Covering finite groups with proper subgroups

Martino Garonzi
PhD student
Supervisor: Andrea Lucchini
University of Padova

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Let G be a non-cyclic finite group. Define:

- a cover of G to be a family $\mathcal{H}$ of proper subgroups of G such that $\bigcup_{H \in \mathcal{H}} H = G$;
- a normal cover of G to be a cover $\mathcal{H}$ of G such that $g^{-1}Hg \in \mathcal{H}$ for every $g \in G$, $H \in \mathcal{H}$;
- a minimal cover of G to be a cover of G of minimal possible size;
- $\sigma(G)$ to be the size of a minimal cover of G;
- a minimal normal cover of G to be a normal cover of G with the smallest possible number of conjugacy classes;
- $\gamma(G)$ to be the number of conjugacy classes of a minimal normal cover of G.

We set $\sigma(G) = \gamma(G) = \infty$ if G is cyclic.

Remark

There always exist minimal (normal) covers consisting of maximal subgroups. In particular, if all maximal subgroups of G are normal, i.e. if G is nilpotent, then $\sigma(G) = \gamma(G)$.

Let G be a finite group.

$\sigma \geq 3$

Since no finite group is the union of two proper subgroups, $\sigma(G) \geq 3$. For example $\sigma(C_2 \times C_2) = 3$.

According to a theorem of Scorza, a group G has $\sigma(G) = 3$ if and only if there exists $N \trianglelefteq G$ such that $G/N \cong C_2 \times C_2$.

$\gamma \geq 2$

Since no finite group is the union of one single conjugacy class of proper subgroups, $\gamma(G) \geq 2$. For example $\gamma(S_3) = 2$.

More in general, if G is a solvable group such that G/G' is cyclic then $\gamma(G) = 2$.

We will now deal with the covering number, $\sigma(G)$. Let us start with a couple of easy remarks. Let G be a non-cyclic group.

- If $\mathcal{H}$ is a cover of G then $\sigma(G) \leq |\mathcal{H}|$.
- If G is any non-cyclic group then $\sigma(G) < |G|$. Indeed, G is always covered by its non-trivial cyclic subgroups.
- If $N \trianglelefteq G$ and $\mathcal{H}$ is a cover of G/N then letting $\pi : G \to G/N$ be the canonical projection, the family

  $$\{\pi^{-1}(H) \ : \ H \in \mathcal{H}\}$$

  is a cover of G of size $|\mathcal{H}|$. It follows that

  $$\sigma(G) \leq \sigma(G/N).$$

- $\sigma(G) = \sigma(G/\Phi(G))$.

Solvable groups were considered by Tomkinson in 1994.

Theorem (Tomkinson, [4])

Let G be a finite solvable group. Then

$$\sigma(G) = |H/K| + 1$$

where H/K is the smallest chief factor of G with more than one complement in G/K.

In particular, if G is solvable then $\sigma(G) - 1$ is always a prime power. This is false for non-solvable groups, for example $\sigma(S_6) - 1 = 12$.

Moreover, there exist non-solvable groups G such that $\sigma(G) - 1$ is a prime power, for example $\sigma(A_5) - 1 = 9$.

Proposition (Maximal Complement Argument)

If a maximal subgroup M of G complements a non-solvable normal subgroup N of G then $\sigma(G) = \sigma(G/N)$.

Proposition

Let G be a group. Then G is simple if and only if

$$\Delta_G := \{(g, g) \mid g \in G\}$$

is a maximal subgroup of $G \times G$.

$\sigma(S \times S) = \sigma(S)$

Let S be a non-abelian simple group. Note that $S \times \{1\}$ is a non-solvable normal subgroup of $S \times S$ and that Δ_S is a complement of $S \times \{1\}$. It follows from the Maximal Complement Argument that

$$\sigma(S \times S) = \sigma(S).$$

Definition

A finite group G is said to be "primitive" if it admits a core-free maximal subgroup.

Denote $\mathrm{soc}(G)$ the socle of the group G, the subgroup generated by the minimal normal subgroups of G. If G is a primitive group, $\mathrm{soc}(G)$ can only be of one of the following types:

- (Type I) An abelian minimal normal subgroup of G;
- (Type II) A non-abelian minimal normal subgroup of G;
- (Type III) The product of exactly two non-abelian minimal normal subgroups of G.

In primitive groups of type I or III the minimal normal subgroups have a common complement, which is a maximal subgroup.

Remark

Note that if G is a primitive group of type III then the Maximal Complement argument implies that $\sigma(G) = \sigma(G/N)$ for every minimal normal subgroup N of G.

Theorem (Scorza 1926)

Let G be a finite group. $\sigma(G) = 3$ if and only if there exists $N \trianglelefteq G$ such that $G/N \cong C_2 \times C_2$.

This is a finiteness condition.

Definition (σ-elementary, γ-elementary groups)

A non-cyclic finite group G is called σ-elementary (resp. γ-elementary) if $\sigma(G) < \sigma(G/N)$ (resp. $\gamma(G) < \gamma(G/N)$) whenever $\{1\} \neq N \trianglelefteq G$.

Scorza's theorem says that the only σ-elementary group G with $\sigma(G) = 3$ is $C_2 \times C_2$.

Proposition

For every integer n there are only finitely many σ-elementary groups G with $\sigma(G) = n$.

Let us give some examples.

Examples

- Non-abelian simple groups are σ-elementary.
- If p is a prime, $C_p \times C_p$ is σ-elementary.
- The dihedral group D_{2p} of order $2p$ is σ-elementary.
- If S is a non-abelian simple group and $n \geq 1$ is an integer such that S^n is σ-elementary then $n = 1$. Indeed, $\sigma(S^n) = \sigma(S)$.
- If $n \geq 3$ is an integer and $n \neq 4$ then S_n is σ-elementary: its only proper quotient is C_2. S_4 is not σ-elementary: it admits S_3 as homomorphic image and $\sigma(S_4) = \sigma(S_3) = 4$.
- If G/N is cyclic whenever $\{1\} \neq N \trianglelefteq G$ then G is σ-elementary. The converse is true for solvable groups but false in general. An example is $I \rtimes A_p$ where $I = \{(x_1, \ldots, x_p) \in \mathbb{F}_2^p : \ \sum_{i=1}^p x_i = 0\}$ and p is a prime not of the form $\frac{q^n-1}{q-1}$ with q a prime power.

Let us list some properties of σ-elementary groups.

Proposition

Let G be a σ-elementary group. Then:

- *the Frattini subgroup of G is trivial:* $\Phi(G) = \{1\}$;
- *either G is abelian or* $Z(G) = \{1\}$;
- *if G is nilpotent then* $G \cong C_p \times C_p$ *for some prime p;*
- *G has at most one abelian minimal normal subgroup;*
- *any proper solvable quotient of G is cyclic.*

Theorem (G 2009)

All σ-elementary groups G with $\sigma(G) \leq 25$ are known.

3	$C_2 \times C_2$
4	$C_3 \times C_3, \mathrm{Sym}(3)$
5	$\mathrm{Alt}(4)$
6	$C_5 \times C_5, D_{10}, AGL(1,5)$
7	$\emptyset$
8	$C_7 \times C_7, D_{14}, 7:3, AGL(1,7)$
9	$AGL(1,8)$
10	$3^2:4, AGL(1,9), \mathrm{Alt}(5)$
11	$\emptyset$
12	$C_{11} \times C_{11}, 11:5,$ $D_{22}, AGL(1,11)$
13	$\mathrm{Sym}(6)$
14	$C_{13} \times C_{13}, D_{26}, 13:3,$ $13:4, 13:6, AGL(1,13)$

15	$SL(3,2)$
16	$\mathrm{Sym}(5), \mathrm{Alt}(6)$
17	$2^4:5, AGL(1,16)$
18	$C_{17} \times C_{17}, D_{34}, 17:4,$ $17:8, AGL(1,17)$
19	$\emptyset$
20	$C_{19} \times C_{19}, AGL(1,19),$ $D_{38}, 19:3, 19:6, 19:9$
21	$\emptyset$
22	$\emptyset$
23	M_{11}
24	$C_{23} \times C_{23}, D_{46},$ $23:11, AGL(1,23)$
25	$\emptyset$

In my Ph.D thesis I mainly deal with the following conjecture.

Conjecture (A. Lucchini, E. Detomi)

Any non-abelian σ-elementary group is ***monolithic****, i.e. admits only one minimal normal subgroup.*

Theorem (Lucchini A., G 2010 [5])

Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_1. *In this case* $\sigma(G) = \sigma(H_1)$.
2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_2. *In this case* $\sigma(G) = \sigma(H_2)$.
3. *There exist* $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ *with* $H_1/N_1 \cong H_2/N_2 \cong C_p$ *and* $\mathcal{M}$ *consists of the maximal subgroups of* $H_1 \times H_2$ *containing* $N_1 \times N_2$. *In this case* $\sigma(G) = p + 1$.

Definition (G-groups)

Let G be a group. A G-group is a group A endowed with a homomorphism $G \to \mathrm{Aut}(A)$. The G-group A is said to be irreducible if it does not have proper G-invariant non-trivial subgroups.

Example

The conjugation action of G on $N \trianglelefteq G$ endow N with the structure of G-group. N is irreducible if and only if it is a minimal normal subgroup of G. The chief factors of G are irreducible G-groups.

Definition (G-homomorphism)

Let A, B be two G-groups. A G-homomorphism $\varphi : A \to B$ is a G-invariant group homomorphism.

Example

Let V be a vector space and let $H \leq GL(V)$. Let $G = (V \times V) \rtimes H$ with $(v_1, v_2)^h = (v_1^h, v_2^h)$ whenever $v_1, v_2 \in V$, $h \in H$. Then $V \times \{0\}$ and $\{0\} \times V$ are G-isomorphic via $(v, 0) \mapsto (0, v)$.

Example

Let S be a non-abelian simple group, and let $G = S \times S$. Then $S \times \{1\}$ and $\{1\} \times S$ are not G-isomorphic. Indeed, their centralizers are different. However, they are *G-equivalent*.

Definition (G-equivalent G-groups)

Let A, B be two G-groups. A, B are said to be G-equivalent (written $A \sim_G B$) if there exist isomorphisms

$$\varphi : A \longrightarrow B \,, \quad \Phi : G \ltimes A \longrightarrow G \ltimes B$$

such that the following diagram commutes:

$$\begin{tikzcd}
\{1\} \arrow[r] & A \arrow[r] \arrow[d, "\varphi"] & G \ltimes A \arrow[r] \arrow[d, "\Phi"] & G \arrow[r] \arrow[d, equal] & \{1\} \\
\{1\} \arrow[r] & B \arrow[r] & G \ltimes B \arrow[r] & G \arrow[r] & \{1\}
\end{tikzcd}$$

P. Jiménez-Seral and J. P. Lafuente [SerLaf] proved the following very interesting and useful result:

Proposition

Let A, B be two chief factors of a group G, with the structure of G-groups given by conjugation. They are G-equivalent if and only if

- *either A, B are G-isomorphic between them,*
- *or A, B are G-isomorphic to the two minimal normal subgroups of a primitive epimorphic image of type III of G.*

Now, recall the Maximal Complement Argument and deduce that:

Proposition

Let G be a non-abelian σ-elementary group. Then the minimal normal subgroups of G are pairwise non-G-equivalent.

Let us state this in another way. Let $A = H/K$ be a chief factor of a group G. Recall that H/K is called “Frattini” if $H/K \subseteq \Phi(G/K)$.

Denote by:

- $I_G(A)$ the set of elements of G which induce by conjugation an inner automorphism of A;
- $R_G(A)$ the intersection of the normal subgroups N of G contained in $I_G(A)$ with the property that $I_G(A)/N$ is non-Frattini and G-equivalent to A.

The quotient $I_G(A)/R_G(A)$ is called the A-**crown** of G.

Proposition

Let G be a non-abelian σ-elementary group, and let N be a minimal normal subgroup of G. Then $X_N := G/R_G(N)$ is primitive monolithic, and its minimal normal subgroup is isomorphic to N. X_N will be called the ***primitive monolithic group associated to*** *N.*

Let G be a non-abelian σ-elementary group, and write

$$\mathrm{soc}(G) = N_1 \times \cdots \times N_t,$$

with $N_1, \dots, N_t$ minimal normal subgroups of G.
Let $X_i := G/R_G(N_i)$ for $i = 1, \dots, t$.

Theorem

The canonical homomorphism $G \to X_1 \times \cdots \times X_t$ is injective. In particular, G is a subdirect product of primitive monolithic groups.

The conjecture is that $t = 1$ always.

Definition (Sigma star)

Let X be a primitive monolithic group, and let N be its unique minimal normal subgroup. If Ω is an arbitrary union of cosets of N in X define $\sigma_{\Omega}(X)$ to be the smallest number of supplements of N in X needed to cover Ω. If $\Omega = \{Nx\}$ we will write $\sigma_{Nx}(X)$ instead of $\sigma_{\{Nx\}}(X)$. Define

$$\sigma^*(X) := \min\{\sigma_{\Omega}(X) \mid \Omega = \bigcup_i N\omega_i,\ \langle \Omega \rangle = X\}.$$

Let $\ell_X(N)$ be the smallest index of a proper supplement of N in X. Then

$$\ell_X(N) \leq \sigma^*(X).$$

Proposition (Lucchini, Detomi [3] Proposition 16)

Let H be a non-abelian σ-elementary group with socle $N_1 \times \cdots \times N_\ell$,

$$H \leq_{subd} X_1 \times \ldots \times X_\ell$$

as above. For $i = 1, \ldots, \ell$ let $\ell_{X_i}(N_i)$ be the smallest primitivity degree of X_i, i.e. the smallest index of a proper supplement of N_i in X_i. Then $\ell_{X_i}(N_i) \leq \sigma^*(X_i)$ for $i = 1, \ldots, \ell$ and

$$\sum_{i=1}^{\ell} \ell_{X_i}(N_i) \leq \sum_{i=1}^{\ell} \sigma^*(X_i) \leq \sigma(H).$$

Let us admire the following inequalities:

$$\sigma^*(X_1) + \ldots + \sigma^*(X_k) \leq \sigma(G) \leq \min\{\sigma(X_1), \ldots, \sigma(X_k)\}.$$

Proposition (Reduction)

Let G be a σ-elementary group, with socle $N_1 \times \cdots \times N_k$, and $G \leq_{subd} X_1 \times \cdots \times X_k$ as before. Let $i \in \{1, \ldots, k\}$ be such that N_i is non-abelian and $\sigma^*(X_i) \leq \sigma^*(X_j)$ whenever $j \in \{1, \ldots, k\}$ is such that N_j is non-abelian. If

$$\sigma(X_i) < 2\sigma^*(X_i)$$

then $k = 1$, i.e. G is monolithic.

Corollary

If $\sigma(X) < 2\sigma^*(X)$ for every primitive monolithic group X with non-abelian socle then every non-abelian σ-elementary group is monolithic.

Let G be a primitive monolithic group with non-abelian socle. In attempting to prove the inequality $\sigma(G) < 2\sigma^*(G)$ our strategy is the following:

- find good upper bounds for $\sigma(G)$,
- find good lower bounds for $\sigma^*(G)$.

Let G be a primitive monolithic group with non-abelian socle.

- Let N be the socle of G and write $N = S_1 \times \cdots \times S_k$ where $S_1 \cong \cdots \cong S_m \cong S$ are non-abelian isomorphic simple groups.
- Let $X := N_G(S_1)/C_G(S_1)$. X is an almost-simple group with socle isomorphic to S.
- Let $\rho : G \to \mathrm{Sym}(m)$ be the homomorphism induced by the conjugation action of G on $\{S_1, \dots, S_m\}$.
- Let $K := \rho(G)$. K is a transitive subgroup of $\mathrm{Sym}(m)$. It is well known that G can be embedded in the wreath product $X \wr K$.

Let L be the subgroup of X generated by the following set:

$$S \cup \{x_1 \cdots x_m \mid \exists k \in K : \ (x_1, \dots, x_m)k \in G\}.$$

Let T be a normal subgroup of X containing S and contained in L with the property that L/T has prime order if $L \neq S$, and $T = L$ if $L = S$.

Theorem (An upper bound)

Assume that X/S is abelian. Let $\mathcal{M}$ be a set of maximal subgroups of X supplementing S and such that $\bigcup_{M \in \mathcal{M}} M$ contains a coset $xS \subseteq L$ with the property that $\langle x, T \rangle = L$. Then

$$\sigma(G) \leq 2^{m-1} + \sum_{M \in \mathcal{M}} |S : S \cap M|^{m-1}.$$

Proof.

If $L \neq S$ let

$$R := \{(x_1, \ldots, x_m)k \in G \mid x_1 \cdots x_m \in T\}.$$

Since X/S is abelian, $R < G$.

- Every element $(x_1, \ldots, x_m)\delta \in G$ such that δ is an m-cycle and $\langle x_1 x_{\delta(1)} \cdots x_{\delta^{m-1}(1)}, T \rangle = L$ belongs to a subgroup of G of the form

  $$N_G((M \cap S) \times (M \cap S)^{a_2} \times \cdots \times (M \cap S)^{a_m})$$

  where $M \in \mathcal{M}$ and $a_2, \ldots, a_m \in S$.
- Every element $(x_1, \ldots, x_m)\delta \in G$ such that $x_{\delta(1)} \cdots x_{\delta(m)} \in T$ for some $\delta \in \mathrm{Sym}(m)$ belongs to R.
- Every element $(x_1, \ldots, x_m)\delta \in G$ such that δ is not an m-cycle belongs to one of $2^{m-1} - 1$ intransitive subgroups of K.

We deduce $\sigma(G) \leq \sum_{M \in \mathcal{M}} |S : S \cap M|^{m-1} + 1 + (2^{m-1} - 1)$. $\square$

In many cases this estimate is not far from the true value of $\sigma(G)$.

Theorem

Let G be a non-abelian σ-elementary group such that $\sigma(G) \leq 56$. Then G admits a unique minimal normal subgroup. Moreover, this minimal normal subgroup is either abelian or simple. Moreover

$$\sigma(A_5 \wr C_2) = 1 + 4 \cdot 5 + 6 \cdot 6 = 57,$$

and the unique minimal normal subgroup of $A_5 \wr C_2$ is $A_5 \times A_5$.

A minimal cover of $A_5 \wr C_2$ consists of the socle $A_5 \times A_5$ and the subgroups of the form $N_{A_5 \wr C_2}(M \times M^a)$ where $M \in \mathcal{M}$ and $a \in A_5$, where $\mathcal{M}$ is a minimal cover of A_5 consisting of four point stabilizers (which have index 5) and the six normalizers of the Sylow 5-subgroups (which have index 6).

Theorem (Maróti, [8])

Let $n > 1$ be an integer.

- *If n is odd then*
  - $\sigma(\mathrm{Alt}(n)) \sim \frac{n!}{(n/p)!^p p!}$ *if n is not a prime and p is the smallest prime divisor of n.*
  - $(p-2)! \leq \sigma(\mathrm{Alt}(p)) \leq (p-2)! + \sum_{i=1}^{[p/3]} \binom{p}{i}$ *for infinitely many primes p.*
- *If $n > 2$ is even then $\sigma(\mathrm{Alt}(n)) \sim 2^{n-2}$.*
- *If n is odd and $n \neq 9$ then $\sigma(\mathrm{Sym}(n)) = 2^{n-1}$.*
- *If $n > 2$ is even then $\sigma(\mathrm{Sym}(n)) \sim \frac{1}{2}\binom{n}{n/2}$.*

Let G, N, S, X be as above. Suppose G/N is cyclic.

Theorem (Maróti, G 2010 [7])

Suppose $X = S = \mathrm{Alt}(n)$. Then the following holds.

1. *If $12 < n \equiv 2 \mod (4)$ then*

$$\sigma(G) = \omega(m) + \sum_{i=1,\ i\ odd}^{(n/2)-2} \binom{n}{i}^m + \frac{1}{2^m}\binom{n}{n/2}^m.$$

2. *If $12 < n \not\equiv 2 \mod (4)$ then*

$$\omega(m) + \frac{1}{2}\sum_{i=1,\ i\ odd}^{n} \binom{n}{i}^m \leq \sigma(G).$$

3. *Suppose n has a prime divisor at most $\sqrt[3]{n}$. Then*

$$\sigma(G) \sim \omega(m) + \min_{\mathcal{M}} \sum_{M \in \mathcal{M}} |S : M|^{m-1} \text{ as } n \to \infty.$$

Theorem (G 2011 [9])

Suppose $X = \mathrm{Sym}(n)$. *Then the following holds.*

1. *Suppose that* $n \geq 7$ *is odd and* $(n, m) \neq (9, 1)$. *Then*

$$\sigma(G) = \omega(2m) + \sum_{i=1}^{(n-1)/2} \binom{n}{i}^m.$$

2. *Suppose that* $n \geq 8$ *is even. Then*

$$\left(\frac{1}{2}\binom{n}{n/2}\right)^m \leq \sigma(G) \leq \omega(2m) + \left(\frac{1}{2}\binom{n}{n/2}\right)^m + \sum_{i=1}^{[n/3]} \binom{n}{i}^m.$$

In particular $\sigma(G) \sim \left(\frac{1}{2}\binom{n}{n/2}\right)^m$ *as* $n \to \infty$.

Typically the way we obtain lower bounds is via the following notion, introduced by Maróti in [8].

Definition (Definite unbeatability)

Let X be a group. Let $\mathcal{H}$ be a set of proper subgroups of X, and let $\Pi \subseteq X$. Suppose that the following four conditions hold for $\mathcal{H}$ and Π.

1. $\Pi \cap H \neq \emptyset$ *for every* $H \in \mathcal{H}$;
2. $\Pi \subseteq \bigcup_{H \in \mathcal{H}} H$;
3. $\Pi \cap H_1 \cap H_2 = \emptyset$ *for every* $H_1 \neq H_2$ *in* $\mathcal{H}$;
4. $|\Pi \cap K| \leq |\Pi \cap H|$ *for every* $H \in \mathcal{H}$ *and* $K < X$ *with* $K \notin \mathcal{H}$.

Then $\mathcal{H}$ is said to be definitely unbeatable on Π.

For $\Pi \subseteq X$ let $\sigma_X(\Pi)$ be the least cardinality of a family of proper subgroups of X whose union contains Π.

Lemma

If $\mathcal{H}$ is definitely unbeatable on Π then $|\mathcal{H}| = \sigma_X(\Pi) \leq \sigma(X)$.

A. Maróti used this notion to prove bounds for the covering number of alternating and symmetric groups.

Symmetric groups of odd degree

Let $n \geq 11$ be an odd integer, and let $G := \mathrm{Sym}(n)$.

- Let $\mathcal{H}$ be the family consisting of $\mathrm{Alt}(n)$ and the maximal intransitive subgroups of $\mathrm{Sym}(n)$.
- Let Π be the subset of $\mathrm{Sym}(n)$ consisting of the permutations which are product of at most two disjoint cycles.

Then $\mathcal{H}$ is a cover of $\mathrm{Sym}(n)$ which is definitely unbeatable on Π, hence

$$\sigma(\mathrm{Sym}(n)) = |\mathcal{H}| = 2^{n-1}.$$

Theorem (A lower bound)

Let G, N, X, S be as above. Suppose that G/N is cyclic, generated by γN for $\gamma \in G$. Let $x \in X$ be such that $X = \langle x, S \rangle$. Without loss of generality assume that $\rho(\gamma) = \delta = (1, \ldots, m)$ and that $\gamma = (1, \ldots, 1, x)\delta$. Let $\mathcal{X}$ be a family of maximal subgroups of X supplementing S and let $\mathcal{M} := \{V \cap S \mid V \in \mathcal{X}\}$, $\Pi \subseteq xS$. Denote by $\sigma_{N\gamma}(G)$ the smallest number of supplements of N in G needed to cover $N\gamma$. If $\mathcal{M}$ satisfies some "unbeatability conditions" on Π then

$$\sum_{M \in \mathcal{M}} |S : M|^{m-1} \leq \sigma_{N\gamma}(G) \leq \sigma(G).$$

Remark

We would like to say that $\sigma_{N\gamma}(G) \leq \sigma^*(G)$ but it is not clear whether this should always hold true: the minimum in the definition of σ^* could be realized by a family of generators which does not include $N\gamma$. Let $X := \mathrm{Aut}(PSL(2, 27))$. Is it true that $\sigma(X) < 2\sigma^*(X)$?

Let G, N, S, X be as above, with G/N not necessarily cyclic, and X/S abelian. Assume that for every $x \in L$ such that $\langle T, x \rangle = L$ there exists a family $\mathcal{M}_x$ of maximal subgroups of X supplementing S which satisfies some "unbeatability conditions" on some $\Pi_x \subset xS$.

Lemma

There exists $x \in L$ such that $\langle x, T \rangle = L$ and $\sigma^*(G) \geq \sum_{M \in \mathcal{M}_x} |S : M|^{m-1}$.

Proof.

Fix a family Ω of generators of $G/\operatorname{soc}(G)$ such that $\sigma_\Omega(G) = \sigma^*(G)$. By definition of L, Ω must contain a coset Ng such that, writing $g = (x_1, \ldots, x_m)k$ with $k \in K$, $\langle T, x_1 \cdots x_m \rangle = L$. Let $x := x_1 \cdots x_m$. By some technical reasons related to the unbeatability of $\mathcal{M}_x$, we may assume that k is an m-cycle. We have

$$\sigma^*(G) \geq \sigma_{Ng}(G) \geq \sigma_{Ng}(N\langle g \rangle) \geq \sum_{M \in \mathcal{M}_x} |S : M|^{m-1}.$$

□

Theorem

Let G, N, X, S be as above. Assume that X/S is abelian. Suppose that if $m \geq 2$ then for every $x \in X$ such that $L = \langle T, x \rangle$ there exist two families $\mathcal{M}_x$, $\mathcal{J}_x$ of maximal subgroups of X supplementing S such that:

- *$\mathcal{M}_x$ satisfies some unbeatability conditions on some $\Pi_x \subseteq xS$;*
- *$\mathcal{M}_x \cup \mathcal{J}_x$ covers xS;*
- $\sum_{M \in \mathcal{J}_x} |S : M|^{m-1} + 2^{m-1} < \sum_{M \in \mathcal{M}_x} |S : M|^{m-1}$.

Then there exists $x \in L$ such that $\langle x, T \rangle = L$ and

$$\sigma(G) < 2^{m-1} + \sum_{M \in \mathcal{M}_x \cup \mathcal{J}_x} |S : M|^{m-1} \leq 2 \cdot \sigma^*(G).$$

Theorem

Let G, N, X, S be as above. Suppose that $S = \mathrm{Alt}(n)$ for some integer $n \geq 5$. Suppose that $m \geq 3$ and $m \neq 4$. Then $\sigma(G) < 2\sigma^*(G)$ in each of the following cases:

1. *$n \geq 15$ is odd and $X = \mathrm{Sym}(n)$.*
2. *$n \geq 30$ is even and $X = \mathrm{Sym}(n)$.*
3. *$n \geq 5$ is odd with a prime divisor at most $\sqrt[4]{n}$ and $X = \mathrm{Alt}(n)$.*
4. *$n \geq 20$ is even and $X = \mathrm{Alt}(n)$.*

The following result follows as a corollary.

Theorem

Let H be a non-abelian σ-elementary group, and suppose that all the non-abelian minimal subnormal subgroups of H are either isomorphic to M_{11} or to $\mathrm{Alt}(n)$ where $n \geq 30$ admits a prime divisor smaller than or equal to $\sqrt[4]{n}$. Then H is monolithic.

Let G be a finite non-cyclic group. Define $\gamma(G)$ to be the smallest number of conjugacy classes of subgroups of G needed to cover G.

$\gamma \geq 2$

Since no finite group is the union of one single conjugacy class of proper subgroups, $\gamma(G) \geq 2$. For example $\gamma(S_3) = 2$. More in general, if G is any solvable group such that G/G' is cyclic then $\gamma(G) = 2$.

Proposition

Let $f(X) \in \mathbb{Z}[X]$ be a separable monic polynomial without integer roots such that $f(x) \equiv 0 \mod (p)$ admits an integer solution for every prime p. Then the irreducible factors of $f(X)$ in $\mathbb{Z}[X]$ are at least $\gamma(G)$, where G is the Galois group of $f(X)$ over $\mathbb{Q}$.

Theorem

Let H_1, H_2 be two non-trivial groups, and suppose that $|H_1/H_1'|$ and $|H_2/H_2'|$ are coprime. Then

$$\gamma(H_1 \times H_2) = \min(\gamma(H_1), \gamma(H_2)).$$

Let us sketch the proof of this result. In order not to get lost in technicalities, let us assume that $H_1 = H_2 = S$ is a non-abelian simple group. The argument we use is the same to prove the analogous result for $\sigma(H_1 \times H_2)$.

1. We know that the maximal subgroups of $S \times S$ are of the following three types:

$$(1)\ K \times S,\ (2)\ S \times K,\ (3)\ \Delta_\varphi := \{(x, \varphi(x)) \mid x \in S\},$$

where K is a maximal subgroup of S and $\varphi \in \mathrm{Aut}(S)$.

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2. Let $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ be a minimal cover of $S \times S$, where $\mathcal{M}_i$ consists of subgroups of type (i). Let $\mathcal{M}_i^*$ denote the set of conjugacy classes of members of $\mathcal{M}_i$.

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3. Let $\Omega := S \times S - \bigcup_{M \in \mathcal{M}_1 \cup \mathcal{M}_2} M = \Omega_1 \times \Omega_2$, where $\Omega_1 = S - \bigcup_{K \times S \in \mathcal{M}_1} K$ and $\Omega_2 = S - \bigcup_{S \times K \in \mathcal{M}_2} K$.

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4. We prove that $\Omega = \emptyset$. Suppose $\Omega \neq \emptyset$. Let $\omega \in \Omega_1$. Then $\{K < S \mid S \times K \in \mathcal{M}_2\} \cup \{\langle \varphi(\omega) \rangle \mid \Delta_\varphi \in \mathcal{M}_3\}$ covers S.

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5. It follows that

$$|\mathcal{M}_1^*| + |\mathcal{M}_2^*| + |\mathcal{M}_3^*| = |\mathcal{M}^*| = \gamma(S \times S) \leq \gamma(S) \leq |\mathcal{M}_2^*| + |\mathcal{M}_3^*|.$$

This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$.

1. We know that the maximal subgroups of $S \times S$ are of the following three types:

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   This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$.
6. This is a contradiction: no member of $\mathcal{M}_3$ contains elements the form $(s, 1) \in S \times S$ with $s \neq 1$.

Theorem

Let G be a non-cyclic finite group, and let p be the largest prime divisor of $|G|$. There exists an absolute constant C such that $\gamma(G) \leq Cp\log(p)$.

Using induction on $|G|$ we may assume that all proper quotients of G are cyclic. G turns out to be a primitive monolithic group with non-abelian socle. Let N be the socle of G, $N = S_1 \times \cdots \times S_m$, $S_1 \cong \cdots \cong S_m \cong S$ a non-abelian simple group, $X = N_G(S_1)/C_G(S_1)$. Using the upper bound we discussed previously we can show that

$$\gamma(G) \leq \omega(|G/N|) + \gamma(X).$$

This basically reduces the problem to almost-simple groups, which can be handled directly.

CONJECTURE (MARÓTI, G)

Let G be a non-cyclic finite group, and let p be the largest prime divisor of $|G|$. Then $\gamma(G) \leq p+1$.

We proved this for solvable groups.

Let $\pi(n)$ denote the number of prime numbers at most n. We reduced Conjecture 2 to the almost-simple case in the following sense.

PROPOSITION (MARÓTI, G)

Suppose that whenever X is an almost-simple group, $S \leq X \leq Aut(S)$, with X/S cyclic, and p denotes the largest prime divisor of $|X|$ we have $\gamma(X) \leq p+1-\pi(p)$. Then Conjecture 2 holds.

We did this for alternating and sporadic groups.

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