# Factorizing a Group with Conjugate Subgroups

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## A MOTIVATION

Let $S$ be a simple non-abelian group and let $G := S \times S$, $\Delta := \{(s,s) \ : \ s \in S\}$. $\Delta$ is a maximal subgroup of $G$.

Let $x^S$ denote the conjugacy class of $x$ in $S$.
The following are equivalent.

- There exists an element $x \in S$ such that $(x^{-1})^S x^S = S$.
- $G$ is the product of three conjugates of $\Delta$.

This is linked to the celebrated Thompson conjecture that says that if $S$ is a nonabelian simple group there exists $x \in S$ such that $(x^S)^2 = S$. Indeed if this is the case then $x^S = (x^{-1})^S$ hence $(x^{-1})^S x^S = S$.

Guralnick and Malle proved ([2] 2012) that there are elements $a, b \in S$ such that $a^S b^S \cup \{1\} = S$. This implies that $S \times S$ is the product of four conjugates of $\Delta$.

The result presented in this talk is the following.

**Theorem (D. Levy, G)**

*Let $G$ be a non-solvable group. Then $G$ is the product of at most 36 conjugates of a proper subgroup.*

*Moreover there is a family of solvable groups (the family of dihedral groups) which are not the product of boundedly many conjugates of a proper subgroup.*

For a finite **non-nilpotent** group $G$ define $\gamma_{cp}(G)$ to be the smallest positive integer $n$ such that there exists a proper subgroup $H$ of $G$ and $n$ elements $x_1, \ldots, x_n \in G$ with the property that

$$H^{x_1} \cdots H^{x_n} = G.$$

Note that obviously if $G = H^{x_1} \cdots H^{x_n}$ then $|G| \leq |H|^n$.

It is easy to prove that if $N \trianglelefteq G$ then $\gamma_{cp}(G) \leq \gamma_{cp}(G/N)$.

**LEMMA**

*Let $A, B$ be subgroups of $G$ such that $G = AB$. Then for every $x, y \in G$, $A^x B^y = G$.*

This implies that $\gamma_{cp}(G) > 2$ always.

Indeed if $A \leq G$ is such that $G = A^x A^y$ for some elements $x, y \in G$ then by the lemma $G = AA = A$.

Let us look at **alternating** and symmetric groups.

**Lemma**

*If $G$ is a 2-transitive non-nilpotent permutation group then $\gamma_{cp}(G) = 3$.*

So in particular $\gamma_{cp}(A_n) = 3$ for all $n \geq 4$ and $\gamma_{cp}(S_n) = 3$ for all $n \geq 3$. For example we can choose $H$ to be a point stabilizer.

**Corollary**

*Let $X$ be an almost-simple group with alternating socle (i.e. symmetric, alternating, or one of the automorphism groups of $A_6$). Then $\gamma_{cp}(X) = 3$.*

What about the other simple groups?
Concerning groups of **Lie type**, Liebeck and Pyber proved the following [1]

### Theorem (Liebeck, Pyber, 2001)

*Let $T$ be a simple group of Lie type in characteristic $p$. Then $T$ is the product of at most 25 Sylow $p$-subgroups.*

Let $X$ be an almost-simple group with socle $T$. We know that there are 25 Sylow $p$-subgroups $P_1, \dots, P_{25}$ of $T$ with $P_1 \cdots P_{25} = T$. Using the Frattini Argument we can show that $N_X(P_1) \cdots N_X(P_{25}) = X$. Indeed

$$N_X(P_1) \cdots N_X(P_{25}) = N_X(P_1) \cdot N_X(P_1) \cdots N_X(P_{25})$$

$$\supseteq N_X(P_1) P_1 \cdots P_{25} = N_X(P_1) T = X.$$

### Corollary

*Let $X$ be an almost-simple group with socle of Lie type. Then $\gamma_{cp}(X) \leq 25$.*

The argument used to treat **sporadic** groups is the following.

We have mentioned that 2-transitive permutation groups $G$ (i.e. transitive permutation groups of rank 2) verify $\gamma_{cp}(G) = 3$. The following result is a generalization of this fact.

**Theorem**

*Let $M$ be a maximal non-normal subgroup of $G$ and consider the transitive action of $G$ by right multiplication on $\{Mx \ : \ x \in G\}$. Let $r$ be its rank, i.e. the number of $M$-orbits. Then $\gamma_{cp}(G) \leq r+1$.*

If $X$ is an almost-simple sporadic group then there exists a maximal subgroup $M$ of $X$ for which the rank of the associated action is at most 35. The number $36 = 35+1$ is attained by a multiplicity-free representation of $\mathrm{Aut}(O'N)$ of rank 35.

**Corollary**

*Let $X$ be an almost-simple sporadic group. Then $\gamma_{cp}(X) \leq 36$.*

Now let $G$ be any non-solvable group. We want to bound $\gamma_{cp}(G)$ from above. Using $\gamma_{cp}(G) \leq \gamma_{cp}(G/N)$ we can reduce to the case where $G$ has a unique minimal normal subgroup (i.e. $G$ is **monolithic**), and $G$ turns out to be a **primitive** permutation group.

Let $T_1 \times \cdots \times T_m = T^m$ be the socle of $G$ and let $X := N_G(T_1)/C_G(T_1)$. $X$ is an almost-simple group with socle $T_1 C_G(T_1)/C_G(T_1) \cong T_1 = T$ and $G$ embeds into the wreath product $X \wr S_m = X^m \rtimes S_m$.

## The trick

Suppose that $U \leq X$ satisfies $UT = X$ and $(U_1 \cap T) \cdots (U_h \cap T) = T$ where $U_1, \ldots, U_h$ are $h$ conjugates of $U$ in $X$. Then

$$G = N_G((U_1 \cap T)^m) \cdots N_G((U_h \cap T)^m).$$

In particular if $1 < U \cap T < T$ then $\gamma_{cp}(G) \leq h$.

This, up to being able to "lift" the needed subgroups, says that the bound for the almost-simple groups provides a bound also for the monolithic primitive groups. Hence, with the setting above, we deduce $\gamma_{cp}(G) \leq 36$.

Let $p$ be an odd prime number and let $G = D_p$ the dihedral group of degree $p$, so that $|G| = 2p$.

**PROPOSITION**

$\gamma_{cp}(G) = \lceil \log_2 p \rceil + 1$. *Moreover for every integer $n \geq 3$ there exists an odd prime $p$ such that $\gamma_{cp}(D_p) = n$.*

The idea to prove this is the following. The non-normal maximal subgroups of $G$ have order 2, hence if $G = K_1 \cdots K_h$ with $K_i$ conjugate maximal subgroups of $G$ then $|K_i| = 2$ for all $i$. Now,

$$2p = |G| = |K_1 \cdots K_h| \leq |K_1| \cdots |K_h| = 2^h.$$

Taking logarithms we find $h \geq \log_2(2p) = \log_2 p + 1$ and $\geq$ follows by taking ceilings. Now using a technical combinatorial argument we can actually show that $\gamma_{cp}(D_p) = \lceil \log_2 p \rceil + 1$.

When $p$ varies among the odd primes, $\gamma_{cp}(G) = \lceil \log_2 p \rceil + 1$ takes any possible integer value $n \geq 3$. This is an easy application of the Bertrand Postulate (choose $p$ such that $2^{n-2} < p < 2^{n-1}$).

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