COVERING DIRECT PRODUCTS WITH PROPER SUBGROUPS

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OUTLINE

1. HISTORY, DEFINITIONS AND EASY RESULTS
2. SIGMA-ELEMENTARY GROUPS
3. COVERING DIRECT PRODUCTS

EXERCISE

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These elementary considerations led Cohn in 1994 to define for every group G:

$\sigma(G)$ Sum of G: the least cardinality of a cover of G consisting of proper subgroups.

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If G is a cyclic group then it is not a union of proper subgroups, because the generators of G do not lie in proper subgroups. In this case we make the convention $\sigma(G) = \infty$, with $n < \infty$ for every integer n.

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If p is a prime number then $\sigma(C_p \times C_p) = p + 1$. Indeed, $C_p \times C_p$ has exactly $p + 1$ maximal subgroups, all of them isomorphic to C_p and pairwise intersecting in the identity subgroup, so they cover $1 + (p - 1)(p + 1) = p^2$ elements.

The following result is due to Tomkinson (1997):

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Let G be a solvable non-cyclic group. Then $\sigma(G) = |S/K| + 1$ where $|S/K|$ is the least order of a chief factor of G with more than one complement.

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Example

If q is a prime power then $\sigma(\mathbb{F}_q \rtimes \mathbb{F}_q{}^*) = q + 1$.

REMARK

If N is a normal subgroup of G then $\sigma(G) \leq \sigma(G/N)$, because every cover of G/N corresponds to a cover of G.

This suggests to study the quotients G/N such that $\sigma(G) = \sigma(G/N)$, and leads to the following:

DEFINITION (σ-ELEMENTARY GROUPS)

A group G is said to be "σ-elementary" if $\sigma(G) < \sigma(G/N)$ for every $1 \neq N \trianglelefteq G$. We say that G is "n-elementary" if G is σ-elementary and $\sigma(G) = n$.

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Example

If the σ-elementary group G is abelian then $G \cong C_p \times C_p$ for some prime p (cf. [2], Theorem 3).

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The answer is no: if $T_1, ..., T_k$ are non-abelian simple groups then

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This is the question we answered in [1], and the answer is again no, with the exception of $C_p \times C_p$.

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- We know that the maximal subgroups of $S \times S$ are of the following three types:

$$(1)\ K \times S,\ (2)\ S \times K,\ (3)\ \Delta_{\varphi} := \{(x, \varphi(x)) \mid x \in S\},$$

where K is a maximal subgroup of S and $\varphi \in \mathrm{Aut}(S)$.

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- Let $\Omega := S \times S - \bigcup_{M \in \mathcal{M}_1 \cup \mathcal{M}_2} M = \Omega_1 \times \Omega_2$, where $\Omega_1 = S - \bigcup_{K \times S \in \mathcal{M}_1} K$ and $\Omega_2 = S - \bigcup_{S \times K \in \mathcal{M}_2} K$.

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- We prove that $\Omega = \emptyset$. Suppose $\Omega \neq \emptyset$. Let $\omega \in \Omega_1$. Notice that $\{K < S \mid S \times K \in \mathcal{M}_2\} \cup \{\langle \varphi(\omega) \rangle \mid \Delta_\varphi \in \mathcal{M}_3\}$ covers S.

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- It follows that

$$|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = |\mathcal{M}| = \sigma(S \times S) \leq \sigma(S) \leq |\mathcal{M}_2| + |\mathcal{M}_3|.$$

  This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$.

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  This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$.
- Since $S = \bigcup_{S \ni s \neq 1} \langle s \rangle$ and $|M| = |S|$ for every $M \in \mathcal{M}_3$,

$$|S| - 1 \geq \sigma(S) \geq \sigma(S \times S) = |\mathcal{M}| = |\mathcal{M}_3| \geq |S|, \text{ contradiction.}$$

In fact there is a much more general conjecture ([3]):

Conjecture (Lucchini, Detomi)

Every non-abelian σ-elementary group is monolithic.

There are partial results supporting this conjecture. Let G be a non-abelian σ-elementary group. Then:

- ([3], Corollary 14) G has at most one abelian minimal normal subgroup. In particular if it is solvable, it is monolithic.
- ([3], Corollary 14) G is a subdirect product of monolithic primitive groups.
- ([3], Proposition 21) Any solvable proper quotient of G is cyclic.
- ([3], Theorem 24) Suppose G has no abelian minimal normal subgroups. Then either G is a primitive monolithic group and $G/\mathrm{soc}(G)$ is cyclic, or $G/\mathrm{soc}(G)$ is non-solvable and all the non-abelian composition factors of $G/\mathrm{soc}(G)$ are alternating groups of odd degree.
- ([5], Lemma 3) If $\sigma(G) \leq 33$ then G is monolithic.

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Theorem

Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_1*. In this case* $\sigma(G) = \sigma(H_1)$.
2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_2*. In this case* $\sigma(G) = \sigma(H_2)$.
3. *There exist* $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ *with* $H_1/N_1 \cong H_2/N_2 \cong C_p$ *and* $\mathcal{M}$ *consists of the maximal subgroups of* $H_1 \times H_2$ *containing* $N_1 \times N_2$*. In this case* $\sigma(G) = p + 1$.

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3. *There exist $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ with $H_1/N_1 \cong H_2/N_2 \cong C_p$ and $\mathcal{M}$ consists of the maximal subgroups of $H_1 \times H_2$ containing $N_1 \times N_2$. In this case* $\sigma(G) = p + 1$.

Remark

This theorem can be re-stated in the general case $G = H_1 \times ... \times H_n$.

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Proposition (Maximal subgroups of a direct product)

Let $G = H_1 \times H_2$ be the direct product of two finite groups. A maximal subgroup of G is called "(of) ***standard*** *(type)" if it is of the form $M \times H_2$ with M a maximal subgroup of H_1 or $H_1 \times M$ with M a maximal subgroup of H_2, it is called "(of)* ***diagonal*** *(type)" if it is of the form $\{(x, y) \in G \mid \varphi(xN_1) = yN_2\}$ where N_i is a maximal normal subgroup of H_i for $i = 1, 2$ and $\varphi : H_1/N_1 \to H_2/N_2$ is an isomorphism. It is well known that every maximal subgroup of G is either of standard type or of diagonal type.*

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For a proof see [7], Chap. 2, (4,19).

Step 1. Let $\mathcal{M}$ be a minimal cover of $G = H_1 \times H_2$ consisting of maximal subgroups. Assume that $\mathcal{M}$ contains no subgroup of diagonal type whose index is a prime number. We want to show that in this case either $H_1 \times 1$ or $1 \times H_2$ is contained in $\bigcap_{M \in \mathcal{M}} M$.

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  $$\mathcal{M}_3 := \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_2),$$
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- Let K_i be intersection of the maximal normal subgroups of H_i, for $i = 1, 2$. There are simple groups $S_1, ..., S_\alpha, T_1, ..., T_\beta$ such that
$$H_1/K_1 = \prod_{1 \leq a \leq \alpha} S_a, \qquad H_2/K_2 = \prod_{1 \leq b \leq \beta} T_b.$$

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- $\rho_i : H_i \to H_i/K_i$, $\qquad \Delta_i := \rho_i(\Omega_i)$,
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- To any $M \in \mathcal{M}_3$ we may associate a triple (a, b, ϕ) with $1 \le a \le \alpha$, $1 \le b \le \beta$ and $\phi : S_a \to T_b$ a group isomorphism such that $M = M(a, b, \varphi)$ equals

$$\{(h_1, h_2) \in H_1 \times H_2 \mid \phi(\pi_{1,a}(h_1)) = \pi_{2,b}(h_2)\}.$$

By the hypothesis, if $M(a, b, \phi) \in \mathcal{M}_3$ then $S_a \cong T_b$ is non-abelian.

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- Fix $\omega \in \Omega_1$. $\rho_1(\omega) =: (s_1, ..., s_\alpha) \in \Delta_1$. For $M(a, b, \phi) \in \mathcal{M}_3$ let

$$U(a, b, \phi) := \{h \in H_2 \mid \pi_{2,b}(h) \in \langle \phi(s_a) \rangle\}.$$

T_b is non-abelian $\Rightarrow \langle \phi(s_a) \rangle \neq T_b \Rightarrow U(a, b, \phi) < H_2$.

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- The following family of proper subgroups of H_2 covers H_2:
  $\{K < H_2 \mid H_1 \times K \in \mathcal{M}_2\} \cup \{U(a, b, \phi) \mid M(a, b, \phi) \in \mathcal{M}_3\}$.

- $H_1/K_1 = \prod_{1 \leq a \leq \alpha} S_a$, $\quad H_2/K_2 = \prod_{1 \leq b \leq \beta} T_b$.
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- Fix $\omega \in \Omega_1$. $\rho_1(\omega) =: (s_1, ..., s_\alpha) \in \Delta_1$. For $M(a, b, \phi) \in \mathcal{M}_3$ let

$$U(a, b, \phi) := \{h \in H_2 \mid \pi_{2,b}(h) \in \langle \phi(s_a) \rangle \}.$$

  T_b is non-abelian $\Rightarrow \langle \phi(s_a) \rangle \neq T_b \Rightarrow U(a, b, \phi) < H_2$.
- The following family of proper subgroups of H_2 covers H_2:
  $\{K < H_2 \mid H_1 \times K \in \mathcal{M}_2\} \cup \{U(a, b, \phi) \mid M(a, b, \phi) \in \mathcal{M}_3\}$.
- It follows that

$$|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = |\mathcal{M}| = \sigma(H_1 \times H_2) \leq \sigma(H_2) \leq |\mathcal{M}_2| + |\mathcal{M}_3|.$$

  This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$.

Now the conclusion of Step 1 follows easily. For sake of exposition, we will state and use the following (quite useful) technical lemma.

Lemma

Let G be a finite group, let N be a proper normal subgroup of G, and let $U_1, ..., U_h, V_1, ..., V_k$ be proper subgroups of G such that $U_1, ..., U_h$ contain N, $V_1, ..., V_k$ supplement N, and $\beta_1 \leq ... \leq \beta_k$, where $\beta_i = |G : V_i|$ for $i = 1, ..., k$.

If $U_1 \cup ... \cup U_h \cup V_1 \cup ... \cup V_k = G$ and $U_1 \cup ... \cup U_h \neq G$ then $\beta_1 \leq k$.

Moreover, if $\beta_1 = k$ then $\beta_1 = ... = \beta_k = k$ and $V_i \cap V_j \leq U_1, ..., U_h$ for every $i \neq j$ in $\{1, ..., k\}$.

Apply this lemma with $N = H_1 \times \{1\} = U_1$, $h = 1$, $\{V_1, ..., V_k\} = \mathcal{M}_3$. The index of $V_1 \in \mathcal{M}_3$ is the order of a simple non-abelian group S which is an epimorphic image of G.

$$|S| = |G : V_1| = \beta_1 \leq k = |\mathcal{M}_3| = |\mathcal{M}| = \sigma(G) \leq \sigma(S) \leq |S| - 1,$$

contradiction.

Step 2. Assume now that there exists $M \in \mathcal{M}$ of diagonal type and index p, say $M = \{(x,y) \in G \mid \varphi(xN_1) = yN_2\}$ for some $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ with $H_i/N_i \cong C_p$ for $i = 1, 2$. We prove that then $\mathcal{M}$ consists of normal subgroups of index p.

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- Note that $\sigma(G) \leq \sigma(H_1 \times H_2/N_1 \times N_2) = \sigma(C_p \times C_p) = p + 1$. Induction: assume that no non-trivial normal subgroup N of G is contained in H_i and in every element of $\mathcal{M}$, for $i = 1, 2$.

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- Let $N = M_1 \cap M_2$ and $\{U_1, ..., U_h, V_1, ..., V_k\} = \mathcal{M}$. Applying the lemma we get $h = 1$, $M_1 \cap ... \cap M_p \cap K = M_1 \cap M_2 = N$. Therefore $\mathcal{M}$ corresponds to the unique cover of $G/N \cong C_p \times C_p$.

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- We only discuss the case in which N is a non-abelian simple group. In this case C_p is isomorphic to a chief factor of a subgroup of $\mathrm{Out}(N)$ hence $p \leq |\mathrm{Out}(N)|$.
- $\beta = |G : M| = |N : M \cap N|$ is the index of a proper subgroup of N, therefore $\beta > 2p$ (this follows from [6], Lemma 2.7, which relies on the CFSG). Then $p + 1 \geq \beta > 2p$, contradiction.

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