COVERS AND NORMAL COVERS OF FINITE GROUPS

MARTINO GARONZI AND ANDREA LUCCHINI

Abstract. For a finite non cyclic group G, let $\gamma(G)$ be the smallest integer k such that G contains k proper subgroups $H_1, \dots, H_k$ with the property that every element of G is contained in H_i^g for some $i \in \{1, \dots, k\}$ and $g \in G$. We prove that if G is a noncyclic permutation group of degree n, then $\gamma(G) \leq (n+2)/2$. We then investigate the structure of the groups G with $\gamma(G) = \sigma(G)$ (where $\sigma(G)$ is the size of a minimal cover of G) and of those with $\gamma(G) = 2$.

1. Introduction

Let G be a non-cyclic finite group. A collection $\mathcal{C}$ of proper subgroups of G is a cover of G if $\cup_{H \in \mathcal{C}} H = G$; it is a minimal cover if $|\mathcal{C}|$ is as small as possible. A normal cover has the property that $H^g \in \mathcal{C}$ for all $H \in \mathcal{C}$, $g \in G$. The covering number of G, denoted $\sigma(G)$, is the size of a minimal cover, and the normal covering number, denoted $\gamma(G)$, is the smallest number of conjugacy classes of subgroups in a normal cover of G. If G is cyclic we pose $\sigma(G) = \gamma(G) = \infty$, with the convention that $n < \infty$ for every integer n.

The first question on finite covers was posed by Scorza in 1926 [28] who settled the question which groups are the union of three proper subgroups. Cohn's 1994 paper [12] brought Scorza's original question again to the forefront of research in group theory and got the attention of many researchers (see for example [3], [6], [8], [12], [16], [21], [22] [23], [26], [29]).

The study of normal covers is an off-shoot of the finite covering problem and relatively new ([5], [7], [10], [11]). The first available results seem to indicate that the arguments used to investigate $\sigma(G)$ fail when applied to the study of $\gamma(G)$ and this second invariant seems more difficult to be estimated. For example, by the main result in Tomkinson's paper [29, Theorem 2.2.], if G is a finite soluble group then $\sigma(G) = |W| + 1$, where W is a chief factor of G with least order among chief factors of G with multiple complements; in particular $\sigma(G) - 1$ is a prime power. A similar formula for $\gamma(G)$ when G is soluble is missing and in any case $\gamma(G)$ has a surprisingly different behavior: for every $n \geq 2$, there exists a finite soluble group G with $\gamma(G) = n$ [14].

In this paper we address two questions related to the behavior of $\gamma(G)$. We study the groups G with $\sigma(G) = \gamma(G)$ and those with $\gamma(G) = 2$.

In order to deal with the first question we start recalling a lower bound for $\sigma(G)$, proved by Cohn. Let $\mu(G)$ be the least integer k such that G has more than one maximal subgroup of index k. Then we have:

Proposition 1 (Cohn [12], Corollary after Lemma 8). *If G is a finite group, then $\sigma(G) \geq \mu(G) + 1$.*

1991 *Mathematics Subject Classification.* 20F05.

Research partially supported by MIUR-Italy via PRIN Group theory and applications.

On the other hand it turns out that the same value $\mu(G)+1$ represents an upper bound for $\gamma(G)$. Indeed we prove:

Proposition 2. *If G is a finite group, then $\gamma(G) \leq \mu(G)+1$. Moreover $\gamma(G) = \mu(G)+1$ if and only if $\mu(G)$ is a prime, G contains at least two normal subgroups of index $\mu(G)$ and $\gamma(G) = \gamma(G/G')$.*

Corollary 3. *Suppose that G is a noncyclic finite group. If $\sigma(G) = \gamma(G)$, then $p = \sigma(G)-1$ is a prime and G has a minimal cover consisting of normal subgroups of index p. In particular $\gamma(G) = \gamma(G/G') = \sigma(G/G')$.*

Proposition 2 is a consequence of a more general result, bounding $\gamma(G)$ when G is a noncyclic permutation group.

Theorem 4. *If G is a noncyclic permutation group of degree n, then $\gamma(G) \leq (n+2)/2$.*

We may complete the previous statement noticing that the upper bound is reached infinitely often: if p is any prime and G is a subgroup of $\mathrm{Sym}(2p)$ generated by two disjoint p-cycles then $G \cong C_p \times C_p$ so $\gamma(G) = p+1 = (2p+2)/2$.

It is interesting to study the groups for which σ or γ takes the smallest possible value. No finite group can be expressed as a union of two proper subgroups or as a union of conjugates of a proper subgroup; so $\sigma(G) \geq 3$ and $\gamma(G) \geq 2$. Scorza's Theorem says that $\sigma(G) = 3$ if and only if G is the union of three subgroups of index 2; this is equivalent to say that if $\sigma(G) = 3$ but $\sigma(G/N) > 3$ for every nontrivial normal subgroup N of G, then $G \cong C_2 \times C_2$. One could expect that, in a similar way, there are only few groups G such that $\gamma(G) = 2$ but $\gamma(G/N) > 2$ for every nontrivial normal subgroup N of G, however it is not precisely like that. Indeed we will give many different examples of groups G with $\gamma(G) = 2$. However some restrictions on the structure of these groups can be proved.

Theorem 5. *Assume that $\gamma(G) = 2$ but $\gamma(G/N) > 2$ for every nontrivial normal subgroup N of G. Then G has a unique minimal normal subgroup N. Moreover if G is covered with the conjugates of two maximal subgroups then either one of these two subgroups contains $\mathrm{soc}(G)$ or G is an almost simple group.*

On the other hand, as we will recall in Section 3, there are several different examples of almost simple groups G with $\gamma(G) = 2$. Moreover in the same section we will construct infinite families of examples of groups G with a unique minimal normal subgroup N, covered by the conjugates of two maximal subgroups H and K, in which H contains N but the intersection of K with N has different behaviors: trivial (when N is abelian), of diagonal type, of product type. The conclusion is that there are several different ways in which a finite group can be covered by the conjugates of two proper subgroups and a complete classification is quite difficult.

Acknoledgements. We would like to thank Attila A. Maróti and Pablo Spiga for fruitful discussions and valuable and helpful comments.

2. Groups G with $\gamma(G) = \sigma(G)$

We start this section with some preliminary results.

Lemma 6. *Let G be a finite soluble noncyclic group such that G/G' is cyclic. Then* $\gamma(G) = 2$.

Proof. We make induction on the order of G. Since $\gamma(G) \leq \gamma(G/N)$ for every normal subgroup N of G, we may assume that every proper quotient of G is cyclic. Together with the fact that G/G' is cyclic, this implies that G contains a unique minimal normal subgroup, say N, and N has a cyclic complement M. Moreover M has precisely $|N|$ conjugates in G. Let K be a conjugate of M in G, with $K \neq M$. Since M is cyclic, so is K and $K \cap M \trianglelefteq \langle K, M \rangle = G$. Since $K \cap M \not\supseteq N$ it follows $K \cap M = 1$. The $|N|$ conjugates of M together with N cover in total

$$|N| + (|G : N| - 1)|N| = |G|$$

elements of G. It follows that $\gamma(G) = 2$. □

Denote by $m(G)$ the smallest index of a proper subgroup of G. The following consequence of the classification of the finite simple groups plays a crucial role in our proof.

Proposition 7. *Let X be an almost simple group. If $X \neq \mathrm{Aut}(\mathrm{Alt}(6))$ then $\gamma(X) < m(\mathrm{soc}(X))/2$. Moreover $\gamma(\mathrm{Aut}(\mathrm{Alt}(6)) = 3$.*

Proof. Let $S = \mathrm{soc}(X)$. For the value of $m(S)$ we refer to [25, Table 5.2.A] and [15, Table 1].

If S is an alternating group of degree $n \geq 5$ then $\gamma(X) < n/2 = m(S)/2$ [11] unless $n = 6$. Moreover it is easy to check using [20] that if $S = \mathrm{Alt}(6)$ then $\gamma(X) \leq 3$ with equality only if $X = \mathrm{Aut}(\mathrm{Alt}(6))$.

Suppose that S is a sporadic simple group. It can be deduced from [19, Table 1] that $\gamma(M_{11}) = 2$, $\gamma(M_{12}) \leq 3$, $\gamma(S) \leq 9$ if S is not the Monster group M and $\gamma(\mathrm{M}) \leq 14$: this is sufficient to conclude $\gamma(S) < m(S)/2$. If X is not simple then $X/S \cong C_2$ and X has at most six conjugacy classes of involutions, and precisely 3 conjugacy classes of involutions if $X = \mathrm{Aut}(M_{12})$. Since every element of X of odd order lies in S and every element of X of even order centralizes an involution, $\gamma(X) \leq 6 + 1 = 7 < m(S)/2$ if $S \neq M_{12}$, $\gamma(\mathrm{Aut}(M_{12})) \leq 1 + 3 = 4$.

Suppose that S is a simple group of Lie type. Denote by $q = p^f$ the size of the base field F, where p is the characteristic. Since X is the union of the centralizers of the nontrivial elements of S [16, Proposition 7], $\gamma(X) \leq k^*(S)$, the number of conjugacy classes of elements in S of prime order. In the case $S \neq A_m(q)$ we will prove that $k^*(S) < k(S) \leq m(S)/2$, by using the bounds for the number $k(S)$ of conjugacy classes in S proved in [18, Corollary 1.2 and Tables 1 and 2]. Suppose that S is of classical type and let n be the dimension of the natural module over F. In [1], eight collections $\mathcal{C}_1, \ldots, \mathcal{C}_8$ of natural subgroups of X are defined, and each cyclic subgroup of X is contained in one of these subgroups. So X is covered by the maximal subgroups of X belonging to these Aschbacher classes. In the particular case when $S = A_m(q)$, we have $n = m + 1$, $S \cong PSL(n+1, q)$ and the number of conjugacy classes of subgroups of type $\mathcal{C}_1, \ldots, \mathcal{C}_8$ is at most $2 \cdot n + 3 \cdot d(n) + \log n + \log f + 5 \leq 5(n+1) + \log n + \log q$ where $\log = \log_2$, $d(n)$ is the number of divisors of n and $\omega(f)$ is the number of prime divisors of f [24, p. 69]. In the case $S = A_m(q)$ we will prove that

$$5(n+1) + \log n + \log q < m(S)/2 \tag{1}$$

with finitely many exceptions. We are now ready to start our case by case analysis.

- $S = A_m(q)$, $n = m+1$, $m \geq 1$, $(n,q) \neq (2,2),(2,3)$. We have $m(S) = \frac{q^n-1}{q-1}$ if $(n,q) \neq (2,5),(2,7),(2,9),(2,11),(4,2)$, $m(A_1(5)) = 5$, $m(A_1(7)) = 7$, $m(A_1(9)) = 6$, $m(A_1(11)) = 11$, $m(A_3(2)) = 8$. By [9] $\gamma(PSL(2,q)) = \gamma(PGL(2,q)) = 2$, so me way assume that if $n = 2$ then q is not a prime. Moreover $PSL(2,4) \cong \mathrm{Alt}(5)$, $PSL(3,2) \cong PSL(2,7)$, $PSL(2,9) \cong \mathrm{Alt}(6)$ and $PSL(4,2) \cong \mathrm{Alt}(8)$. In the remaining cases inequality (1) holds except for $(n,q) \in \{(6,2),(5,2),(4,3),(3,3),(3,4),(3,5),(2,8),(2,16),(2,25),(2,27)\}$. On the other hand $k^*(PSL(5,2)) = 13$, $k^*(PSL(4,3)) = 11$, $k(PSL(3,4)) = 10$, $k^*(PSL(3,5)) = 14$, $k^*(PSL(2,8)) = 4$, $k^*(PSL(2,25)) = 10$ and $k^*(PSL(2,27)) = 12$ (see [13]). Suppose $S \in \{PSL(2,16),\ PSL(3,3)\}$: by [9] $\gamma(X) = 2$ if $X = S$, otherwise X/S is a non-trivial 2-group, so every elements in $X \setminus S$ centralizes an involution and since X contains 2 conjugacy classes of involution we deduce that $\gamma(X) \leq 3$. Finally $\gamma(PSL(6,2)) \leq [2 \cdot 6 + 3 \cdot d(6) + 5 + \log 6] = 31 < 63/2 = m(PSL(6,2))/2$.
- $S = B_m(q)$, q odd, $m > 1$. We have $m(S) = \frac{q^{2m}-1}{q-1}$ if $q > 3$, $m(S) = \frac{1}{2}3^m(3^m-1)$ if $q = 3$ and $m > 2$, $m(B_2(3)) = 27$. Moreover $k(S) \leq 7.3 \cdot q^m$ and $k(B_2(q)) \leq q^2 + 12q$ if q is odd. This is enough to deduce that $k(S) < m(S)/2$, except in the three cases $B_2(3)$, $B_2(5), B_3(3)$. However, it follows from [13], that $k^*(B_2(3)) = 7 < m(B_2(3))/2 = 27/2$, $k(B_2(5)) = 34 < m(B_2(3))/2 = 78$ and $k(B_3(3)) = 58 < m(B_3(3))/2 = 351/2$.
- $S = C_m(q)$, $m > 2$. We have $m(S) = \frac{q^{2m}-1}{q-1}$ if $q > 2$, $m(S) = 2^{m-1}(2^m-1)$ if $q = 2$. Moreover $k(S) \leq 15.2 \cdot q^m$. It is easy to see that $15.2 \cdot q^m < m(S)/2$, except for $(m,q) \in \{(3,2),(3,3),(3,4),(3,5),(4,2),(4,3),(5,2)\}$. On the other hand by [18, Table 3] $k(C_3(4)) \leq 4^3 + 5 \cdot 4^2 < m(C_3(4))/2 = 1365/2$, $k(C_3(5)) \leq 5^3 + 12 \cdot 5^2 < m(C_3(5))/2 = 1953$ and it follows from [13] that $k^*(C_3(2)) = 9 < m(C_3(2))/2 = 14$, $k(C_3(3)) = 74 < m(C_3(3))/2 = 182$, $k^*(C_4(2)) = 15 < m(C_4(2))/2 = 60$, $k(C_4(3)) = 278 < m(C_4(3))/2 = 1640$ and $k(C_5(2)) = 198 < m(C_5(2))/2 = 248$.
- $S = D_m(q)$, $m > 3$. We have that $m(S) = \frac{(q^m-1)(q^{m-1}+1)}{q-1}$ if $q > 2$ and $m(S) = 2^{m-1}(2^m-1)$ if $q = 2$. Moreover $k(S) \leq 6.8 \cdot q^m$ and it is easy to see that $6.8 \cdot q^m < m(S)/2$ except for $(m,q) \in \{(4,2),(4,3)\}$. On the other hand it follows from [13] that $k(D_4(2)) = 53 < m(D_4(2))/2 = 60$ and $k(D_4(3)) = 114 < m(D_4(3))/2 = 520$.
- $S = {}^2A_m(q)$, $m > 1$. We have $m(S) = \frac{(q^{m+1}-(-1)^{m+1})(q^m-(-1)^m)}{q^2-1}$ if $m \geq 4$ and $m+1$ is not divisible by 6 when $q = 2$, $m(S) = 2^m(2^{m+1}-1)/3$ if $q = 2$ and m is divisible by 6, $m({}^2A_3(q)) = (q+1)(q^3+1)$, $m({}^2A_2(q)) = q^3+1$ if $q \neq 2,5$, $m({}^2A_2(5)) = 50$. Moreover $k(S) \leq 8.26 \cdot q^m$ and $k({}^2A_m(q)) \leq q^{n-1} + 7q^{n-2}$ if $q > 2$. This is enough to deduce that $k(S) < m(S)/2$, except when $m = 2$ and $q \leq 7$, $m = 3$ and $q \leq 5$ or $(m,q) = (4,2)$. However $k({}^2A_2(3)) = 14 \leq m({}^2A_2(3))/2 = 14$, $k({}^2A_2(4)) = 22 < m({}^2A_2(4))/2 = 65/2$, $k({}^2A_2(5)) = 14 < m({}^2A_2(5))/2 = 25$, $k({}^2A_2(7)) = 58 < m({}^2A_2(7)) = 172$, $k^*({}^2A_3(2)) = 7 < m({}^2A_3(2))/2 = 27/2$, $k({}^2A_3(3)) = 20 < m({}^2A_3(3))/2 = 66$, $k({}^2A_3(4)) = 94 < m({}^2A_3(4))/2 = 325/2$, $k({}^2A_3(5)) = 97 < m({}^2A_3(5))/2 = 378$ and $k({}^2A_4(2)) = 47 < m({}^2A_4(2))/2 = 165/2$.

- $S = {}^2D_m(q)$, $m > 3$. We have $m(S) = \frac{(q^m+1)(q^{m-1}-1)}{q-1}$. Moreover $k(S) \leq 6.8 \cdot q^m < m(S)$ except when $(m,q) \in \{(4,2),(4,3),(5,2)\}$. Moreover it follows from [13] that $k({}^2D_4(2)) = 39 < m({}^2D_4(2))/2 = 119/2$, $k({}^2D_4(3)) = 114 < m({}^2D_4(3))/2 = 533$ and $k({}^2D_5(2)) = 115 < m({}^2D_5(2))/2 = 495/2$.

Now suppose that S is a Lie group of exceptional type. The bound $k(S) \leq 15.2q^r$ (where r is the rank) in [18, Corollary 1.2] compared with [15, Table 1] implies that $\gamma(X) < m(S)/2$ if S is one of the groups $F_4(q)$, ${}^2F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, ${}^3D_4(q)$, $E_7(q)$, $E_8(q)$. Suppose this is not the case. We will use [18, Table 1] and [15, Table 1].

- $S = G_2(q)$. We have $k(S) \leq q^2 + 2q + 9$ and $q^2 + 2q + 9 \leq 3q^2 < q^5/2 \leq m(S)/2$.
- $S = {}^2G_2(q)$, $p = 3$, $f = 2m+1$, $m \geq 1$. We have $k(S) \leq q + 8$ and $q + 8 \leq (q^3+1)/2 = m(S)/2$, since $q \geq 27$.
- $S = {}^2B_2(q)$, $p = 2$, $f = 2m+1$, $m \geq 1$. We have $k(S) \leq q + 3$ and $q + 3 < m(S)/2 = (q^2+1)/2$, since $q \geq 8$.
- $S = {}^2F_4(2)'$. In this case $k(S) = 22 < 2^5 \cdot 5^2 = m(S)/2$.

This concludes our proof. □

Proposition 8. *Let G be a group with a unique minimal normal subgroup N and assume that N is nonabelian and G/N is cyclic. Let $N \cong S^t$ with S a nonabelian simple group. Then $\gamma(G) < t \cdot m(S)/2$.*

Proof. By assumption, $N = S_1 \times \ldots \times S_t$, with $S_i \cong S$ for $i = 1, \ldots, t$. Let ψ be the map from $N_G(S_1)$ to $\mathrm{Aut}(S)$ induced by the conjugacy action on S_1. Set $X = \psi(N_G(S_1))$ and note that X is an almost simple group with socle $S = \mathrm{Inn}(S) = \psi(S_1)$. Then G embeds in the wreath product $X \wr \mathrm{Sym}(t)$ [4, Remarks 1.1.40.13]. Since G/N is cyclic, X/S is also cyclic; more precisely if $h = (y_1, \ldots, y_t)\rho \in G$ generates G modulo N, then ρ is a t-cycle and $y_1 y_{\rho(1)} \cdots y_{\rho^{(t-1)}(1)}$ generates X modulo S.

Now let $g = (x_1, \ldots, x_t)\delta \in G$. If $\langle g, N\rangle \neq G$, then g is contained in one of the $\omega(|G/N|) = \omega(t \cdot |X/S|)$ normal subgroups of prime index containing N. Assume now that $\langle g, N\rangle = G$ and let $y := x_1 x_{\delta(1)} \cdots x_{\delta^{(t-1)}(1)}$. Since $\langle y, S\rangle = X$, there exists a proper subgroup M of X with $y \in M$ and $MS = X$. Choose $a_2, \ldots, a_t \in S$ such that

$$x_1 a_{\delta(1)}^{-1},\ a_{\delta(1)} x_{\delta(1)} a_{\delta^2(1)}^{-1}, \ldots, a_{\delta^{t-2}(1)} x_{\delta^{t-2}(1)} a_{\delta^{t-1}(1)}^{-1},\ a_{\delta^{t-1}(1)} x_{\delta^{t-1}(1)} \in M.$$

It can be easily checked that g normalizes $M \times M^{a_2} \times \ldots \times M^{a_t}$. In other words, if $\mathcal{M}$ is a normal cover of X, then a normal cover of G can be obtained taking the maximal normal subgroups of G containing N and the conjugates of the normalizers $N_G(M \times \cdots \times M)$ with M running in $\mathcal{M}$. It follows that

$$\gamma(G) \leq \omega(t \cdot |X/S|) + \gamma(X).$$

If $t = 1$ then $G = X$ and the result follows from Proposition 7. If $t \geq 2$ then, since $\omega(|X/S|) < m(S)/4$ [2, Lemma 2.7], $\gamma(X) \leq m(S)/2$ and $4\omega(t) \leq 4(2t-3) < m(S)(2t-3)$, we conclude

$$\gamma(G) \leq \omega(t) + \frac{m(S)}{4} + \frac{m(S)}{2} < \frac{t \cdot m(S)}{2}$$

as in our claim. □

Proof of Theorem 4. The proof is by induction on the degree n. If G/G' is not cyclic then $C_p \times C_p$ is an epimorphic image of G for some prime p. Since $G \leq \mathrm{Sym}(n)$, p^2 divides $n!$ so $p \leq n/2$, and we deduce that $\gamma(G) \leq \gamma(C_p \times C_p) = p+1 \leq (n+2)/2$. So from now on we will assume that G/G' is cyclic. If G is soluble, then $\gamma(G) = 2$, by Lemma 6. So we may assume that G is not soluble. First suppose that G is not transitive; let $\Omega_1, \dots, \Omega_t$ be the orbits of G on $\{1, \dots, n\}$ and $G_1, \dots, G_t$ the corresponding transitive constituents. Since G is not soluble and it is a subdirect product of $G_1 \times \cdots \times G_t$, there exists i such that G_i is noncyclic: by induction $\gamma(G) \leq \gamma(G_i) \leq (|\Omega_i| + 2)/2 \leq (n+2)/2$. So we may assume that G is transitive. Suppose that $\{B_1, \dots, B_s\}$ is a system of blocks for G with $|B_i| = r$. Consider $\mathrm{St}_G(B_1)$, the stabilizer in G of the block B_1. Denote by $\alpha : \mathrm{St}_G(B_1) \to \mathrm{Sym}(r)$ the permutation representation induced by the action of $\mathrm{St}_G(B_1)$ on the set B_1 and by $\beta : G \to \mathrm{Sym}(s)$ the permutation representation induced by the action of G on the set of blocks and let $H = \alpha(\mathrm{St}_G(B_1))$, $K = \beta(G)$. We may identify G, as a permutation group, with a subgroup of $H \wr K$ (in its imprimitive representation) in such a way that, for $1 \leq i \leq s$, $\mathrm{St}_G(B_j)$ acts on B_j as the subgroup H of $\mathrm{Sym}(r)$ and G acts on the set $\{B_1, \dots, B_s\}$ as the subgroup K of $\mathrm{Sym}(s)$. We choose $B_1 = \{1, \dots, n\}$ if G is primitive, B_1 to be an imprimitive block of minimal size otherwise. If K is noncyclic, then, by induction, $\gamma(G) \leq \gamma(K) \leq (s+2)/2 \leq (n+2)/2$, so we may assume that K is cyclic. We distinguish three different possibilities:

1) H has a unique minimal normal subgroup N and $N \cong C_p^t$ is an elementary abelian p-group. In this case $r = p^t$ and $H/N \leq GL(t,p) \leq \mathrm{Sym}(r-1)$. Consider the normal subgroup $M \cong N^s$ of $H \wr \mathrm{Sym}(s)$. Notice that $G/(M \cap G) \leq \mathrm{GL}(t,p) \wr \mathrm{Sym}(s)$ has a faithful permutational representation of degree $(r-1)s$. Since G is not soluble, $G/(M \cap G)$ is not cyclic and therefore by induction $\gamma(G) \leq \gamma(G/(M \cap G)) \leq ((r-1)s+2)/2\} \leq (n+2)/2$.

2) H has a unique minimal normal subgroup N and $N \cong S^t$ is the direct product of t isomorphic non abelian simple groups. In particular N is transitive of degree r so $r \geq m(S)^t$ (see [25, Proposition 5.2.7] and the comment afterwards) and $G \leq H \wr \mathrm{Sym}(s) \leq (\mathrm{Aut}(S) \wr \mathrm{Sym}(t)) \wr \mathrm{Sym}(s) \leq \mathrm{Aut}(S) \wr \mathrm{Sym}(t \cdot s)$. Consider the normal subgroup $M \cong S^{t \cdot s}$ of $\mathrm{Aut}(S) \wr \mathrm{Sym}(t \cdot s)$. Notice that $G/(M \cap G) \leq \mathrm{Out}(S) \wr \mathrm{Sym}(t \cdot s)$ has a faithful permutational representation of degree $|\mathrm{Out}(S)| \cdot t \cdot s \leq (2 \cdot m(S) \cdot t \cdot s)/3 < m(S)^t s \leq r \cdot s \leq n$ (indeed $|\mathrm{Out}(S)| \leq 2m(S)/3$ by [2, Lemma 2.7]). If $G/(M \cap G)$ is not cyclic, then by induction $\gamma(G) \leq \gamma(G/(M \cap G)) \leq (n+2)/2$. Assume that $G/(M \cap G)$ is cyclic and let T be a minimal normal subgroup of G contained in $M \cap G$. We have $T \cong S^u$ with $u \leq t \cdot s$; moreover $G/C_G(T)$ has a unique minimal normal subgroup $T^*/C_G(T) \cong T$ and G/T^* is cyclic: by Proposition 8 $\gamma(G) \leq \gamma(G/C_G(T)) < u \cdot m(S)/2 \leq t \cdot s \cdot m(S) \leq n/2$.

3) $\mathrm{soc}\, H = N = N_1 \times N_2$ where N_1 and N_2 are isomorphic non abelian minimal normal subgroups of H. In this case $r = |N_1| = |N_2|$. Let $H^* := H/C_H(N_1) \leq \mathrm{Aut}(N_1) \leq \mathrm{Sym}(r-1)$. We have $G \leq H \wr \mathrm{Sym}(s)$ and the wreath product $H \wr \mathrm{Sym}(s)$ contains a normal subgroup $M \cong C_H(N_1)^s$. Notice that $G/(M \cap G) \leq H^* \wr \mathrm{Sym}(s)$ is a noncyclic permutation group of degree $(r-1)s < n$. So $\gamma(G) \leq \gamma(G/(M \cap G)) \leq (n+2)/2$. □

Proof of Proposition 2. Let $m = \mu(G)$. First assume that G contains a maximal subgroup M of index m, which is not normal in G. In this case G/M_G is a non cyclic permutation group of degree m and $\gamma(G) \leq \gamma(G/M_G) \leq m$ by Theorem 4. Otherwise G contains two normal maximal subgroup of index m. In this case

m is a prime and $C_m \times C_m$ is an epimorphic image of G. In particular $\gamma(G) \leq \gamma(C_m \times C_m) = m + 1$.

Therefore we have proved that $\gamma(G) \leq m + 1$ and $\gamma(G) = m + 1$ only if m is a prime, $C_m \times C_m$ is an epimorphic image of G and $\gamma(G) = \gamma(C_m \times C_m) = \gamma(G/G')$. $\square$

3. Groups G with $\gamma(G) = 2$

Before to stat our discussion, let us introduce a couple of easy observations.

Lemma 9. *Let H be a proper subgroup of a finite group G and let $N \trianglelefteq G$ be such that $HN = G$. We have $\bigcup_{g \in G}(H \cap N)^g \neq N$.*

Proof. Let $X = H \cap N$. Since $X \trianglelefteq H$, we have $G = HN = N_G(X)N$. Hence $\cup_{g \in G} X^g = \cup_{n \in N} X^n \neq N$. $\square$

Lemma 10. *Let H be a proper subgroup of a finite group G and let N_1, N_2 be two different minimal normal subgroups of G. If $HN_1 = HN_2 = G$, then $H \cap N_1 = H \cap N_2 = 1$.*

Proof. Assume $HN_1 = HN_2 = G$. Then $H \cap N_1$ is normalized by H and centralized by N_2 hence $H \cap N_1$ is normalized by $HN_2 = G$. Since N_1 is a minimal normal subgroup of G and $N_1 \not\leq H$, we must have $H \cap N_1 = 1$. $\square$

For the remaining part of this section, G will be a finite group with the following properties:

(1) $\gamma(G) = 2$;
(2) $\gamma(G/N) > 2$ if N is a non trivial normal subgroup of G.

In particular there exists two maximal subgroups H and K with

$$G = (\bigcup_{x \in G} H^x) \bigcup (\bigcup_{y \in G} K^y).$$

Moreover $(H \cap K)_G = 1$, otherwise we would have $\gamma(G/(H \cap K)_G) = 2$. Let

$$M = \mathrm{soc}(G) = N_1 \times \cdots \times N_t$$

be the socle of G with N_i a minimal normal subgroup of G for $1 \leq i \leq t$.

Lemma 11. *$t = 1$ i.e. G contains a unique minimal normal subgroup.*

Proof. We distinguish 2 cases:
a) One of the two subgroups H and K contains M.
Assume for example $M \leq H$. In this case $K_G \cap M \leq (K \cap H)_G = 1$, hence $K_G = 1$ and t is the number of minimal normal subgroups of a primitive permutation group G with point stabilizer K. Assume by contradiction that $t \neq 1$. Then $t = 2$ and (see for example [4, Proposition 1.1.12]) we may assume that there exists a monolithic primitive group L with non abelian socle N and a subgroup T of L with $N \leq T < L$ such that

$$G = \{(l_1, l_2) \in L^2 \mid Nl_1 = Nl_2\}, \quad M = N^2, \quad K = \{(l, l) \mid l \in L\}, \quad H = T^2 \cap G.$$

Let $x \notin \cup_{l \in L} T^l$ and consider the coset $\Omega = (x, x)N^2$. Clearly $\Omega \cap (\cup_{g \in G} H^g) = \varnothing$ hence $\Omega \subseteq \cup_{g \in G} K^g$. Let $R = \{(1, n) \mid n \in N\} \subseteq G$. Since $KR = G$, we have $\cup_{g \in G} K^g = \cup_{r \in R} K^r$, hence

$$\{(x, xn) \mid n \in N\} \subseteq \Omega \subseteq \{(l, l^n) \mid l \in L, n \in N\}.$$

In particular $Nx = \{x^n \mid n \in N\}$ and this implies $C_N(x) = 1$, i.e. N admits a fixed-point free automorphism: by [27] N is a soluble group, a contradiction.
b) $HM = KM = G$. Let us define the following two subsets of $\Omega = \{1, \dots, t\}$:

$$\Omega_H = \{i \in \Omega \mid N_i \cap H = 1\}, \quad \Omega_K = \{i \in \Omega \mid N_i \cap K = 1\}.$$

We claim that $\Omega_H = \Omega_K = \varnothing$. To prove this, assume for example that $\Omega_H = \{1, \dots, u\}$ with $u \neq 0$. By Lemma 10, $N_i \leq H$ for all $i > u$. Moreover if $i \leq u$, then $N_i \cap H^g = (N_i \cap H)^g = 1$, hence $N_i \leq \cup_{g \in G} K^g$. It follows that $N_i = \cup_{g \in G} N_i \cap K^g = \cup_{g \in G} (N_i \cap K)^g$: by Lemma 9 we must have $N_i \leq K$. Since $KM = G$, there exists j such that $KN_j = G$. We have $j > u$ hence $N_j \leq H$. By Lemma 9 there exists $x \in N_j \setminus (\cup_{g \in G} N_j \cap K^g)$. Take $1 \neq y \in N_1$ and consider $z = yx$. We cannot have $z \in H^g$ (since $x \in N_j = N_j^g \subseteq H^g$, $z \in H^g$ would imply $y \in H^g \cap N_1 = 1$). Hence $z = yx \in K^g$ for some g, however $y \in N_1 \leq K^g$ hence $x \in K^g$, a contradiction. So our claim that $\Omega_H = \Omega_K = \varnothing$ has been proved. Combined with Lemma 10 and the fact that $(H \cap K)_G = 1$, this implies that if $t \neq 1$ then $t = 2$ and we may assume $N_1 \leq H$, $N_2 \leq K$ and $N_2 H = N_1 K = G$. By Lemma 9, there exist $x \in N_1 \setminus \cup_{g \in G} K^g$ and $y \in N_2 \setminus \cup_{g \in G} H^g$. Consider $z = xy$. If $z \in H^g$, then since $x \in N_1 = N_1^g \leq H^g$ we would have $y \in H^g$, a contradiction. Similarly, we cannot have $z \in K^g$. This proves that $t = 1$. $\square$

Lemma 12. *If neither H nor K contains* $\mathrm{soc}(G)$*, then G is an almost simple group.*

Proof. Let $M = \mathrm{soc}(G)$ and assume $G = HM = KM$. We have $M \subseteq (\cup_{x \in M} H^x) \cup (\cup_{y \in M} K^y)$ and this implies

$$M = (\cup_{x \in M} (H \cap M)^x) \cup (\cup_{y \in M} (K \cap M)^y). \qquad (*)$$

Together with Lemma 9, this implies $H \cap M \neq 1$ and $K \cap M \neq 1$. In particular, if M is abelian, then $M \leq H \cap K$, a contradiction. Therefore M is a direct product of r copies of a non-abelian simple group S. Assume, by contradiction, that $r \neq 1$. To fix the notation, let $M = S_1 \times \cdots \times S_r$ and $\pi : M \mapsto S$ the map induced by the projection of M on the first component. The maximal subgroups X of G with $XM = G$ and $X \cap M \neq 1$ are of one of the following types:

a) product type: if $1 < \pi(M \cap X) < S$;
b) diagonal type: if $\pi(M \cap X) = S$.

In the first case $X \cap M \cong T_1 \times \cdots \times T_r$ with $1 < T_i < S_i$ and $T_i \cong T_j$ for every $1 \leq i \leq j \leq r$. In the second case there exists a partition Φ of $\{1, \dots, r\}$ such that $X \cap M = \prod_{B \in \Phi} D_B$, where all the blocks have the same cardinality and, for every block $B \in \Phi$, $|B| \neq 1$ (otherwise we would have $X \cap M = M$ hence $X = G$) and D_B is a full diagonal subgroup of $\prod_{j \in B} S_j$ (that is, if $B = \{j_1, \dots j_t\}$, there exist $\phi_2, \dots, \phi_t \in \mathrm{Aut}\, S$ such that $D_B = \{(x, x^{\phi_2}, \dots, x^{\phi_t}) \mid x \in S\} \leq S_{j_1} \times \cdots \times S_{j_t}$). We have three possibilities:

(1) H and K are both of diagonal type. Let $\Delta = \{(s, 1, \dots, 1) \mid s \in S, s \neq 1\} \subseteq M$. By the way in which maximal subgroups of diagonal type are defined, $\Delta \cap H^m = \Delta \cap K^m = \varnothing$ for each $m \in M$, against $(*)$.
(2) H is of product type and K is of diagonal type. We have $H \cap M = T_1 \times \cdots \times T_r$ with $T = T_1 < S$. There exists $s \in S \setminus \cup_{s \in S} T^s$. Consider $m = (s, 1, \dots, 1) : m \notin (\cup_{x \in M} (H \cap M)^x) \cup (\cup_{y \in M} (K \cap M)^y)$, against $(*)$.

(3) H and K are both of product type. Let $H \cap M = T_1 \times \cdots \times T_r$ and $K \cap M = U_1 \times \cdots \times U_r$. Since T_1 and U_2 are proper subgroup of S, there exist $a \in S \setminus \cup_{s \in S} T^s$ and $b \in S \setminus \cup_{s \in S} U_2^s$. Consider $m = (a, b, 1, \dots, 1)$: $m \notin (\cup_{x \in M}(H \cap M)^x) \cup (\cup_{y \in M}(K \cap M)^y)$, against $(*)$.

All the possibilities lead to a contradiction, hence it must be $r = 1$ and G is an almost simple group. $\square$

We recall some results concerning almost simple groups G with $\gamma(G) = 2$. It was shown by H. Dye [17] that the symplectic group $G = \mathrm{Sp}_{2l}(2^f)$ defined over a finite field of characteristic 2 is the union of the two G-conjugacy classes of subgroups isomorphic to $O^+_{2l}(2^f)$ and $O^-_{2l}(2^f)$ embedded naturally. D. Bubboloni, M.S. Lucido and T. Weigel [10] notices the existence of an interesting example in characteristic 3, i.e. in $G = F_4(3^f)$ every element is conjugated to an element of the subgroup $B_4(3^f)$ or of the subgroup $3.^3D_4(3^f)$. In [5] it is proved that $\gamma(\mathrm{Alt}(n)) = 2$ if and only if $4 \leq n \leq 8$, $\gamma(\mathrm{Sym}(n)) = 2$ if and only if $3 \leq n \leq 6$. In [9] it is proved that $\gamma(PSL(n, q)) = \gamma(PGL(n, q)) = 2$ if and only if $2 \leq n \leq 4$. Another example is given by the Mathieu group M_{11} [26, Claim 5.1].

In the remaining part of the section we concentrate our attention in the case when $\mathrm{soc}(G) \leq H$ (and consequently $G = KM$).

Lemma 13. *Assume that H and K are maximal subgroups of a primitive monolithic group G with $M = \mathrm{soc}(G) \leq H$ and $KM = G$. Let $R = K \cap M$. The following are equivalent:*

(1) $G = (\cup_{x \in G} H^x) \bigcup (\cup_{y \in G} K^y)$;
(2) $gM \subseteq \cup_{m \in M} K^m$ *for each* $g \in G \setminus \cup_{x \in K} H^x$;
(3) $gM = \cup_{m \in M} (gR)^m$ *for each* $g \in K \setminus \cup_{x \in G} H^x$;
(4) whenever $g \in K \setminus \cup_{x \in K} H^x$ *and* $m \in M$, *we have* $m \in R$ *if and only if* $(gR)^m = gR$.

Proof. Since $KM = G$ and $M \leq H$, we have $\Gamma = \cup_{x \in G} H^x = \cup_{x \in K} H^x$. Moreover $gM \cap \Gamma \neq \varnothing$ if and only if $gM \subseteq \Gamma$. Equivalently, if $g \notin \Gamma$, then $gM \cap \Gamma = \varnothing$. It follows that (1) holds if and only if $gM \subseteq \cup_{x \in G} K^x = \cup_{m \in M} K^m$ whenever $g \notin \Gamma$, i.e. (1) and (2) are equivalent. Assume that (2) holds. In particular if $g \in K \setminus \Gamma$, then for each $m_1 \in M$, there exists $m_2 \in M$ with $gm_1 \in K^{m_2}$; it follows that $(gm_1)^{m_2^{-1}} = g[g, m_2^{-1}]m_1^{m_2^{-1}} \in K$ hence, since $g \in K$, we have $[g, m_2^{-1}]m_1^{m_2^{-1}} \in K \cap M = R$ and consequently $gm_1 \in (gR)^{m_2}$. Therefore (2) implies (3). Conversely, assume that (3) holds and let $g \notin \Gamma$. Since $KM = G$ and $M \leq H$, there exists $\bar{g} \in K \setminus \Gamma$ with $\bar{g}M = gM$, hence $gM = \bar{g}M = \cup_{m \in M} (\bar{g}R)^m \leq \cup_{m \in M} K^m$. So (3) implies (2). Now let $a = |R|, b = |M : R|$ and let $m_1, \dots, m_b$ be a transversal of R in M. Notice that if $g \in K$, then R is normalized by g and $(gR)^r \subseteq gR$ for all $r \in R$. This implies that (3) is equivalent to

$$gM = \cup_{1 \leq i \leq b} (gR)^{m_i} \text{ for each } g \in K \setminus \cup_{x \in K} H^x.$$

Since $|gM| = a \cdot b$ and $|gR| = a$, the previous condition is satisfied if and only if the subsets $(gR)^{m_i}$ are pairwise disjoint; on the other hand these subsets are disjoint if and only if the only elements m of M with $(gR)^m = gR$ are those of R. Therefore (3) and (4) are equivalent. $\square$

Let us introduce some additional definitions. Let M be an elementary abelian group and K be an irreducible subgroup of $\mathrm{Aut}(M)$. Consider the subset

$$K^* = \{k \in K \mid C_M(k) \neq 1\}.$$

We will say that K is *almost-transitive* if there exists a proper subgroup T of K with $K^* \subseteq \cup_{x \in K} T^x$. If this situation holds, we have that $\gamma(M \rtimes K) = 2$. Indeed if $k \in K$ and $C_M(k) = 1$, then $kM = \{k^m \mid m \in M\}$, hence $M \rtimes K$ can be covered by the conjugates of the two subgroups K and $M \rtimes T$.

Corollary 14. *If* $\mathrm{soc}(G) = M$ *is abelian, then* $G = M \rtimes K$ *and* K *is an almost transitive irreducible subgroup of* $\mathrm{Aut}(M)$.

Proof. Since $KM = G$ and $\mathrm{Frat}(G) = 1$, it must be $G = M \rtimes K$ and M is an irreducible K-module. Let $T = K \cap H$ and assume $g \in K \setminus \cup_{x \in K} H^x$. Since $R = K \cap M = 1$, it follows from Lemma 13 that $g^m = g$ if and only if $m = 1$. This implies $K^* = \{k \in K \mid C_M(k) \neq 1\} \leq K \cap (\cup_{x \in K} H^x) = \cup_{x \in K}(K \cap H^x) = \cup_{x \in K} T^x$. □

In virtue of the previous result, it should be interesting to classify the almost-transitive irreducible groups. There are two extreme situations, one is when K is an irreducible fixed point free subgroup of $\mathrm{Aut}\, M$ (and consequently $G = M \rtimes K$ is a Frobenius group), the other is when K is a transitive irreducible subgroup of $\mathrm{Aut}\, M$ (and consequently $G = M \rtimes K$ is a 2-transitive permutation group of degree $|M|$). However, other possibilities occur, as the following three examples indicate.

(1) Let M be the additive group of the finite field F with 16 elements. The multiplicative group F^* contains a subgroup Q of order 5, which is normalized by the Frobenius automorphism $\sigma : f \to f^2$. The semidirect product $K = Q \rtimes \langle \sigma \rangle$ is an irreducible subgroup of $\Gamma L(1, 16) \leq \mathrm{Aut}(M)$; moreover $K^* = K \setminus Q$ is contained in the union of the conjugates of a Sylow 2-subgroup. Hence K is almost transitive.

(2) Assume that $p > 2$ and q are two prime numbers, with p dividing $q - 1$. The multiplicative group F^* of the field F with q elements contains a cyclic subgroup $X = \langle x \rangle$ of order p. Let Y be the subgroup of $\mathrm{Sym}(p)$ generated by the permutation $\sigma = (1, 2, \ldots, p)$. Consider the following subgroup K of the wreath product $X \wr Y$:

$$K = \{(x^{a_1}, \ldots, x^{a_p})\sigma^i \in X \wr Y \mid \sum_{1 \leq j \leq p} a_j = i \mod p\}.$$

The wreath product $X \wr Y$ acts on a p-dimensional F-vector space M and K is an irreducible subgroup of $X \wr Y \leq \mathrm{Aut}(M)$. Suppose now that $k = (x^{a_1}, \ldots, x^{a_p})\sigma^i \in K^*$. There exists $(0, \ldots, 0) \neq m = (b_1, \ldots, b_p) \in M$ such that

$$(b_1, \ldots, b_p) = m = m^k = (b_1 x^{a_1}, \ldots, b_p x^{a_p})^{\sigma^i}.$$

Since $m \neq 0$, there exists $1 \leq j \leq p$ with $b_j \neq 0$. It must be $i = 0 \mod p$, otherwise the previous equality would imply that $b_r \neq 0$ for each $1 \leq r \leq p$ and $b_1 \cdots b_p = b_1 x^{a_1} \cdots b_p x^{a_p}$, and consequently $0 = \sum_j a_j = i \mod p$. It follows that $K^* \subseteq T = \{(x^{a_1}, \ldots, x^{a_p}) \mid \sum_j a_j = 0 \mod p\}$.

(3) Let F be a field with q^2 elements, $q \equiv 3 \mod 4$, $q \neq 3$, and consider the 2-dimensional vector space $M = F^2$. The multiplicative group F^* contains two cyclic subgroups A and B of orders, respectively, $(q-1)/2$ and $q+1$.

The Frobenius automorphism $\sigma : f \to f^q$ normalizes B and centralizes A. Consider the subgroup K of $\Gamma L(1,q^2) \wr \langle (1,2) \rangle \le \mathrm{Aut}(M)$ defined as follows:

$$K = \{(1,2)^r \sigma^t (ab_1, ab_2) \mid a \in A, b_1, b_2 \in B, 0 \le r,t \le 1\}.$$

Assume $k = (1,2)^r \sigma^t (ab_1, ab_2)$ has a non trivial fixed point $(f_1, f_2) \neq (0,0)$. There are two possibilities:

(a) If $r = 0$ we have $(f_1,f_2) = (f_1,f_2)^k = (f_1^{q^t} ab_1, f_2^{q^t} ab_2)$. There exists $i \in \{1,2\}$ with $f_i \neq 0$ and we must have that $f_i^{q^t} ab_i = f_i$, i.e. $a = f_i^{1-q^t} b_i^{-1}$. Since $f_i^{1-q^t} \in B$, we conclude that $a \in A \cap B = 1$.

(b) If $r = 1$ we have $(f_1,f_2) = (f_1,f_2)^k = (f_2^{q^t} ab_1, f_1^{q^t} ab_2)$. We must then have $f_1 \neq 0$, $f_2 \neq 0$, $f_2 = f_1^{q^t} ab_2$, $f_1 = f_2^{q^t} ab_1 = (f_1^{q^t} ab_2)^{q^t} ab_1 = f_1 a^{q^t+1} b_2^{q^t} b_1 = f_1 a^2 b_2^{q^t} b_1$, hence $a^2 = (b_2^{q^t} b_1)^{-1} \in A \cap B = 1$; since $|A| = (q-1)/2$ is odd, we conclude that $a = 1$.

It follows that $K^* \subseteq T = \{(1,2)^r \sigma^t (b_1, b_2) \mid b_1, b_2 \in B, 0 \le r,t \le 1\}$.

We conclude this section with discussing some examples in which $M = \mathrm{soc}(G)$ is nonabelian, $M \le H$ and $MK = G$.

Let S be a finite non abelian simple group and let p be a prime which does not divide $|S|$. Consider the wreath product $G = S \wr \langle \sigma \rangle$ with $\sigma = (1,2,\dots,p) \in \mathrm{Sym}(p)$. We claim that $\gamma(G) = 2$. More precisely let $M = S^p$ be the base of the wreath product and let $H = \{(s,\dots,s)\sigma^i \mid s \in S,\ 0 \le i \le p-1\}$ be a maximal subgroup of G of diagonal type. We prove that $G = M \cup (\cup_{m \in M} H^m)$. Indeed consider for example $(t_1,\dots,t_p)\sigma \in G$. We look for $s, x_1, \dots, x_p \in S$ such that

$$(t_1,\dots,t_p)\sigma = ((s,\dots,s)\sigma)^{(x_1,\dots,x_p)} = (x_1^{-1} s x_2, x_2^{-1} s x_3, \dots, x_p^{-1} s x_1)\sigma.$$

We can take

$$\begin{aligned}
x_1 &= 1\\
x_2 &= s^{-1} t_1\\
x_3 &= s^{-2} t_1 t_2\\
&\cdots\cdots\cdots\\
x_p &= s^{-(p-1)} t_1 t_2 \cdots t_{p-1}\\
s^p &= t_1 t_2 \cdots t_p
\end{aligned}$$

where the existence of s is ensured from the fact that p does not divide $|S|$.

We want to discuss the existence of examples in which $M = \mathrm{soc}(G) = S^n$, with S a nonabelian simple group, $M \le H$ and K is a maximal subgroup of G of product type. We have $M = S^n \le G \le \mathrm{Aut}(S) \wr \mathrm{Sym}(n)$ and it is not restrictive to assume that $R = K \cap M = T^n$ with $T < S$. There exists $g \in K \setminus \cup_{x \in G} H^x$; we can write g in the form $g = (h_1,\dots,h_n)\sigma$ with $\sigma \in \mathrm{Sym}(n)$ and $h_i \in \mathrm{Aut}\, S$. Since g normalizes R, we have that h_i normalizes T for each $1 \le i \le n$. Let $\Omega \subseteq \{1,\dots,n\}$ be the σ-orbit containing 1. It is not restrictive to assume that $\Omega = \{1,\dots,r\}$ and $\sigma = \rho\tau$ where $\rho = (1,2,\dots,r)$ and τ fixes pointwise the elements of Ω (we don't exclude the possibility $r = 1$). Let $U = S^r$, $V = T^r$ and let $y = (h_1,\dots,h_r)\rho \in \mathrm{Aut}(S) \wr \mathrm{Sym}(r)$.

By Lemma 13 (3), we must have

$$yU = \cup_{u \in U} (yV)^u.$$

Recall that if $u = (y_1, \dots, y_r) \in U$ then

$$y^u = (y_1, \dots, y_r)^{-1}(h_1, \dots, h_r)\rho(y_1, \dots, y_r) = (y_1^{-1}h_1y_2, y_2^{-1}h_2y_3, \dots, y_r^{-1}h_ry_1)\rho.$$

In particular, given $s \in S$, there exist $x_1, \dots, x_r \in T$ and $y_1, \dots, y_r \in S$ such that

$$(h_1, \dots, h_rs)\rho = (y_1^{-1}h_1x_1y_2, y_2^{-1}h_2x_2y_3, \dots, y_r^{-1}h_rx_ry_1)\rho$$

and this implies

$$h_1 \cdots h_rs = (y_1^{-1}h_1x_1y_2)(y_2^{-1}h_2x_2y_3) \cdots (y_r^{-1}h_rx_ry_1) = y_1^{-1}h_1x_1 \cdots h_rx_ry_1.$$

But then, setting $h = h_1 \cdots h_r \in \mathrm{Aut}(S)$ we must have

$$hS = \cup_{s \in S} (hT)^s \quad (**).$$

The previous equality cannot occur if $h \in S$; otherwise we would have $hS = S = \cup_{s \in S} (\langle h \rangle T)^s$, which implies $S = \langle h \rangle T$, and consequently, since $h \in N_S(T)$, $T \trianglelefteq S$. For some choices of S, it is impossible to find $h \in \mathrm{Aut}(S) \setminus S$ and $T < S$ satisfying $(**)$. Assume for example $S = \mathrm{Alt}(n)$, with $n \neq 6$. If $(**)$ holds, since $h \notin S = \mathrm{Alt}(n)$ we would have $h\,\mathrm{Alt}(n) = (1,2)\,\mathrm{Alt}(n) \subseteq \cup_{s \in S}(\langle h \rangle T)^s$. In particular $\langle h \rangle T$ would be a proper subgroup of $\mathrm{Sym}(n)$ containing at least one conjugate of every odd permutation. The situation is different for $S = \mathrm{Alt}(6)$. In this case consider $G = M_{10} \leq \mathrm{Aut}(S)$. $G \setminus S$ consists of three conjugacy classes whose representatives have orders respectively 4, 8, 8. So $G \setminus S$ is covered by the Sylow 2-subgroups and $\gamma(G) = 2$. But we may consider also the group $\Gamma = (S \times S)\langle \gamma \rangle$ with $\gamma = (g, 1)\epsilon$, where $\epsilon = (1, 2)$ and $g \in M_{10} \setminus S$. This group Γ contains a normal subgroup $M = S^2$ of index 4: we claim that if $x \in \Gamma \setminus M$ then $|x|$ divides 16. Indeed one of the following holds:

(1) $x = (gs_1, s_2)\epsilon$ for some $s \in S$. Then $x^2 = (gs_1s_2, s_2gs_1)$ has either order 4 or 8.
(2) $x = (s_1, gs_2)\epsilon$ for some $s \in S$. Then $x^2 = (s_1g_2s_2, gs_2s_1)$ has either order 4 or 8.
(3) $x = (gs_1, gs_2)$ for $s_1, s_2 \in S$. Then $|x|$ divides 8.

But then any element of Γ belongs either to M or to a Sylow 2-subgroup, hence $\gamma(\Gamma) = 2$.

A more general family of examples can be obtained in the following way. Let $S = \mathrm{SL}(2, 2^p)$ with $p \geq 5$ a prime and let $A = \mathrm{Aut}\, S = S\langle \phi \rangle$ with ϕ the Frobenius automorphism. Since $p \neq 3$ we have that $(|S|, p) = 1$. In particular if $a \in A \setminus S$, then $|a|$ is divisible by p hence a centralizes a Sylow p-subgroup of A. This implies that $A \setminus S \subseteq \cup_{s \in S} H^s$ where $H = C_A(P)$ and P is a Sylow p-subgroup of A. Consider now the group $G = S^p\langle x \rangle \leq A \wr \langle \sigma \rangle$, where $\sigma = (1, 2, \dots, p)$ and $x = (\phi, 1, \dots, 1)\sigma$. Let $M = S^p$. Notice that G/M is cyclic of order p^2. In particular if $g \in G \setminus M$, then p divides $|g|$ hence $g \in C_G(K)$ for a cyclic subgroup K of order p. On the other hand, the Sylow p-subgroups of G are cyclic of order p^2 and $K = \langle x^p \rangle^m$ for some $m \in M$. This implies that $g \in H^m$, for $H = C_G\langle x^p \rangle$. It follows that $\gamma(G) = 2$.

References

1. M. Aschbacher, On the maximal subgroups of the finite classical groups. Invent. Math. 76 (1984), no. 3, 469-514.
2. M. Aschbacher and R. Guralnick, On abelian quotients of primitive groups, Proc. Amer. Math. Soc., 107 (1989) 89–95.
3. S. Blackburn, Sets of permutations that generate the symmetric group pairwise, J. Combin. Theory Ser. A 113 (2006), no. 7, 1572–1581.
4. A. Ballester-Bolinches and L. M. Ezquerro, Classes of finite groups, Mathematics and Its Applications (Springer), vol. 584, Springer, Dordrecht, 2006.
5. R. Brandl and D. Bubboloni and I. Hupp, Polynomials with roots mod p for all primes p, J. Group Theory 4 (2001) 233-239.
6. J. R. Britnell, A. Evseev, R. M. Guralnick, P. E. Holmes and A. Maróti, Sets of elements that pairwise generate a linear group, J. Combin. Theory Ser. A. 115 (2008), no. 3, 442-465.
7. J. R. Britnell and A. Maróti, Normal Coverings of Linear Groups, Algebra Number Theory, to appear
8. R. A Bryce, V. Fedri and L. Serena, Subgroup coverings of some linear groups, Bull. Austral Math. Soc. 60, (1999), no. 2, 227-238.
9. D. Bubboloni and M. S. Lucido, Coverings of linear groups, Comm. Algebra 30 (2002), no. 5, 2143-2159.
10. D. Bubboloni, M. S. Lucido and T. Weigel, Generic 2-coverings of finite groups of Lie type, Rend. Semin. Mat. Univ. Padova 115, 209–252 (2006).
11. D. Bubboloni and C. E. Praeger, Normal Coverings of Symmetric and Alternating Groups; Journal of Combinatorial Theory, Series A, 2011.
12. J. H. E. Cohn, On n-sum groups, Math. Scand. 75 (1) (1994) 44–58.
13. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and Wilson, R. A, Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray, Oxford University Press, Eynsham, 1985. xxxiv+252 pp.
14. E. Crestani and A. Lucchini, Normal coverings of soluble groups, Arch. Math. 98 (2012), no. 1, 13-18.
15. E. Damian and A. Lucchini, On the Dirichlet polynomial of finite groups of Lie type, Rend. Sem. Mat. Univ. Padova 115 (2006), 51-69.
16. E. Detomi and A. Lucchini, On the Structure of Primitive n-Sum Groups, CUBO A Mathematical Journal 10 (2008), 195–210 .
17. R. H. Dye, Interrelations of Symplectic and Orthogonal Groups in Characteristic Two, J. Algebra 59 (1979), 202–221.
18. J. Fulman and R. Guralnick, Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements, Trans. Am. Math. Soc. 364, No. 6 (2012), , 3023–3070.
19. P. E. Holmes and A. Maróti, Pairwise generating and covering sporadic simple groups, J. Algebra 324 (2010), 25–35.
20. The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008, (`http://www.gap-system.org`).
21. M. Garonzi, Finite Groups that are the union of at most 25 proper subgroups, J. Algebra Appl. Vol. 12, No. 4 (2013) 1350002
22. M. Garonzi and A. Lucchini, Direct products of finite groups as unions of proper subgroups, Arch. Math. 95 (2010), no. 3, 201-206.
23. M. Garonzi and A. Maróti, Covering certain wreath products with proper subgroups, J. Group Theory 14 (2011), no. 1, 103-125.
24. W. M. Kantor and A. Lubotzky, The probability of generating a finite classical group, Geom. Dedicata 36 (1990), no. 1, 67-87.
25. P. Kleidman and M. Liebeck, The Subgroup Structure of the Finite Classical Groups, Cambridge University Press.
26. A. Maróti, Covering the symmetric groups with proper subgroups, J. Combin. Theory Ser. A 110 (2005), no. 1, 97–111.
27. P. Rowley, Finite groups admitting a fixed-point-free automorphism group, J. Algebra 174 (1995), no. 2, 724–727.

28. G. Scorza, I gruppi che possono pensarsi come somma di tre loro sottogruppi, Boll. Un. Mat. Ital 5 (1926) 216–218.
29. M. J. Tomkinson, Groups as the union of proper subgroups, Math. Scand. 81 (2) (1997) 191–198.

Dipartimento di Matematica, Via Trieste 63, 35121 Padova, Italy.
E-mail address: mgaronzi@gmail.com

Dipartimento di Matematica, Via Trieste 63, 35121 Padova, Italy.
E-mail address: lucchini@math.unipd.it