Ideas in finite group theory

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Invertible matrices

Let F be a field (for example $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$). Let

$$GL(n, F) \qquad \text{(General Linear Group)}$$

denote the set of $n \times n$ invertible matrices with entries in the field F. The usual row-column multiplication gives $GL(n, F)$ the structure of a (non-abelian!) group.

$F^* := F - \{0\}$ is a group with respect to multiplication.
It is abelian (commutative).

$\det : GL(n, F) \to F^*$ is a **group homomorphism**. Its kernel $\ker(\det)$ is denoted

$$SL(n, F) \qquad \text{(Special Linear Group).}$$

It consists of the $n \times n$ matrices with entries in F and determinant 1.

Permutations

The Symmetric Group: $\mathrm{Sym}(n) =$ the group of bijections

$$\{1,\dots,n\} \to \{1,\dots,n\}$$

with the operation given by the **composition**. An element of $\mathrm{Sym}(n)$ is called permutation of $\{1,\dots,n\}$. The order of $\mathrm{Sym}(n)$ (its size as a set) is $n! = 1 \cdot 2 \cdots n$. For example

$$(123)(4567): \qquad 1 \mapsto 2 \mapsto 3 \mapsto 1, \qquad 4 \mapsto 5 \mapsto 6 \mapsto 7 \mapsto 4.$$

$$(123\cdots k): \qquad 1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto k \mapsto 1 \qquad k\text{-cycle}.$$

Composition goes as follows:

$$(12)(234)(13) = (234), \qquad (143)(1352)(4312) = (13)(45).$$

Note that disjoint cycles always commute.

$$(123)(4567) = (4567)(123).$$

Remark

Every permutation can be written uniquely (up to reordering) as **product of disjoint cycles**.

Definition

A permutation is called "even" if it can be written as the product of an even number of 2-cycles, and "odd" otherwise. For example $(12)(25)(13)(35)$ *is even,* $(13)(26)(43)$ *is odd.*
A product of disjoint cycles is an even permutation if and only if the number of cycles of even length is even.

Example:

$$(123)(4567), \qquad (12)(3456)(78) \qquad ODD,$$

$$(123)(45)(67), \qquad (123)(4567)(89) \qquad EVEN.$$

- All elements of cycle structure $(3,4)$, $(2,2,4)$ are odd.
- All elements of cycle structure $(2,2,3)$, $(2,3,4)$ are even.

Definition (Alternating group)

Even permutations are said to have sign 1.
$\sigma \in Sym(n)$ *is even if* $sgn(\sigma) = 1$.

Odd permutations are said to have sign -1.
$\sigma \in Sym(n)$ *is odd if* $sgn(\sigma) = -1$.

$sgn : Sym(n) \to \{-1, 1\} = C_2$ *is a group homomorphism.*

Its kernel $\ker(sgn)$ *is denoted* $Alt(n)$ *or* A_n*: it is the Alternating group of degree* n*. It is a normal subgroup of* $Sym(n)$*. Its order is* $n!/2$.

For **example** Alt(4) consists of the following elements.

$$\begin{array}{cccc} 1, & (12)(34), & (13)(24), & (14)(23), \\ (123), & (132), & (124), & (142), \\ (234), & (243), & (134), & (143). \end{array}$$

To each polynomial $f(X) \in \mathbb{Q}[X]$ without multiple roots can be attached a finite group G_f, called the Galois group of the polynomial (named after **Evariste Galois, 1811 - 1832**). It is defined as follows: if $a_1, \ldots, a_n \in \mathbb{C}$ denote the distinct roots of $f(X)$ then

$$G_f = \mathrm{Aut}(\mathbb{Q}(a_1, \ldots, a_n)).$$

That is, G_f is the group of ring isomorphisms

$$\mathbb{Q}(a_1, \ldots, a_n) \to \mathbb{Q}(a_1, \ldots, a_n).$$

The group G_f permutes the roots of f, and the only element of G_f which fixes all the roots is the identity. In other words, G_f can be described (or better, "represented") as a subgroup of $\mathrm{Sym}(n)$.

For **example** the Galois group of $X^2 - 2$ is the automorphism group of the field $\mathbb{Q}(\sqrt{2})$, so it consists of two elements: the identity and the automorphism $\tau : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ which sends $a + b\sqrt{2}$ to $a - b\sqrt{2}$. The Galois group of $X^2 - 2$ is cyclic of order 2. $G_f \cong \mathrm{Sym}(2) \cong C_2$.

The Galois group encodes the factorization patterns modulo prime numbers.

Let us be more precise. Let $f(X)$ be an irreducible polynomial of $\mathbb{Z}[X]$ of degree n. According to a theorem of Frobenius,

- if for a prime p for which $f(X)$ mod p does not admit multiple irreducible factors (such prime is usually called "unramified") the factorization pattern of $f(X)$ mod p is $(n_1, \ldots, n_t)$ (meaning that there are t irreducible factors of degrees $n_1, \ldots, n_t$)
- then the Galois group of $f(X)$, seen as a (transitive) subgroup of $\mathrm{Sym}(n)$, contains an element of cycle structure $(n_1, \ldots, n_t)$.

Let us consider the following **examples**:

- $X^3 + X^2 + X + 3$ (discriminant $-204 = -2^2 \cdot 3 \cdot 17$);
- $X^3 - 3X + 1$ (discriminant $81 = 3^4$).

p	X^3+X^2+X+3	p	X^3+X^2+X+3
2	$(X+1)^3$	53	$(X+43)(X^2+11X+5)$
3	$X(X+2)^2$	59	$(X+12)(X^2+48X+15)$
5	X^3+X^2+X+3	61	$(X+6)(X^2+56X+31)$
7	$(X+4)(X^2+4X+6)$	67	$(X+23)(X+52)(X+60)$
11	X^3+X^2+X+3	71	$(X+38)(X+52)(X+53)$
13	X^3+X^2+X+3	73	$(X+34)(X^2+40X+28)$
17	$(X+5)(X+15)^2$	79	$(X+74)(X^2+6X+31)$
19	X^3+X^2+X+3	83	$(X+45)(X^2+39X+72)$
23	X^3+X^2+X+3	89	$(X+32)(X^2+58X+14)$
29	$(X+11)(X+23)(X+25)$	97	$(X+59)(X^2+39X+28)$
31	$(X+15)(X^2+17X+25)$	101	$(X+75)(X^2+27X+97)$
37	$(X+25)(X^2+13X+9)$	103	X^3+X^2+X+3
41	X^3+X^2+X+3	107	X^3+X^2+X+3
43	X^3+X^2+X+3	113	X^3+X^2+X+3
47	$(X+31)(X^2+17X+38)$	127	X^3+X^2+X+3

$\mathrm{Sym}(3)=\{1,(12),(13),(23),(123),(132)\}$.

p	X^3-3X+1	p	X^3-3X+1
2	X^3+X+1	53	$(X+18)(X+39)(X+49)$
3	$(X+1)^3$	59	$X^3+56X+1$
5	X^3+2X+1	61	$X^3+58X+1$
7	X^3+4X+1	67	$X^3+64X+1$
11	X^3+8X+1	71	$(X+16)(X+25)(X+30)$
13	$X^3+10X+1$	73	$(X+14)(X+25)(X+34)$
17	$(X+3)(X+4)(X+10)$	79	$X^3+76X+1$
19	$(X+10)(X+12)(X+16)$	83	$X^3+80X+1$
23	$X^3+20X+1$	89	$(X+12)(X+36)(X+41)$
29	$X^3+26X+1$	97	$X^3+94X+1$
31	$X^3+28X+1$	101	$X^3+98X+1$
37	$(X+14)(X+28)(X+32)$	103	$X^3+100X+1$
41	$X^3+38X+1$	107	$(X+7)(X+40)(X+60)$
43	$X^3+40X+1$	113	$X^3+110X+1$
47	$X^3+44X+1$	127	$(X+53)(X+87)(X+114)$

$\mathrm{Alt}(3)=\{1,(123),(132)\}\subset \mathrm{Sym}(3)$.

The most famous open problem in group theory is probably the Inverse Galois Problem.

Inverse Galois Problem

Is it true that for any finite group G there exists a polynomial $f(X) \in \mathbb{Q}[X]$ with $G_f \cong G$?

This problem has been solved for abelian groups (even solvable groups), but the answer in general is not known.

Let us list the important results of "elementary" finite group theory.

- the **"order" of an element** $g \in G$, denoted $o(g)$, is the smallest positive integer n such that $g^n = 1$;
- the **"order" of a subgroup** $H \leq G$, denoted $|H|$, is its size.
- It turns out that $|\langle g \rangle| = o(g)$.

Theorem (Lagrange (1736 - 1813))

*Let G be a finite group, and let $H \leq G$. Then $|H|$ divides $|G|$. The integer $|G|/|H| = |G : H|$ is called the "**index**" of H in G.*

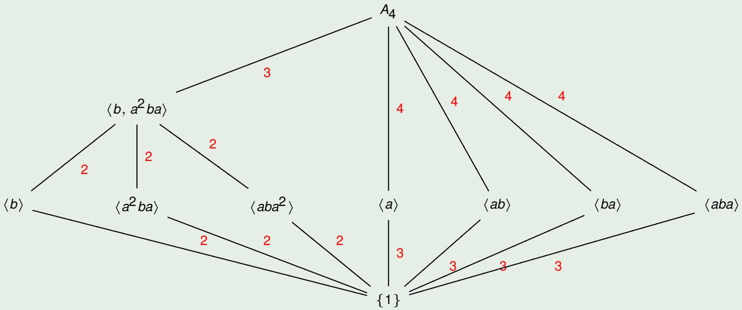
Not every divisor of $|G|$ equals the size of a subgroup of G, but...

Theorem (Cauchy (1789 - 1857))

Let G be a finite group, and let p be a prime dividing $|G|$. Then there exists $g \in G$ of order p.

Example: The alternating group of degree 4

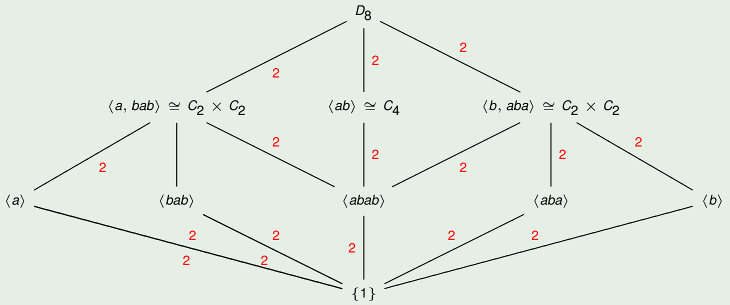
$A_4 = \langle a, b \rangle$ where $a = (123)$ and $b = (12)(34)$.
$|A_4| = 4!/2 = 12 = 2^2 \cdot 3$. Subgroup lattice:



$A_4 = \{1, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (234), (243), (134), (143)\}$.
Note that A_4 has no subgroups of order 6.

Example: The dihedral group of order 8

$D_8 = \langle a, b \mid a^2 = b^2 = 1,\ (ab)^4 = 1 \rangle$. $|D_8| = 8 = 2^3$. Subgroup lattice:



Note that D_8 has subgroups of order $1, 2, 4, 8$.

Suppose $|G|$ is divisible by a prime-power p^k. Can we always find a subgroup $H \leq G$ with $|H| = p^k$?

Theorem (Sylow (1832 - 1918))

Let G be a finite group and write $|G| = mp^n$ where p is a prime and m is not divisible by p.

- *G contains a subgroup P of order p^n. P is called "Sylow p-subgroup" of G.*
- *G contains a subgroup of order p^k for every $0 \leq k \leq n$.*
- *If P, Q are two Sylow p-subgroups of G then they are conjugated: there exists $g \in G$ such that $g^{-1}Pg = Q$.*
- *The number of Sylow p-subgroups of G is congruent to $1 \mod p$.*
- *If H is a subgroup of G such that $|H|$ is a power of p then there exists a Sylow p-subgroup P of G such that $H \leq P$.*

Example

Let $F = \mathbb{Z}/5\mathbb{Z}$ and let

$$G := \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \ : \ a, b, c \in F, \ a, c \neq 0 \},$$

$$H := \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in G \ : \ b = 0 \},$$

$$K := \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in G \ : \ a = c = 1 \}.$$

- G is a group (with respect to multiplication) of order $4^2 \cdot 5 = 2^4 \cdot 5$,
- $|H| = 4^2 = 2^4 \Rightarrow H$ is a Sylow 2-subgroup of G and
- $|K| = 5 \Rightarrow K$ is a Sylow 5-subgroup of G.

Theorem (Cayley (1821 - 1895))

Let G be a group. Then the application

$$G \to \mathrm{Sym}(G), \qquad g \mapsto (x \mapsto gx)$$

is an injective homomorphism.
In particular, G is isomorphic with a subgroup of $\mathrm{Sym}(G)$.

Corollary

Let G be a finite group. There exists a positive integer n such that G is isomorphic with a subgroup of $\mathrm{Sym}(n)$.

Cayley's theorem says we may choose $n = |G|$. But sometimes we can choose a smaller n.

For **example**, if G_f is the Galois group of the polynomial $f(X) \in \mathbb{Q}[X]$ with n distinct roots, then the permutation action of G_f on the n roots gives an injective homomorphism $G_f \to \mathrm{Sym}(n)$.

Simple group G is **simple** if $\{1\}$ and G are the only normal subgroups of G.

Abelian simple groups

Abelian simple groups are the cyclic groups of prime order, $C_p = \{g, g^2, \ldots, g^{p-1}, g^p = 1\} \cong (\mathbb{Z}/p\mathbb{Z}, +) = \{1, 2, \ldots, p-1, p = 0\}$.

Alternating Groups

If $n \geq 5$ is an integer, $\mathrm{Alt}(n)$ is a non-abelian simple group.

Projective Linear Groups

Let F be a field, and let $GL(n, F)$ be the group of invertible matrices over the field F. Let $SL(n, F)$ be the subgroup of $GL(n, F)$ consisting of matrices of determinant 1. Let Z be the subgroup of $GL(n, F)$ consisting of scalar matrices. If $n \geq 2$ and $|F| \geq 4$, the quotient

$$PSL(n, F) := SL(n, F)/Z \cap SL(n, F)$$

(projective linear group) is an example of non-abelian simple group.

Given a finite group G, we can costruct longest possible chains of subgroups of the form

$$\{1\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_k = G.$$

Maximality of k implies that the factor groups G_i/G_{i-1} are all simple groups. Such chain is called "**composition series**" and its factors G_i/G_{i-1} are called "**composition factors**".

Theorem (Jordan-Holder)

Any two composition series of a given finite group have the same length and the same composition factors (up to reordering and isomorphism).

Cyclic groups

The composition factors of a cyclic group C_n correspond to the prime divisors of n, counted with multiplicity. If $n = 60 = 2^2 \cdot 3 \cdot 5$,

$$1 \lhd \langle g^{30} \rangle (\cong C_2) \lhd \langle g^{15} \rangle (\cong C_4) \lhd \langle g^5 \rangle (\cong C_{12}) \lhd \langle g \rangle = C_{60}.$$

Solvable group If the composition factors of the finite group G are all abelian (hence cyclic of prime order) then G is said to be **solvable**.

Evariste Galois proved that the zeros of a polynomial $f(X) \in \mathbb{Q}[X]$ can be expressed by starting from the elements of $\mathbb{Q}$ and performing sums, differences, products, divisions, and root extractions if and only if the Galois group G_f is solvable. In this case $f(X)$ is said to be "**solvable by radicals**".

Degrees $2, 3, 4$

Polynomials of degree 2, 3, 4 are solvable by radicals.

Indeed, all subgroups of Sym(4) are solvable.

Example

The Galois group of $f(X) = X^4 - 4X + 2 \in \mathbb{Z}[X]$ is S_4, so $f(X)$ is solvable by radicals. Indeed, S_4 is solvable:

$$\{1\} \xrightarrow{C_2} \langle (12)(34) \rangle \xrightarrow{C_2} O_2(S_4) \xrightarrow{C_3} A_4 \xrightarrow{C_2} S_4$$

The composition factors of S_4 are C_2 (three times) and C_3.
$|S_4| = 24 = 2^3 \cdot 3$.

Example

The symmetric group S_n is not solvable when $n \geq 5$:

$$\{1\} \xrightarrow{A_n} A_n \xrightarrow{C_2} S_n$$

The composition factors of S_n are A_n (not abelian) and C_2.

- Any subgroup of a solvable group is solvable.
- Any quotient of a solvable group is solvable.
- Any finite direct product of solvable groups is solvable.
- If $N \trianglelefteq G$ and G/N are solvable then G is solvable.

EXAMPLE

Abelian groups are solvable.

EXAMPLE

Suppose the order of the finite group G is a prime-power, $|G| = p^n$, where p is a prime. Then G is solvable.

EXAMPLE

A direct product of finitely many groups of prime-power order is solvable.

A non-solvable polynomial

Consider the polynomial $f(X) = X^5 - 2X - 2$, irreducible in $\mathbb{Q}[X]$. Its discriminant is $41808 = 2^4 \cdot 3 \cdot 13 \cdot 67$.

- Reduction modulo 5: $(X+2)(X^4+3X^3+4X^2+2X+4)$.
- Reduction modulo 7: X^5+5X+5.
- Reduction modulo 17: $(X^2+X+6)(X^3+16X^2+12X+11)$.
- The Galois group G_f of $f(X)$ is a subgroup of $\mathrm{Sym}(5)$ containing elements of cycle structures (4), (5) and $(3,2)$.
- Deduce that $G_f = \mathrm{Sym}(5)$.
- $\mathrm{Sym}(5)$ contains the non-abelian simple group $\mathrm{Alt}(5)$, hence G_f is non-solvable.
- $f(X) = X^5 - 2X - 2$ is not solvable by radicals.

A finite group is said to be **nilpotent** if it is a direct product of finitely many groups of prime-power order. Equivalently, a finite group is nilpotent if and only if its Sylow subgroups are all normal.

- Cyclic groups are abelian. $C_2 \times C_2$ is abelian but not cyclic.
- Abelian groups are nilpotent. D_8 is nilpotent but not abelian.
- Nilpotent groups are solvable. S_3 is solvable but not nilpotent.
- A_5 is a non-abelian simple group. In particular it is not solvable.

Cyclic $\subset$ *Abelian* $\subset$ *Nilpotent* $\subset$ *Solvable*.

When facing a problem concerning finite groups, one usually goes through all these steps before dealing with the general case.

Feit and Thompson proved the following very deep result.

Theorem (Feit, Thompson, 1962-1963)

Any finite group of odd order is solvable.

This result is called the "Odd Order Theorem". It implies that **non-abelian finite simple groups have even order**.

With this as a starting point, all finite non-abelian simple groups have been classified:

Theorem

Let S be a non-abelian simple group. Then one of the following holds.

- $S \cong C_p$ *for some prime* p.
- $S \cong Alt(n)$ *for some integer* $n \geq 5$.
- S *is a group of Lie type.*
- S *is one of 26 sporadic groups.*

Consider the symmetric group $G = \mathrm{Sym}(n)$, and a subgroup $H \leq G$. G consists of permutations of $I_n = \{1, \ldots, n\}$, so if $g \in G$ and $i \in I_n$ it makes sense to consider $g(i) \in I_n$. Given $i \in I_n$ consider

$$O_H(i) := \{h(i) \ : \ h \in H\} \qquad \text{(The } H\text{-}\textbf{orbit} \text{ of } i\text{).}$$

If H has exactly one orbit it is called **transitive** (e.g. $\langle(1234)\rangle < S_4$), otherwise it is called **intransitive** (e.g. $\langle(12),(34)\rangle < S_4$).

If H has exactly two orbits then H is said to be "**maximal intransitive**". If $A \cap B = \emptyset$, $\mathrm{Sym}(A) \times \mathrm{Sym}(B) < \mathrm{Sym}(A \cup B)$.

We can easily establish wheter an element of G lies in an intransitive subgroup. For example $(123)(45)(67) \in \mathrm{Sym}(7)$ lies in the (maximal) intransitive subgroup whose orbits are $\{1,2,3,4,5\}$ and $\{6,7\}$, and also in the (maximal) intransitive subgroup whose orbits are $\{1,2,3\}$ and $\{4,5,6,7\}$. (1234567) does not lie in intransitive subgroups.

The elements of $\mathrm{Sym}(n)$ which do not lie in intransitive subgroups are precisely the n-cycles.

Note that **a maximal intransitive subgroup is determined by the choice of its two orbits**.
Therefore, there are exactly $\binom{n}{k}$ maximal intransitive subgroups with orbits of sizes k, $n-k$.

Suppose $n > 1$ is odd. The total number of maximal intransitive subgroups of $\mathrm{Sym}(n)$ is

$$\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{(n-1)/2} = 2^{n-1} - 1.$$

Since any n-cycle belongs to $\mathrm{Alt}(n)$ (n being odd), $\mathrm{Sym}(n)$ can be covered by 2^{n-1} proper subgroups. It turns out that it actually cannot be covered with less if $n \neq 9$. The case $n = 9$ is not known.

We express this by saying that if $n > 1$ is odd and $n \neq 9$ then the **covering number** of $\mathrm{Sym}(n)$ is 2^{n-1}: $\sigma(\mathrm{Sym}(n)) = 2^{n-1}$.

PROPOSITION

Let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree $n > 1$. Suppose that $f(X)$ admits a root modulo every prime. Then $f(X)$ is reducible.

PROOF.

There is a correspondence between factorization patterns modulo unramified primes and cycle structures of elements of the Galois group of $f(X)$ viewed as a transitive subgroup of S_n.

If $f(X)$ admits a root modulo every prime then every $\sigma \in G_f \leq S_n$ has at least a fixed point in $\{1, \dots, n\}$. Since G_f is transitive, the point stabilizers are pairwise conjugated.

But **a finite group is never equal to the union of one single conjugacy class of subgroups**. $\square$

REMARK

No group is union of two proper subgroups.

THEOREM (SCORZA 1926)

A group G is union of three proper subgroups if and only if it there exists $N \trianglelefteq G$ such that $G/N \cong C_2 \times C_2$.

Remark

No group is union of two proper subgroups.

Theorem (Scorza 1926)

A group G is union of three proper subgroups if and only if it there exists $N \trianglelefteq G$ such that $G/N \cong C_2 \times C_2$.

These considerations led Cohn in 1994 to define for every group G:

$\sigma(G)$ Covering number of G: the smallest cardinality of a covering of G, i.e. family of proper subgroups of G whose union equals G.

If G is cyclic we pose $\sigma(G) = \infty$.

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Theorem (Neumann 1954)

*If G is an **infinite** group with finite covering number then there exists $N \trianglelefteq G$ such that G/N is finite and $\sigma(G) = \sigma(G/N)$.*

Cyclic groups do not admit coverings!

Proposition

A group G is cyclic if and only if G is not the union of its proper subgroups.

Proof.

Suppose G is cyclic generated by $g \in G$. Then

$$g \notin \bigcup_{H<G} H \qquad \Rightarrow \qquad \bigcup_{H<G} H \neq G.$$

Conversely, if G is not cyclic then

$$G = \bigcup_{g \in G} \langle g \rangle \subseteq \bigcup_{H<G} H \qquad \Rightarrow \qquad \bigcup_{H<G} H = G.$$

$\square$

A couple of easy remarks:

- If $\mathcal{H}$ is a covering of G then $\sigma(G) \leq |\mathcal{H}|$.
- If G is any non-cyclic group then $\sigma(G) < |G|$. Indeed, G is always covered by its non-trivial cyclic subgroups.
- If $N \trianglelefteq G$ and $\mathcal{H}$ is a covering of G/N then letting $\pi : G \to G/N$ be the canonical projection, the family

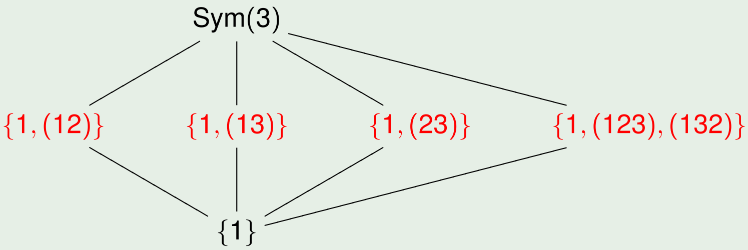
$$\{\pi^{-1}(H) \ : \ H \in \mathcal{H}\}$$

is a covering of G of size $|\mathcal{H}|$. It follows that $\sigma(G) \leq \sigma(G/N)$.
- Let $\Phi(G)$, **the Frattini subgroup of** G, be **the intersection of the maximal subgroups of** G. Then $\Phi(G) \trianglelefteq G$ and

$$\sigma(G) = \sigma(G/\Phi(G)).$$

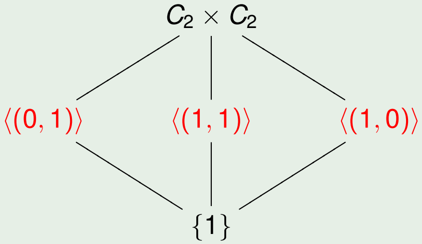
The symmetric group of degree 3

$\mathrm{Sym}(3) = \{1, (12), (13), (23), (123), (132)\}$. $\sigma(\mathrm{Sym}(3)) = 4$.
Subgroup lattice:



The Klein 4 group

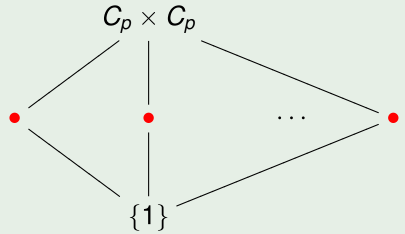
$C_2 \times C_2 = \{(a,b) \ : \ a,b \in C_2\}$. $\sigma(C_2 \times C_2) = 3$. Subgroup lattice:



Let us consider a very important **abelian** non-cyclic type of group.

The non-cyclic group of order p^2

Let p be a prime. $\sigma(C_p \times C_p) = p + 1$. Subgroup lattice:



Consider the following matrices of $GL(2,\mathbb{C})$.

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ K = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The group they generate is **finite**. It has eight elements. It is called the quaternion group of order 8.

$$Q_8 = \langle 1, I, J, K \rangle = \{1, -1, I, -I, J, -J, K, -K\}.$$

We have the following relations:

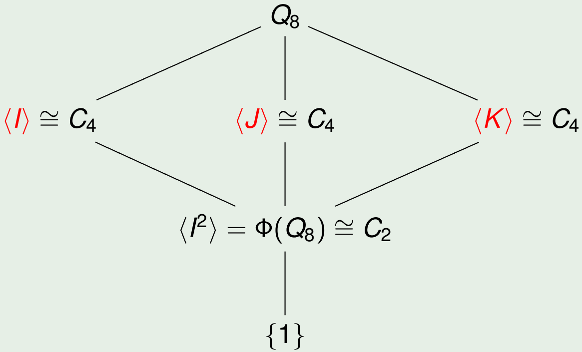
$$IJ = K = -JI, \qquad I^2 = J^2 = K^2 = -1.$$

The group Q_8 is an example of a finite **non-abelian group all of whose subgroups are normal** (Hamiltonian group).
Hamilton (1805 - 1865) was the discoverer of quaternions.

Hamiltonian groups have been classified. The finite Hamiltonian groups are precisely the groups of the form $Q_8 \times {C_2}^k \times A$ with $k \geq 0$ an integer and A an abelian group of odd order.

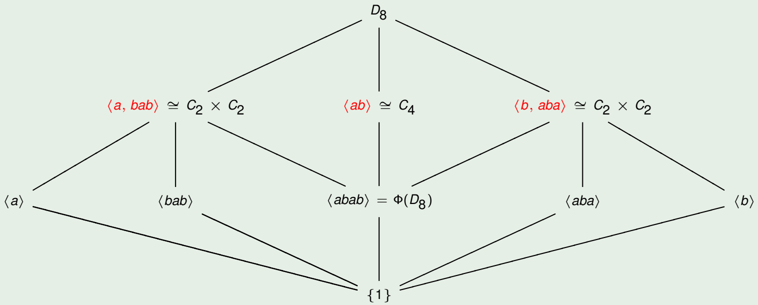
The Quaternion group of order 8

$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ with $IJ = K = -JI$, $I^2 = J^2 = K^2 = -1$.
$|Q_8| = 8$. $\sigma(Q_8) = 3$. Subgroup lattice:



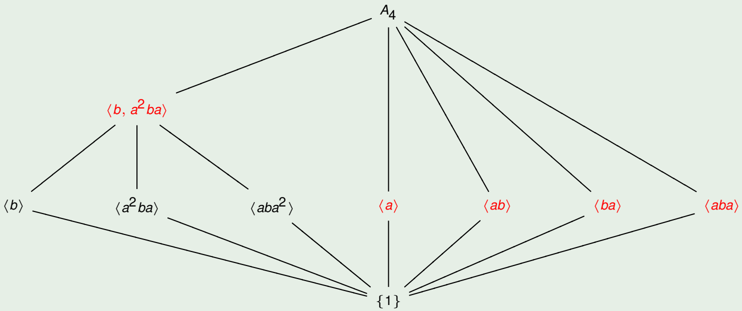
The dihedral group of order 8

$D_8 = \langle a, b \mid a^2 = b^2 = 1,\ (ab)^4 = 1 \rangle$. $\sigma(D_8) = 3$. Subgroup lattice:



The alternating group of degree 4

$A_4 = \langle a, b \rangle$ where $a = (123)$ and $b = (12)(34)$. $|A_4| = 4!/2 = 12$.
$\sigma(A_4) = 5$. Subgroup lattice:



$A_4 = \{1, (12)(34), (13)(24), (14)(23), (123), (132),$
$(124), (142), (234), (243), (134), (143)\}$.

Let us prove a very easy lower bound.

Proposition (The Minimal Index Lower Bound)

Let G be a non-cyclic group, and write $G = H_1 \cup \cdots \cup H_n$ as union of $n = \sigma(G)$ proper subgroups. Let $\beta_i := |G : H_i| := |G|/|H_i|$ for $i = 1, \dots, n$. Then $\min\{\beta_1, \dots, \beta_n\} < \sigma(G)$.

Proof.

We may assume that $\beta_1 \leq \cdots \leq \beta_n$. Since $1 \in H_1 \cap \ldots \cap H_n$ the union $H_1 \cup \ldots \cup H_n$ is not disjoint and hence

$$|G| < \sum_{i=1}^{n} |H_i| = |G| \sum_{i=1}^{n} \frac{1}{\beta_i} \leq \frac{|G|n}{\beta_1}.$$

Therefore $\beta_1 < n$. $\square$

p-groups

Consider the group $G = {C_p}^n = C_p \times \cdots \times C_p$ ($n \geq 2$ times), where p is a prime. $|G| = p^n$.

- G projects onto $C_p \times C_p$, so $\sigma(G) \leq \sigma(C_p \times C_p) = p + 1$.
- By the minimal index lower bound, $p < \sigma(G)$.

It follows that $\sigma(G) = p + 1$.

Let $d(G)$ denote the smallest number of generators of the group G. If G is any finite p-group then

$$\sigma(G) = \sigma(G/\Phi(G)) = ({C_p}^{d(G)}) = p + 1$$

as long as G is non-cyclic, i.e. $d(G) > 1$.

Nilpotent groups (direct products of finitely many groups of prime-power order).

Proposition

If A and B are two finite groups of coprime order then

$$\sigma(A \times B) = \min\{\sigma(A), \sigma(B)\}.$$

It follows that if G is a nilpotent group $P_1 \times \cdots \times P_t$, where $P_1, \ldots, P_t$ are finite groups of prime-power pairwise coprime order, then

$$\sigma(G) = \min\{\sigma(P_1), \ldots, \sigma(P_t)\}.$$

We can re-state this as follows:

Proposition

Let G be a finite nilpotent group.
Then $\sigma(G) = p + 1$ where p is the smallest prime divisor of G such that the Sylow p-subgroup of G is non-cyclic.

A "**chief factor**" of a group G is a minimal normal subgroup of a quotient of G.

If $K \trianglelefteq G$ and H/K is a chief factor of G, a complement of the chief factor H/K is a subgroup M/K of G/K such that $(M/K) \cap (H/K) = K/K$ and $(M/K)(H/K) = G/K$.

Solvable groups were considered by Tomkinson in 1994.

Theorem (Tomkinson, [4])

Let G be a finite solvable group. Then

$$\sigma(G) = |H/K| + 1$$

where H/K is the smallest chief factor of G with more than one complement in G/K.

In particular, if G is solvable then $\sigma(G) - 1$ is always a prime power. This is false for non-solvable groups, for example $\sigma(S_6) - 1 = 12$.

Lemma (Intersection argument)

Let K be a maximal subgroup of a group G and let $\mathcal{H}$ be a minimal cover of G consisting of maximal subgroups.
If $\sigma(G) < \sigma(K)$ then $K \in \mathcal{H}$.
Equivalently, if $K \notin \mathcal{H}$ then $\sigma(K) \leq \sigma(G)$.

$\sigma(\mathrm{Sym}(6)) = 13$

For example let $G = \mathrm{Sym}(6)$, K a subgroup of G isomorphic to $\mathrm{Sym}(5)$. Since $\sigma(\mathrm{Sym}(5)) = 2^{5-1} = 16$ and G can be covered with 13 subgroups (Alt(6) together with the twelve subgroups isomorphic to Sym(5)) we have $\sigma(G) \leq 13 < 16 = \sigma(\mathrm{Sym}(5)) = \sigma(K)$.

Applying the lemma we find that $\mathcal{H}$ contains the twelve subgroups of G isomorphic to Sym(5). Since they are not enough to cover G and $\sigma(G) \leq 13$, we find $\sigma(G) = 13$.

- If $A, B \leq G$ let $AB := \{ab \ : \ a \in A, \ b \in B\}$.
- Let $N \trianglelefteq G$. A **complement** of N in G is $H \leq G$ such that

$$NH = G \qquad \text{and} \qquad N \cap H = \{1\}.$$

- If H complements N in G then $G/N \cong H$.

Example

Any two 1-dimensional vector subspaces of $\mathbb{R}^2$ complement each other as additive groups.

Example

Consider the group G of isometries of the plane $\mathbb{R}^2$. We certainly have the subgroup T of translations and the subgroup O of linear isometries. T is a normal subgroup of G complemented by O.

Proposition (Maximal Complement Argument)

If a maximal subgroup M of G complements a non-solvable normal subgroup N of G then $\sigma(G) = \sigma(G/N)$.

Proposition

Let G be a group. Then G is simple if and only if

$$\Delta_G := \{(g, g) \mid g \in G\}$$

is a maximal subgroup of $G \times G$.

$\sigma(S \times S) = \sigma(S)$

Let S be a non-abelian simple group. Note that $S \times \{1\}$ is a non-solvable normal subgroup of $S \times S$ and that Δ_S is a complement of $S \times \{1\}$. It follows from the Maximal Complement Argument that

$$\sigma(S \times S) = \sigma(S).$$

One of the main results in my thesis is the following.

Theorem (Lucchini A., G 2010 [5])

Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_1. *In this case* $\sigma(G) = \sigma(H_1)$.
2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ *where* $\mathcal{X}$ *is a minimal cover of* H_2. *In this case* $\sigma(G) = \sigma(H_2)$.
3. *There exist* $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ *with* $H_1/N_1 \cong H_2/N_2 \cong C_p$ *and* $\mathcal{M}$ *consists of the maximal subgroups of* $H_1 \times H_2$ *containing* $N_1 \times N_2$. *In this case* $\sigma(G) = p + 1$.

Suppose we want to compute $\sigma(G)$. If $N \trianglelefteq G$ is such that $\sigma(G) = \sigma(G/N)$ then we are reduced to compute $\sigma(G/N)$.

Definition (σ-elementary groups)

A non-cyclic group G is said to be σ-elementary if $\sigma(G) < \sigma(G/N)$ whenever $\{1\} \neq N \trianglelefteq G$.

It is easy to prove that:

Proposition

Let G be a finite non-cyclic group. Then there exists $N \trianglelefteq G$ such that:

- $\sigma(G) = \sigma(G/N)$;
- *G/N is σ-elementary.*

Examples

- Non-abelian simple groups are σ-elementary.
- If p is a prime, $C_p \times C_p$ is σ-elementary.
- The dihedral group D_{2p} of order $2p$ is σ-elementary.
- If S is a non-abelian simple group and $n \geq 1$ is an integer such that S^n is σ-elementary then $n = 1$. Indeed, $\sigma(S^n) = \sigma(S)$.
- If $n \geq 3$ is an integer and $n \neq 4$ then S_n is σ-elementary: its only proper quotient is C_2. S_4 is not σ-elementary: it admits S_3 as homomorphic image (quotient) and $\sigma(S_4) = \sigma(S_3) = 4$.
- If G/N is cyclic whenever $\{1\} \neq N \trianglelefteq G$ then G is σ-elementary. The converse is true for solvable groups but false in general. An example is $I \rtimes A_p$ where $I = \{(x_1, \ldots, x_p) \in \mathbb{F}_2^p : \ \sum_{i=1}^p x_i = 0\}$ and p is a prime not of the form $\frac{q^n-1}{q-1}$ with q a prime power.

Proposition

Let G be a σ-elementary group.

- *Since $\sigma(G) = \sigma(G/\Phi(G))$, $\Phi(G) = \{1\}$.*
- *If G is non-abelian then it has trivial center: $Z(G) = \{1\}$.*
- *If G is abelian then $G \cong C_p \times C_p$ for some prime p.*
- *Scorza's theorem: if $\sigma(G) = 3$ then $G \cong C_2 \times C_2$.*
- *Scorza's theorem revisited: if $\sigma(G) = p + 1$ with p the smallest prime divisor of $|G|$ then $G \cong C_p \times C_p$.*
- *Let n be a positive integer. There are only finitely many σ-elementary groups G with $\sigma(G) = n$.*
- *If $H_1 \times H_2$, a direct product of two non-trivial groups, is σ-elementary then $H_1 \cong H_2 \cong C_p$ for some prime p.*
- *If G is σ-elementary, $\{1\} \neq N \trianglelefteq G$ and G/N is solvable then G/N is cyclic. In particular G/G' is cyclic.*

Theorem (G 2009)

All σ-elementary groups G with $\sigma(G) \leq 25$ are known.

3	$C_2 \times C_2$
4	$C_3 \times C_3, \mathrm{Sym}(3)$
5	$\mathrm{Alt}(4)$
6	$C_5 \times C_5, D_{10}, AGL(1,5)$
7	$\emptyset$
8	$C_7 \times C_7, D_{14}, 7:3, AGL(1,7)$
9	$AGL(1,8)$
10	$3^2:4, AGL(1,9), \mathrm{Alt}(5)$
11	$\emptyset$
12	$C_{11} \times C_{11}, 11:5, D_{22}, AGL(1,11)$
13	$\mathrm{Sym}(6)$
14	$C_{13} \times C_{13}, D_{26}, 13:3, 13:4, 13:6, AGL(1,13)$
15	$SL(3,2)$
16	$\mathrm{Sym}(5), \mathrm{Alt}(6)$
17	$2^4:5, AGL(1,16)$
18	$C_{17} \times C_{17}, D_{34}, 17:4, 17:8, AGL(1,17)$
19	$\emptyset$
20	$C_{19} \times C_{19}, AGL(1,19), D_{38}, 19:3, 19:6, 19:9$
21	$\emptyset$
22	$\emptyset$
23	M_{11}
24	$C_{23} \times C_{23}, D_{46}, 23:11, AGL(1,23)$
25	$\emptyset$

Definition (Minimal normal subgroups)

A minimal normal subgroup of a group G is a non-trivial normal subgroup N of G which does not contain any non-trivial normal subgroup of G different from N.

Examples

- If p is a prime, $C_p \times C_p$ has $p+1$ minimal normal subgroups.
- If S is a simple group, it is its unique minimal normal subgroup.
- If $n \geq 3$ is an integer and $n \neq 4$ then the unique minimal normal subgroup of S_n is A_n.
- The unique minimal normal subgroup of S_4 is $V = \{1, (12)(34), (13)(24), (14)(23)\}$.
- If $k \geq 1$ is an integer and S is a non-abelian simple group then the minimal normal subgroups of $S \times \cdots \times S = S^k$ are its k direct factors, $S \times \{1\} \times \cdots \times \{1\}, \ldots, \{1\} \times \cdots \times \{1\} \times S$.
- If F is a field with at least 4 elements and $n \geq 2$, the unique minimal normal subgroup of $PGL(n, F)$ is $PSL(n, F)$.

Given a finite group G denote by $mn(G)$ the **number of minimal normal subgroups** of G.

The main problem I dealt with in my Ph.D thesis is the following.

Conjecture (A. Lucchini, E. Detomi)

Let G be a non-abelian σ-elementary group. Then $mn(G) = 1$.

If $mn(G) = 1$ we usually say that G is monolithic.

Here is what I can say when the covering number is “small”.

Theorem

Let G be a non-abelian σ-elementary group such that $\sigma(G) \leq 56$. Then G is monolithic. Moreover, its minimal normal subgroup is either simple or abelian.

$\sigma(A_5 \wr C_2) = 57$, $A_5 \wr C_2$ is monolithic and its minimal normal subgroup is $A_5 \times A_5$, not simple and not abelian.

Here is what I can say when I “block” pieces of the group.

Theorem

Let H be a non-abelian σ-elementary group, and suppose that whenever S is a non-abelian simple subgroup of H and there exists a chain

$$\{1\} \lhd S \lhd H_2 \lhd \cdots \lhd H_n = H,$$

S is isomorphic to an alternating group $Alt(n)$ with n large enough and even. Then H is monolithic.

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