \cap \Omega|$ for every maximal subgroup H of G of product type out of $\mathcal{H}$ and for every prime divisor r of $2m$. Let L be the transitive subgroup of A_n such that $H = N_G(L \times L^{a_2} \times \cdots \times L^{a_m})$. Note that

$$|H \cap \Omega| = |H \cap (\Omega_1 \cup \Omega_2)| = |L|^{m-1} \cdot (|N_{S_n}(L) \cap \Pi| + |L \cap C|).$$

Suppose first that $r \neq 2$ or m is even. All we have to prove is that

$$\frac{2}{(n-1)(n-2)}|A_n|^m = |\Omega_r| = |H_r \cap \Omega| \geq |H \cap (\Omega_1 \cup \Omega_2)| =$$
$$= |L|^{m-1} \cdot (|N_{S_n}(L) \cap \Pi| + |L \cap C|).$$

This is easily seen to be true for $n \in \{5, 7, 9\}$. Suppose $n \geq 11$. It suffices to show that $\frac{2}{(n-1)(n-2)}|A_n|^m \geq 2|R|^m$ for any maximal transitive subgroup R of S_n different from A_n, i.e. $(|S_n : R|/2)^m \geq (n-1)(n-2)$, and this is true by Corollary 1, being true for $m = 1$: $|S_n : R|/2 \geq \binom{n}{5}/2 > (n-1)(n-2)$ since $n > 8$.

Assume now that $r = 2$ and m is odd. All we have to prove is that

$$|C| \cdot |A_n|^{m-1} = |\Omega_2| = |H_2 \cap \Omega| \geq |H \cap (\Omega_1 \cup \Omega_2)| =$$
$$= |L|^{m-1} \cdot (|N_{S_n}(L) \cap \Pi| + |L \cap C|).$$

It suffices to prove that for every transitive subgroup R of S_n not containing A_n we have $|C| \cdot |A_n|^{m-1} \geq 2|R|^m$, i.e. $(|S_n : R|/2)^m \geq |S_n|/|C|$. If $n > 5$ this follows from $|S_n : R| \geq n$, if $n = 5$ this follows from $|C| \geq 12$.

Now we prove that if $(n, m) \neq (5, 3)$ then $|H_r \cap \Omega| \geq |H \cap \Omega|$ for every prime divisor r of $2m$ and every maximal subgroup H of G of diagonal type. Notice that $|H| \leq 2m|A_n|^{m/2}$, hence if $r \neq 2$ or m is even we are reduced to prove that $2|A_n|^m/((n-1)(n-2)) \geq 2m|A_n|^{m/2}$, and this is clearly true for every m and $n \geq 5$. If $r = 2$ and m is odd we have to prove that $(2/n)|A_n|^m \geq 2m|A_n|^{m/2}$, and this is clearly true for every m and $n \geq 5$. □

Note that Proposition 2 implies Theorem 1 if $n > 5$ is odd.

Observation 1. *Let $\mathcal{K}$ be a minimal cover of the finite group X, so that $|\mathcal{K}| = \sigma(X)$, and let $\mathcal{K}_1$ be a subset of $\mathcal{K}$. Let Ω be a subset of $X - \bigcup_{K \in \mathcal{K}_1} K$. Then $|\mathcal{K}_1| + \sigma_X(\Omega) \leq \sigma(X)$, where $\sigma_X(\Omega)$ denotes the least number of proper subgroups of X needed to cover Ω.*

Suppose that $n = 5$ and all the prime divisors of m belong to $\{2, 3\}$. Fix a minimal cover $\mathcal{K}$ of G. Let $\mathcal{K}_0$ be the family of the maximal subgroups of G of the form $N_G(M \times M^{a_2} \times \cdots \times M^{a_m})$ with $a_2, \ldots, a_m \in A_5$ and M an intransitive maximal subgroup of A_5 of type $(3, 2)$. Since the $(3, 2)$-cycles are not of the form x^2 or x^3 for $x \in S_5$, by Lemma 7 the only maximal subgroups of G which contain elements of the form $(x_1, \ldots, x_m)\gamma$ where $x_1 \cdots x_m \tau$ is a $(3, 2)$-cycle are the subgroups in $\mathcal{K}_0$. In particular $\mathcal{K}_0 \subset \mathcal{K}$. In the following we use Notations 1, with A the set of the $(3, 2)$-cycles, B the set of the 4-cycles and C the set of the 5-cycles.

Suppose that m is even, and let $\mathcal{K}_1 := \mathcal{K}_0$. For every $K \in \mathcal{K}_1$ we have $\Omega_1 \cap K = \Omega_2 \cap K = \Omega_3 \cap K = \emptyset$, thus by Observation 1 and Proposition 2 $|\mathcal{K}_1| + |\mathcal{H}| \leq \sigma(G)$, and we have equality since $\mathcal{K}_1 \cup \mathcal{H}$ covers G.

Suppose that m is a power of 3, and let $\mathcal{K}_1 := \mathcal{K}_0 \cup \{H_2, H_3\}$. If either $H_2 \notin \mathcal{K}$ or $H_3 \notin \mathcal{K}$ then in order to cover $\Omega_2 \cup \Omega_3$ we need at least

$$\frac{\min\{|\Omega_2|, |\Omega_3|\}}{|N_G(\Delta)|} = \frac{|A| \cdot |B| \cdot |A_5|^{m-1}}{2m \cdot |A_5|^{m/3}} = (5/m) \cdot 60^{2m/3-1}$$

subgroups, where $N_G(\Delta)$ is a maximal subgroup of G of diagonal type. Since $\sigma(G) \leq 2 + 5^m + 10^m$, we obtain that $10^m + (5/m)60^{2m/3-1} \leq 2 + 5^m + 10^m$, contradiction. Therefore $\mathcal{K}_1 \subset \mathcal{K}$. Since $\Omega_1 \cap K = \emptyset$ for every $K \in \mathcal{K}_1$, by Observation 1 and Proposition 2 we obtain that $2 + 5^m + 10^m \leq \sigma(G)$, thus we have equality.

Assume now that n is any positive integer at least 5. The following observation follows easily from the proof of Proposition 2.

Observation 2. *Let $\mathcal{A}$ be a family of proper subgroups of A_n, and let*

$$\mathcal{K} := \{N_G(M \times M^{a_2} \times \cdots \times M^{a_m}) \mid a_2, \ldots, a_m \in A_n,\ M \in \mathcal{A}\}.$$

Let Π be a subset of S_n such that $\mathcal{A}$ is definitely unbeatable on Π. Let

$$\Omega := \{(x_1, \ldots, x_m)\gamma \in G \mid x_1 \cdots x_m \tau \in \Pi\}.$$

Suppose that the following two conditions hold:

(1) $|M| \geq |K|$ for every $M \in \mathcal{A}$ and every maximal subgroup K of A_n such that $N_{S_n}(K) \cap \Pi \neq \emptyset$.

(2) $|M|^{m-1} \cdot |N_{S_n}(M) \cap \Pi| \geq |H \cap \Omega|$ *for every* $M \in \mathcal{A}$ *and every maximal subgroup* H *of* G *of diagonal type. Note that this is true if*

$$|M|^{m-1} \cdot |N_{S_n}(M) \cap \Pi| \geq 2m|A_n|^{m/p},$$

where p *is the smallest prime divisor of* m *such that there exists a maximal subgroup of* G *of diagonal type* p *whose intersection with* Ω *is non-empty.*

Then the family $\mathcal{K}$ *of subgroups of* G *is definitely unbeatable on* Ω*. In particular* $|\mathcal{K}| \leq \sigma(G)$*.*

Let us apply this observation to the cases we are left with.

Let $n = 5$. Let $\mathcal{A}$ be the set of the intransitive maximal subgroups of A_5 of type $(3, 2)$ and let Π be the set of the $(3, 2)$-cycles in S_5. Condition (1) of Observation 2 is clearly verified. Let us prove condition (2). By Lemma 7 we may assume $p \geq 5$ (the elements of Π have no square roots nor cubic roots in S_5). The inequality $6^{m-1} \cdot 2 \geq 2m \cdot 60^{m/p}$ is then true. We obtain $\sigma(G) \geq 10^m$.

Let $n = 6$. Fix a minimal cover $\mathcal{M}$ of G consisting of maximal subgroups. Let $\mathcal{K}_0$ be the family of the maximal subgroups of G of the form $N_G(M \times M^{a_2} \times \cdots \times M^{a_m})$ where M is a subgroup of A_6 isomorphic to A_5, so that $|\mathcal{K}_0| = 12 \cdot 6^{m-1}$. Let us use Notations 1. Let $\mathcal{K}_1$ be the set consisting of the subgroups in $\mathcal{K}_0$ and the subgroups H_r for r a prime divisor of m. Since $S_6 - A_6$ is covered by the two conjugacy classes of maximal subgroups of S_6 isomorphic to S_5, $\mathcal{K}_1 \cup \{H_2\}$ covers G, in particular $\sigma(G) \leq \omega(2m) + 2 \cdot 6^m$. It is easy to see that $H \cong A_5^m \rtimes C_{2m}$ for every $H \in \mathcal{K}_0$, therefore

$$\sigma(H) \geq 10^m > \omega(2m) + 2 \cdot 6^m \geq \sigma(G).$$

By Lemma 1 in [7] we deduce that $\mathcal{K}_0 \subset \mathcal{M}$. Let A be the set of the $(3, 2)$-cycles in S_6, let B be the set of the 6-cycles in S_6, and let C be the set of the 3-cycles in S_6. Since no subgroup of S_6 intersects both A and B, $H \cap \Omega_r = \emptyset$ for every prime divisor r of m and every maximal subgroup H of G of product type. If H is a maximal subgroup of G of diagonal type (in particular m is not a power of 2 by Lemma 7) then $|H \cap \Omega_r| \leq |H \cap \mathrm{soc}(G)|$. Therefore if r is a prime divisor of m and $H_r \notin \mathcal{M}$ then in order to cover Ω_r we need at least

$$\frac{\min_r |\Omega_r|}{|H \cap \mathrm{soc}(G)|} \geq \frac{40 \cdot 360^{m-1}}{360^{m/2}} = 40 \cdot 360^{m/2-1}$$

subgroups. Since $m \geq 3$, this contradicts $\sigma(G) \leq \omega(2m) + 2 \cdot 6^m$. Therefore $\mathcal{K}_1 \subseteq \mathcal{M}$. If m is even then $\mathcal{K}_1$ covers G, thus $\mathcal{K}_1 = \mathcal{M}$ and we are done. Suppose m is odd. Since the subgroups of S_6 isomorphic to S_5 do not intersect C, the family $\mathcal{K}_1$ does not cover Ω_2. Since $\Omega_2 \subset H_2$ and $\mathcal{K}_1 \cup \{H_2\}$ covers G, we obtain $\sigma(G) = |\mathcal{M}| = \omega(2m) + 2 \cdot 6^m$.

Let $n \geq 8$ be even. Let Π be the set of the n-cycles in S_n, and let $\mathcal{A}$ be the family of the maximal imprimitive subgroups of A_n corresponding to the partitions given by two subsets of $\{1, \ldots, n\}$ of size $n/2$. In [12] (claims 3.3 and 3.4) it is proved that if $n \geq 8$ then $\mathcal{A}$ is definitely unbeatable on Π. Condition (1) of Observation 2 follows from Lemma 1 and the fact that the order of a primitive maximal subgroup of A_n is at most 2.6^n (see [11]). In fact $(n/2)!^2 \geq 2.6^n$ if $n \geq 10$, and all the maximal subgroups of A_8 whose normalizers in S_8 contain 8-cycles belong to $\mathcal{A}$. We now prove condition (2). We may assume that m is not a power of 2 by Lemma 7. If $n \in \{8, 10\}$ then $|K|^{m-1} \cdot |N_{S_n}(K) \cap \Pi| \geq 2m|A_n|^{m/2}$ whenever $K \in \mathcal{A}$. Suppose $n \geq 12$. Using Lemma 3 we see that $|K|^{3/2} \geq |A_n|$ for every $K \in \mathcal{A}$.

Therefore since $m \geq 2$ is not a power of 2, if p is the smallest prime divisor of m then $|K|^{m-1} \geq |A_n|^{(2/3)(m-1)} \geq 2m|A_n|^{m/p}$ for $K \in \mathcal{A}$ (if $m \geq 5$ this follows from $p \geq 2$). Applying Observation 2 we obtain that

$$\left(\frac{1}{2}\binom{n}{n/2}\right)^m \leq \sigma(G) \leq \omega(2m) + \left(\frac{1}{2}\binom{n}{n/2}\right)^m + \sum_{i=1}^{[n/3]} \binom{n}{i}^m .$$

The upper bound is obtained by observing that the non-n-cycles of S_n are covered by the maximal intransitive subgroups of S_n of type $(i, n-i)$ for $1 \leq i \leq [n/3]$.

4. Proof of Theorem 2

In this whole section we will call $G := A_5 \wr C_2$, the semidirect product $(A_5 \times A_5) \rtimes \langle \varepsilon \rangle$ where ε, of order 2, acts on $A_5 \times A_5$ exchanging the two variables. Recall that the maximal subgroups of G are of the following five types:

- The socle $N = A_5 \times A_5$.
- Type 'r': $N_G(M \times M^l)$ where $l \in A_5$ and M is a point stabilizer.
- Type 's': $N_G(M \times M^l)$ where $l \in A_5$ and M is the normalizer of a Sylow 5-subgroup.
- Type 't': $N_G(M \times M^l)$ where $l \in A_5$ and M is an intransitive subgroup of type $(3, 2)$.
- Type 'd': $N_G(\Delta_\alpha)$ where $\alpha \in S_5$ and $\Delta_\alpha := \{(x, x^\alpha) \mid x \in A_5\}$.

Recall that:

- $N \cap N_G(H) = H$ for every H of the type $M \times M^l$ or Δ_α with M a maximal subgroup of A_5.
- The element $(x, y)\varepsilon$ belongs to $N_G(M \times M^l)$ if and only if $xl^{-1}, ly \in M$. In particular $xy \in M$.
- The element $(x, y)\varepsilon$ belongs to $N_G(\Delta_\alpha)$ if and only if $(\alpha y)^2 = xy$.

Let $\mathcal{M}$ be a family of proper subgroups of G which cover G.

Observation 3. $N \in \mathcal{M}$

Proof. Let $x \in A_5$ be a 5-cycle, and let $y \in A_5$ be a 3-cycle. Then the element (x, y) does not belong to any $N_G(M \times M^l)$ or $N_G(\Delta_\alpha)$ by the remarks above (no maximal subgroup of A_5 has order divisible by 3 and 5). □

Call i the number of subgroups of type i in $\mathcal{M}$ for $i = r, s, t, d$.

The 'type' of an element $(x, y)\varepsilon \in G - N$ is the cyclic structure of the element $xy \in A_5$. The four possible cyclic structures will be denoted by 1, (3), (5), (2, 2).

The only maximal subgroups of G containing elements of type (3) are the ones of type r or t or d. A subgroup of type r contains 96 elements of type (3). A subgroup of type t contains 12 elements of type (3). A subgroup of type d contains 20 elements of type (3). G contains 1200 elements of type (3). In particular $96r + 12t + 20d \geq 1200$, in other words

$$24r + 3t + 5d \geq 300. \tag{1}$$

The only maximal subgroups of G containing elements of type (5) are the ones of type s or d. A subgroup of type s contains 40 elements of type (5). A subgroup of type d contains 24 elements of type (5) if α is even, 0 if α is odd. G contains 1440 elements of type (5). In particular $40s + 24d \geq 1440$, in other words

$$5s + 3d \geq 180. \tag{2}$$

We know that G admits a cover which consists of 57 proper subgroups, with $s=36$, $r=20$, $t=d=0$ (the 20 subgroups of type r are $N_G(M\times M^l)$ where $l\in A_5$ and $M\in\{\mathrm{Stab}(1),\mathrm{Stab}(2),\mathrm{Stab}(3),\mathrm{Stab}(4)\}$).

Suppose by contradiction that $\sigma(G)<57$, and let $\mathcal{M}$ be a cover with 56 proper subgroups. In particular $r+s+t+d+1=56$, i.e. $r+s+t+d=55$.

Observation 4. *$d\leq 33$, $s\geq 17$ and $r\geq 6$.*

Proof. Inequality 2 re-writes as $s\geq 36-\frac{3}{5}d$. Since $r+s+t+d=55$, $r+t=55-s-d\leq 55-36+\frac{3}{5}d-d=19-\frac{2}{5}d$. Combining this with inequality 1 we obtain $24(19-\frac{2}{5}d)+5d\geq 300$, i.e. $d\leq 156\cdot 5/23$, i.e. $d\leq 33$. Therefore $s\geq 36-\frac{3}{5}d=36-\frac{99}{5}>16$.

Inequality 1 re-writes as $21r+2d-3s+3(r+t+d+s)\geq 300$, i.e. $21r+2d-3s\geq 135$. Since $d\leq 33$ and $s\geq 17$, $21r\geq 135+3\cdot 17-2\cdot 33=120$, i.e. $r\geq 6$. $\square$

Observation 5. *$r+t+d\geq 20$ and $s<36$.*

Proof. Consider the following elements of A_5: $a_1:=(243)\in\mathrm{Stab}(1)$, $a_2:=(143)\in\mathrm{Stab}(2)$, $a_3:=(142)\in\mathrm{Stab}(3)$, $a_4:=(132)\in\mathrm{Stab}(4)$. Let $\mathcal{X}$ be the set of elements of G of the form $(x,y)\varepsilon$ with $xy=a_i$ for an $i\in\{1,2,3,4\}$ and $x\in J_i$, where J_i is a fixed set of representatives of the right cosets of $\mathrm{Stab}(i)$, which will be specified later. Let $\mathcal{H}$ be the set of the 20 subgroups $N_G(M\times M^l)$ of G of type r with M the stabilizer of i for $i\in\{1,2,3,4\}$. Notice that every element of $\mathcal{X}$ lies in exactly one element of $\mathcal{H}$. Now observe that if a subgroup $N_G(K\times K^l)$ of type t contains an element $(x,y)\varepsilon\in\mathcal{X}$ then K is determined by $a_i=xy$ - use this to label the K's as K_i for $i\in\{1,2,3,4\}$ -, so that the only freedom is in the choice of the coset K_il. We will choose the sets J_i in such a way that any two elements of J_i lie in different right cosets of K_i. This implies that for every subgroup $N_G(K\times K^l)$ of G of type t we have $|\mathcal{X}\cap N_G(K\times K^l)|\leq 1$. Let us choose the J_i's in such a way that for every subgroup $N_G(\Delta_\alpha)$ of type d we have $|\mathcal{X}\cap N_G(\Delta_\alpha)|\leq 1$. Choose:

$$J_1=\{(452),(12534),(13425),(14)(35),(23)(15)\},$$
$$J_2=\{(134),(245),(123),(152),(125)\},$$
$$J_3=\{(142),(132),(134),(153),(135)\},$$
$$J_4=\{(132),(142),(243),(154),(145)\}.$$

We have that for any $i=1,2,3,4$ any two elements of J_i lie in different right cosets of K_i. We have to check that every subgroup of the form $N_G(\Delta_\alpha)$ contains at most one element of $\mathcal{X}$. In other words we have to check that if $(x,y)\varepsilon\in\mathcal{X}\cap N_G(\Delta_\alpha)$ then $(x,y)\varepsilon$ is determined. We have $(\alpha y)^2=xy$, so that if α is even then $\alpha=xyx$, if α is odd then $\alpha=\tau_{xy}xyx$, where τ_{xy} is the transposition whose support is pointwise fixed by xy. Let

$$P_i:=\{xyx\mid xy=a_i,\ (x,y)\varepsilon\in\mathcal{X}\}\cup\{\tau_{xy}xyx\mid xy=a_i,\ (x,y)\varepsilon\in\mathcal{X}\}\subset S_5$$

for $i=1,2,3,4$. Clearly $|P_i|=10$ for $i=1,2,3,4$. All we have to show is that the P_i's are pairwise disjoint. This follows from the computation:

$$P_1=\{(25)(34),(12)(35),(135),(14532),(15)(24),$$
$$(125)(34),(1352),(35),(132)(45),(24)\},$$
$$P_2=\{1,(15243),(14)(23),(14352),(14325),$$
$$(25),(1543),(14)(253),(1435),(1432)\},$$

$$P_3 = \{(124), (14)(23), (234), (14253), (14235), \\ (124)(35), (14)(235), (2354), (1425), (1423)\},$$
$$P_4 = \{(123), (13)(24), (12)(34), (13254), (13245), \\ (123)(45), (13)(245), (12)(345), (1325), (1324)\}.$$

Clearly, the subgroups of G of type s do not contain any element of $\mathcal{X}$.

All this implies that $\mathcal{H}$ is definitely unbeatable on $\mathcal{X}$, hence $r+t+d \geq |\mathcal{H}| = 20$. It follows that $56 = |\mathcal{M}| = 1+r+s+t+d > s+20$, i.e. $s < 36$. □

Observation 6. *Let M be the normalizer of a Sylow 5-subgroup of A_5, let $l \in A_5$ and suppose that $N_G(M \times M^l) \notin \mathcal{M}$. Then $N_G(\Delta_\alpha) \in \mathcal{M}$ for every $\alpha \in Ml$. In particular if $\mathcal{L}$ is the family of the cosets Ml where $M < A_5$ is the normalizer of a Sylow 5-subgroup and $N_G(M \times M^l) \notin \mathcal{M}$ then the number of subgroups of type d in $\mathcal{M}$ is at least the size of the union of $\mathcal{L}$.*

Proof. The number of elements of type (5) in $N_G(M \times M^l)$ is 40. Moreover the only maximal subgroup of G of type r, s, t which contains one of these 40 elements is the one we are considering: $xy \in M$ determines M and $x \in Ml$ determines Ml. Let $c \in M$ be a 5-cycle. The element $(x, x^{-1}c)\varepsilon$ belongs to $N_G(\Delta_\alpha)$ if and only if $(\alpha x^{-1}c)^2 = c$, i.e. $\alpha x^{-1} c = c^3$, i.e. $\alpha = c^2 x$. The result follows. □

Lemma 9. *We have the following facts:*

1. *Let k be a positive integer, and let $\mathcal{L}$ be the family of the cosets of the normalizers of the Sylow 5-subgroups of A_5. Then any subfamily of $\mathcal{L}$ consisting of exactly k cosets covers at least $10k - 2\binom{k}{2}$ elements of A_5.*
2. *Let $H \neq K$ be two normalizers of Sylow 5-subgroups of A_5. Then for any $a_1, a_2, a_3, b_1, b_2, b_3 \in A_5$ such that $Ha_1, Ha_2, Ha_3, Kb_1, Kb_2, Kb_3$ are pairwise distinct, the union*
$$Ha_1 \cup Ha_2 \cup Ha_3 \cup Kb_1 \cup Kb_2 \cup Kb_3$$
has size at least 42.

Proof. Let $Ha, Kb \in \mathcal{L}$. If the intersection $Ha \cap Kb$ is non-empty then it contains an element x, so that $Ha = Hx$, $Kb = Kx$, and $Ha \cap Kb = Hx \cap Kx = (H \cap K)x$. It follows that the maximum size of the intersection of two elements of $\mathcal{L}$ equals the maximum size of the intersection of two normalizers of Sylow 5-subgroups, i.e. 2. Maximizing the sizes of the intersections we find that k cosets cover at least $10k - 2\binom{k}{2}$ elements.

We now prove the second statement. Clearly $|Ha_1 \cup Ha_2 \cup Ha_3| = 30$. Since $|Ha_i \cap Kb_j| \leq 2$ for every $i, j = 1, 2, 3$,
$$|Ha_1 \cup Ha_2 \cup Ha_3 \cup Kb_1 \cup Kb_2 \cup Kb_3| \geq 30 + 3 \cdot (10 - 3 \cdot 2) = 42,$$
as we wanted. □

Corollary 3. $s \leq 31$ *and* $d \geq 30$.

Proof. Recall that the subgroups of G of type s are 36. In the following we use Lemma 9 and Observation 6. If $s = 32$ then $d \geq 28$, impossible; if $s = 33$ then $d \geq 24$, impossible; if $s = 34$ then $d \geq 18$, impossible since $r \geq 6$. Assume now $s = 35$, so that $d \geq 10$. Since $r \geq 6$, $6 + 35 + t + d \leq r + s + t + d = 55$, i.e. $t + d \leq 14$. Thus inequality 1 implies that $5 \cdot 14 \geq 300 - 24r$, i.e. $r \geq 10$. Hence $d = r = 10$ and $t = 0$. This contradicts inequality 1. Since $s < 36$, we deduce that $s \leq 31$ and consequently $d \geq 30$. □

Since $d \geq 30$, $r+s+t+30 \leq r+s+t+d = 55$, i.e. $r+s+t \leq 25$. Since $s \geq 17$ we obtain that $r+t \leq 8$. In particular $r \in \{6,7,8\}$.

- $r = 6$. Then by inequality 1 we have $144 + 5(t+d) \geq 24r + 3t + 5d = 24r + 3t + 5d \geq 300$, and we deduce that $t + d \geq 32$. Therefore $55 = r+s+t+d \geq 6+s+32$, i.e. $s \leq 17$. Since $s \geq 17$ we obtain that $s = 17$. Inequality 2 says that $5 \cdot 17 + 3d \geq 180$, i.e. $d \geq 32$, so that $d = 32$ and $t = 0$.
- $r = 7$. Since $d \geq 30$, $7+s+30 \leq r+s+t+d = 55$, i.e. $s \leq 18$.
  - $s = 18$. Then $7+18+t+d = r+s+t+d = 55$, i.e. $t+d = 30$. Since $d \geq 30$ we obtain $d = 30$ and $t = 0$.
  - $s = 17$. Inequality 2 says that $5 \cdot 17 + 3d \geq 180$, i.e. $d \geq 32$, so that $55 = r+s+t+d \geq 7+17+32 = 56$, contradiction.
- $r = 8$. Then since $r+s+t \leq 25$ we obtain $s+t \leq 17$, and since $s \geq 17$ we have $s = 17$, $t = 0$ and $d = 30$. This contradicts inequality 2.

We deduce that either $(r,s,t,d) = (7,18,0,30)$ or $(r,s,t,d) = (6,17,0,32)$.

In both these cases there are at least 18 subgroups of type s outside $\mathcal{M}$. Therefore Observation 6 and Lemma 9(2) imply that $d \geq 42$, a contradiction.

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