Coverings of groups by subgroups

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These considerations led Cohn in 1994 to define for every non-cyclic group G:

$\sigma(G)$ Covering number of G: the smallest cardinality of a family of proper subgroups of G whose union equals G.

If G is cyclic we pose $\sigma(G) = \infty$.

We are interested in groups with finite covering number.

The following result, easy consequence of a well-known result of Neumann of 1954 [2], implies that when studying the covering number we can concentrate on finite groups.

THEOREM

If G is a group with finite covering number then there exists $N \trianglelefteq G$ such that G/N is finite and $\sigma(G) = \sigma(G/N)$.

From now on every considered group is assumed to be finite.

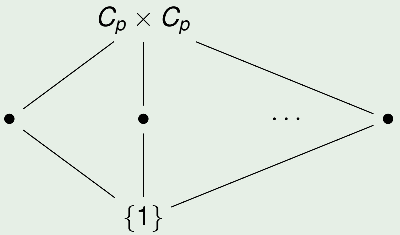
Example

$\sigma(\mathrm{Sym}(3)) = 4$. Subgroup lattice:

$$\begin{tikzcd}
& \mathrm{Sym}(3) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\{1,(12)\} \arrow[dr, dash] & \{1,(13)\} \arrow[d, dash] & \{1,(23)\} \arrow[dl, dash] & \{1,(123),(132)\} \arrow[dll, dash] \\
& \{1\} & &
\end{tikzcd}$$

Example

$\sigma(C_p \times C_p) = p + 1$. Subgroup lattice:



Notation

A "covering" of a finite group G will be a family of proper subgroups of G whose union is G.

A "minimal covering" of a finite group G will be a covering $\mathcal{H}$ of G such that $|\mathcal{H}| = \sigma(G)$.

$\sigma(G)$ is the smallest cardinality of a covering of G.

If $\mathcal{H}$ is any covering of G then by definition $\sigma(G) \leq |\mathcal{H}|$.

Let N be a normal subgroup of the group G. Recall that there is a canonical bijective correspondence between the family of subgroups of G containing N and the family of subgroups of G/N. It is given by

$$K \mapsto i(K) = K/N,$$

the inverse being

$$H \mapsto i^{-1}(H) = \{g \in G \mid gN \in H\}.$$

This implies the **basic inequality**

$$\sigma(G) \leq \sigma(G/N).$$

Indeed, if $\mathcal{H}$ is a minimal covering of G/N then $\{i^{-1}(H) \mid H \in \mathcal{H}\}$ is a covering of G of size $|\mathcal{H}| = \sigma(G/N)$.

Let us prove a very easy lower bound.

Proposition (The minimal index lower bound)

Let G be a non-cyclic group, and write $G = H_1 \cup \cdots \cup H_n$ as union of $n = \sigma(G)$ proper subgroups. Let $\beta_i := |G : H_i| = |G|/|H_i|$ for $i = 1, \ldots, n$. Then $\min\{\beta_1, \ldots, \beta_n\} < \sigma(G)$.

Proof.

We may assume that $\beta_1 \leq \cdots \leq \beta_n$. Since $1 \in H_1 \cap \ldots \cap H_n$ the union $H_1 \cup \ldots \cup H_n$ is not disjoint and hence

$$|G| < \sum_{i=1}^{n} |H_i| = |G| \sum_{i=1}^{n} \frac{1}{\beta_i} \leq \frac{|G|n}{\beta_1}.$$

Therefore $\beta_1 < n = \sigma(G)$. $\square$

The Minimal Index Lower Bound allows us to compute the covering number of p-groups. Note that:

Lemma

If G is a finite non-cyclic group then $\sigma(G) = \sigma(G/\Phi(G))$, where $\Phi(G)$ is the Frattini subgroup of G, the intersection of the maximal subgroups of G. Indeed, each element of a minimal covering of G consisting of maximal subgroups contains the Frattini subgroup.

Let now G be a finite non-cyclic p-group, $|G| = p^n$. Recall that $G/\Phi(G)$ is an elementary abelian p-group, isomorphic to C_p^d where d is the minimal number of generators of G (note that $d > 1$ because G is not cyclic). We are going to prove that $\sigma(G) = p + 1$. By the lemma, it is enough to prove that $\sigma(C_p^d) = p + 1$.

- Since $d > 1$, ${C_p}^d$ projects onto $C_p \times C_p$ so by the basic inequality $\sigma({C_p}^d) \leq \sigma(C_p \times C_p) = p + 1$.
- Since any proper subgroup of ${C_p}^d$ has index at least p, by the Minimal Index Lower Bound we have $p < \sigma({C_p}^d)$.

A finite group G is called **nilpotent** if its Sylow subgroups are all normal. Equivalently, G is nilpotent if it is isomorphic with the direct product of its Sylow subgroups.

It is easy to show that if A and B are two finite groups of coprime order then

$$\sigma(A \times B) = \min\{\sigma(A), \sigma(B)\}.$$

This, with the discussion in the previous slide, shows that:

Proposition

If G is any non-cyclic finite nilpotent group then $\sigma(G) = p + 1$ where p is the smallest prime divisor of $|G|$ with the property that the Sylow p-subgroup of G is not cyclic.

A group G is called **solvable** if it admits a finite chain $1 \lhd N_1 \lhd \ldots \lhd N_t = G$ such that N_i/N_{i-1} is abelian for every $i = 1, \ldots, t$. A **chief factor** of a group G is a minimal normal subgroup of a quotient of G. A **complement** of a normal subgroup N of G is a subgroup H of G such that $HN = G$ and $H \cap N = \{1\}$.

The solvable case has been completely worked out:

Theorem (Tomkinson, [4])

Let G be a finite solvable group. Then

$$\sigma(G) = |H/K| + 1$$

where H/K is the smallest chief factor of G with more than one complement in G/K.

Note that every finite non-cyclic group is the union of its non-trivial cyclic subgroups. Therefore we always have

$$\sigma(G) \leq |G| - 1.$$

In case G admits a non-solvable normal subgroup, we can bound $\sigma(G)$ with the size of this normal subgroup:

Proposition

Let N be a non-solvable normal subgroup of the finite group G. Then by CFSG $\bigcup_{1 \neq n \in N} C_G(n) = G$, *thus* $\sigma(G) \leq |N| - 1$.

This is in some sense the best upper bound we can hope for.

Example

There are infinitely many primes p such that $\sigma(A_p) \geq (p-2)!$ (unfortunately).

The basic inequality $\sigma(G) \leq \sigma(G/N)$ suggests to consider the quotients G/N such that $\sigma(G) = \sigma(G/N)$, and leads to the following:

Definition (σ-elementary groups)

A group G is said to be "σ-elementary" if

$$\sigma(G) < \sigma(G/N)$$

for every $1 \neq N \trianglelefteq G$. We say that G is "n-elementary" if G is σ-elementary and $\sigma(G) = n$.

Clearly if G is any finite group then there exists $N \trianglelefteq G$ such that G/N is σ-elementary and $\sigma(G) = \sigma(G/N)$.

Example

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If the σ-elementary group G is abelian then $G \cong C_p \times C_p$ for some prime p ([1], Theorem 3).

Recall that the "**socle**" of a group is the subgroup generated by its minimal normal subgroups, and a group is called "**monolithic**" if it has only one minimal normal subgroup.

Example

Tomkinson's result implies that if G is a non-abelian solvable σ-elementary group then G is monolithic, $G/\operatorname{soc}(G)$ is cyclic and

$$\sigma(G) = |\operatorname{soc}(G)| + 1.$$

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Theorem (Lucchini, Detomi [3] Corollary 14)

Let G be a non-abelian σ-elementary group and let $N_1, \dots, N_k$ be minimal normal subgroups of G such that $\mathrm{soc}(G) = N_1 \times \cdots \times N_k$.

Then there exist epimorphic images $X_1, \dots, X_k$ of G with the property that X_i is a primitive monolithic group with socle isomorphic to N_i for $i = 1, \dots, k$ and G is a subdirect product of $X_1, \dots, X_k$: the natural homomorphism

$$G \to X_1 \times \ldots \times X_k$$

is injective.

This and other results led Lucchini and Detomi to formulate the following conjecture:

Conjecture (Lucchini, Detomi [3])

Every non-abelian σ-elementary group is monolithic.

In other words, the guess is that the number k in the previous proposition is always equal to 1.

There is a good results which points out how crucial the inequality $\sigma(A_p) \geq (p-2)!$ for infinitely many primes p is.

Theorem ([3], Theorem 24)

Let G be a σ-elementary group with no abelian minimal normal subgroups. Then either G is a primitive monolithic group and $G/\operatorname{soc}(G)$ is cyclic, or $G/\operatorname{soc}(G)$ is non-solvable and all the non-abelian composition factors of $G/\operatorname{soc}(G)$ are alternating groups of odd degree.

When dealing with the conjecture, the easiest possible case to consider is the following. Let S be a non-abelian simple group. We want to prove that $\sigma(S \times S) = \sigma(S)$.

Lemma (The Intersection Argument)

Let $\mathcal{H}$ be a minimal covering of G, and let M be a maximal subgroup of G. If $\sigma(G) < \sigma(M)$ then $M \in \mathcal{H}$.

Proof.

We have $M = M \cap G = M \cap \bigcup_{H \in \mathcal{H}} H = \bigcup_{H \in \mathcal{H}} H \cap M$. Therefore the family $\{H \cap M \mid H \in \mathcal{H}\}$ covers M, and has size $\sigma(G)$. Since $\sigma(G) < \sigma(M)$, one of the subgroups $H \cap M$ of M must be unproper, i.e. there exists $H \in \mathcal{H}$ such that $H \cap M = M$, i.e. $M \subseteq H$. By maximality of M it follows that $M = H$. $\square$

Let now $G = S \times S$. We want to prove that $\sigma(G) = \sigma(S)$. Let

$$M := \{(s,s) \mid s \in S\} < G.$$

M is a maximal subgroup of G isomorphic to S.

Since $S \times \{1\}$ is a quotient of G, $\sigma(G) \leq \sigma(S \times \{1\}) = \sigma(S)$, so now assume by contradiction that $\sigma(G) < \sigma(S) = \sigma(M)$. By the intersection argument all the $|G : M|$ conjugates of M in G belong to every minimal cover of G (they are all maximal subgroups isomorphic to M), therefore

$$|S| = |G : M| \leq \sigma(G) \leq \sigma(S) \leq |S| - 1,$$

a contradiction.

If the non-abelian group G is a direct product of two non-trivial subgroups then it is not σ-elementary. This is a consequence of the following result:

Theorem (Lucchini A., G 2010 [5])

Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_1. *In this case* $\sigma(G) = \sigma(H_1)$.
2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_2. *In this case* $\sigma(G) = \sigma(H_2)$.
3. *There exist* $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ *with* $H_1/N_1 \cong H_2/N_2 \cong C_p$ *and* $\mathcal{M}$ *consists of the maximal subgroups of* $H_1 \times H_2$ *containing* $N_1 \times N_2$. *In this case* $\sigma(G) = p+1$.

Let us describe the idea in the case $(|H_1/H_1'|, |H_2/H_2'|) = 1$. In order not to get lost in technicalities, let us assume that $H_1 = H_2 = S$ is a non-abelian simple group.

1. We know that the maximal subgroups of $S \times S$ are of the following three types:

$$(1)\ K \times S,\ (2)\ S \times K,\ (3)\ \Delta_\varphi := \{(x, \varphi(x)) \mid x \in S\},$$

where K is a maximal subgroup of S and $\varphi \in \mathrm{Aut}(S)$.

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3. Let $\Omega := S \times S - \bigcup_{M \in \mathcal{M}_1 \cup \mathcal{M}_2} M = \Omega_1 \times \Omega_2$, where $\Omega_1 = S - \bigcup_{K \times S \in \mathcal{M}_1} K$ and $\Omega_2 = S - \bigcup_{S \times K \in \mathcal{M}_2} K$.

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4. We prove that $\Omega = \emptyset$. Suppose $\Omega \neq \emptyset$. Let $\omega \in \Omega_1$. Then $\{K < S \mid S \times K \in \mathcal{M}_2\} \cup \{\langle \varphi(\omega) \rangle \mid \Delta_\varphi \in \mathcal{M}_3\}$ covers S.

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5. It follows that

$$|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = |\mathcal{M}| = \sigma(S \times S) \leq \sigma(S) \leq |\mathcal{M}_2| + |\mathcal{M}_3|.$$

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This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$.
6. By the Minimal Index Lower Bound $|S| \leq \sigma(G) \leq \sigma(S) \leq |S| - 1$, contradiction.

Let us examine the validity of the conjecture for small values of $\sigma(G)$.

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3	$C_2 \times C_2$
4	$C_3 \times C_3, \mathrm{Sym}(3)$
5	$\mathrm{Alt}(4)$
6	$C_5 \times C_5, D_{10}, AGL(1,5)$
7	$\emptyset$
8	$C_7 \times C_7, D_{14}, 7:3, AGL(1,7)$
9	$AGL(1,8)$
10	$3^2:4, AGL(1,9), \mathrm{Alt}(5)$
11	$\emptyset$
12	$C_{11} \times C_{11}, 11:5, D_{22}, AGL(1,11)$
13	$\mathrm{Sym}(6)$
14	$C_{13} \times C_{13}, D_{26}, 13:3, 13:4, 13:6, AGL(1,13)$

15	$SL(3,2)$
16	$\mathrm{Sym}(5), \mathrm{Alt}(6)$
17	$2^4:5, AGL(1,16)$
18	$C_{17} \times C_{17}, D_{34}, 17:4, 17:8, AGL(1,17)$
19	$\emptyset$
20	$C_{19} \times C_{19}, AGL(1,19), D_{38}, 19:3, 19:6, 19:9$
21	$\emptyset$
22	$\emptyset$
23	M_{11}
24	$C_{23} \times C_{23}, D_{46}, 23:11, AGL(1,23)$
25	$\emptyset$

By using similar techniques it is possible to prove that:

Theorem

Let G be a non-abelian σ-elementary group, and assume that $\sigma(G) \leq 56$. Then G is either affine or almost simple.

Moreover we have $\sigma(A_5 \wr C_2) = 4 \cdot 5 + 6 \cdot 6 + 1 = 57$.

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Moreover we have $\sigma(A_5 \wr C_2) = 4 \cdot 5 + 6 \cdot 6 + 1 = 57$.

Note that $G := A_5 \wr C_2$ is the smallest monolithic group which is neither affine nor almost simple. A minimal covering of this group consists of the socle plus subgroups of the form

$$N_G(M \times M^a)$$

where M varies in the covering of A_5 consisting of the six normalizers of the Sylow 5-subgroups and four point stabilizers, and a varies in A_5.

Let G be a monolithic group with non-abelian socle

$$S_1 \times \ldots \times S_m = S^m,$$

where $S = \mathrm{Alt}(n)$, $n \geq 5$ and $G/\operatorname{soc}(G)$ is cyclic. Let

$$X := N_G(S_1)/C_G(S_1).$$

X is an almost-simple group with socle isomorphic to S, and G embeds in the wreath product $X \wr \mathrm{Sym}(m)$.

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The coverings we prove to be minimal in the following theorems consist of the subgroups containing the socle together with the subgroups of the form

$$N_G(M \times M^{a_2} \times \ldots \times M^{a_m})$$

where $a_1, \ldots, a_m$ vary in X and M varies in a suitable family $\mathcal{M}$ of subgroups of X which covers xS, where $\langle xS \rangle = X/S$. By $\omega(x)$ we denote the number of prime divisors of the integer x.

Theorem (Maróti, G 2010 [7])

Suppose $X = S = \mathrm{Alt}(n)$. Then the following holds.

1. *If* $12 < n \equiv 2 \mod (4)$ *then*

$$\sigma(G) = \omega(m) + \sum_{i=1,\ i \text{ odd}}^{(n/2)-2} \binom{n}{i}^m + \frac{1}{2^m}\binom{n}{n/2}^m.$$

2. *If* $12 < n \not\equiv 2 \mod (4)$ *then*

$$\omega(m) + \frac{1}{2}\sum_{i=1,\ i \text{ odd}}^{n} \binom{n}{i}^m \leq \sigma(G).$$

3. *Suppose n has a prime divisor at most $\sqrt[3]{n}$. Then*

$$\sigma(G) \sim \omega(m) + \min_{\mathcal{M}} \sum_{M \in \mathcal{M}} |S : M|^{m-1} \text{ as } n \to \infty.$$

Theorem (G 2011 [8])

Suppose $X = \mathrm{Sym}(n)$. Then the following holds.

1. *Suppose that $n \geq 7$ is odd and $(n,m) \neq (9,1)$. Then*
$$\sigma(G) = \omega(2m) + \sum_{i=1}^{(n-1)/2} \binom{n}{i}^m .$$
2. *Suppose that $n \geq 8$ is even. Then*
$$\left(\frac{1}{2}\binom{n}{n/2}\right)^m \leq \sigma(G) \leq \omega(2m) + \left(\frac{1}{2}\binom{n}{n/2}\right)^m + \sum_{i=1}^{[n/3]} \binom{n}{i}^m .$$
In particular $\sigma(G) \sim \left(\frac{1}{2}\binom{n}{n/2}\right)^m$ as $n \to \infty$.

Why is the study of monolithic groups so important when attacking the main conjecture?

Why is the study of monolithic groups so important when attacking the main conjecture?

Proposition (Lucchini, Detomi [3], Proposition 16)

Let X be a monolithic primitive group with socle N. If Ω is an arbitrary union of cosets of N in X define $\sigma_\Omega(X)$ to be the smallest number of supplements of N in X needed to cover Ω. Define

$$\sigma^*(X) := \min\{\sigma_\Omega(X) \mid \Omega = \bigcup_i \omega_i N,\ \langle \Omega \rangle = X\}.$$

Let now G be a non-abelian σ-elementary group,

$$G \leq_{subd} X_1 \times \ldots \times X_k.$$

Then

$$\sigma^*(X_1) + \ldots + \sigma^*(X_k) \leq \sigma(G).$$

Let us admire for a second the following inequalities:

$$\sigma^*(X_1) + \ldots + \sigma^*(X_k) \leq \sigma(G) \leq \min\{\sigma(X_1), \ldots, \sigma(X_k)\}.$$

Recall that the X_i's are monolithic groups. From this it is possible to deduce the following partial reduction to the monolithic case.

Proposition

Let G be a σ-elementary group, with socle $N_1 \times \cdots \times N_k$, and $G \leq_{subd} X_1 \times \cdots \times X_k$ as before. Let $i \in \{1, \ldots, k\}$ be such that N_i is non-abelian and $\sigma^*(X_i) \leq \sigma^*(X_j)$ whenever $j \in \{1, \ldots, k\}$ is such that N_j is non-abelian. If

$$\sigma(X_i) < 2\sigma^*(X_i)$$

then $k = 1$, i.e. G is monolithic.

I have a result of the following type.

Theorem

Let G be a primitive monolithic group with non-abelian socle $T^m = T_1 \times \cdots \times T_m$, where $T = T_1 = \cdots = T_m$ is a simple group. Let $L := N_G(T_1)/C_G(T_1)$. L is an almost-simple group with socle isomorphic to T. If there exists a family of proper subgroups of L with some specified properties then $\sigma(G) < 2\sigma^*(G)$.

This is a partial reduction to the almost simple groups. For example using this it is possible to deduce the following.

Theorem

Let G be a non-abelian σ-elementary group, and assume that all minimal sub-normal subgroups of G are alternating groups of large enough even degree. Then G is monolithic.

Let us sketch the proof of the following fact:

Proposition

Let G be a non-abelian σ-elementary group.
If $\sigma(G) \leq 55$ then G is monolithic.

Assume that G has no abelian minimal normal subgroup. Let N_1, N_2 be two non-abelian minimal normal subgroups of G, and let X_1, X_2 be as above. Let $\ell_{X_i}(N_i)$ be the smallest index of a proper supplement of N_i in X_i, i.e. the smallest primitivity degree of X_i. It is possible to prove that $\ell_{X_i}(N_i) \leq \sigma^*(X_i)$, therefore

$$\ell_{X_1}(N_1) + \ell_{X_2}(N_2) \leq \sigma^*(X_1) + \sigma^*(X_2) \leq \sigma(G) \leq 55,$$

so we may assume for example that $\ell_{X_1}(N_1) \leq 27$.
Let us consider only the case in which X_1/N_1 is cyclic. Then it has prime-power order and hence $\sigma(X_1) \leq \sigma^*(X_1) + 1$. This leads to the contradiction

$$\sigma^*(X_1) + \sigma^*(X_2) \leq \sigma(G) \leq \sigma(X_1) \leq \sigma^*(X_1) + 1.$$

Let X be the set of values $\sigma(G)$ where G is a finite non-cyclic group. The following is an open problem:

(*) Is it true that $\mathbb{N} - X$ is infinite?

I obtained the following result:

Proposition

Let Y be the set of values $\sigma(G)$ where G is a finite primitive monolithic group such that $G/\operatorname{soc}(G)$ is cyclic. Then Y has natural density 0.

This is an example of how proving the main conjecture would help answering questions such as (*).

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