AN INTRODUCTION TO REPRESENTATION THEORY OF GROUPS

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Take a cube and write the numbers $1, 2, 3, 4, 5, 6$ on its faces, in any way you like.

Then perform the following operation on the cube: substitute to the number on each face the arithmetic mean of the numbers written on the (four) adjacent faces. Iterate this.

The question is: what do the numbers on the faces of the cube look like after n iterations, where n is a large number?

For example in the case of a die, from the first iteration onward the value on each face is constantly 3.5, because in a die the sum of the numbers labeling two opposite faces is always 7.

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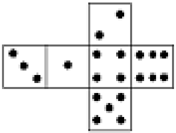

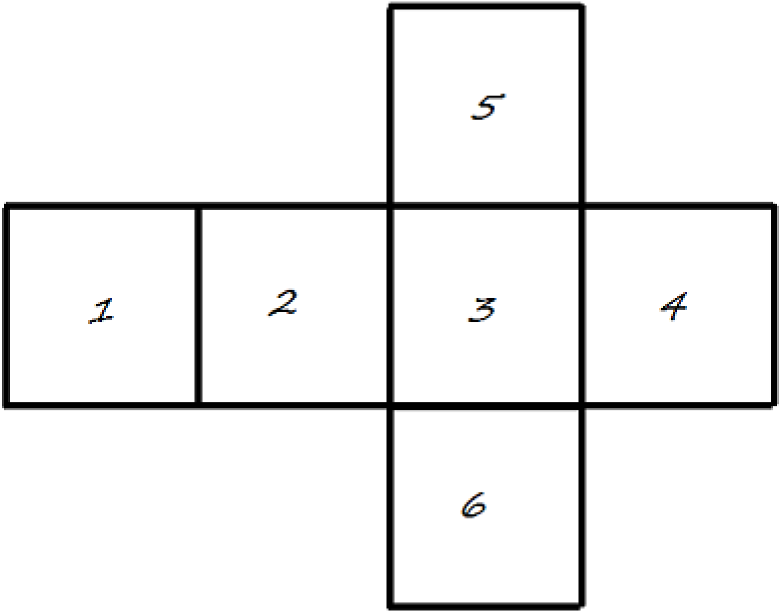
5
1
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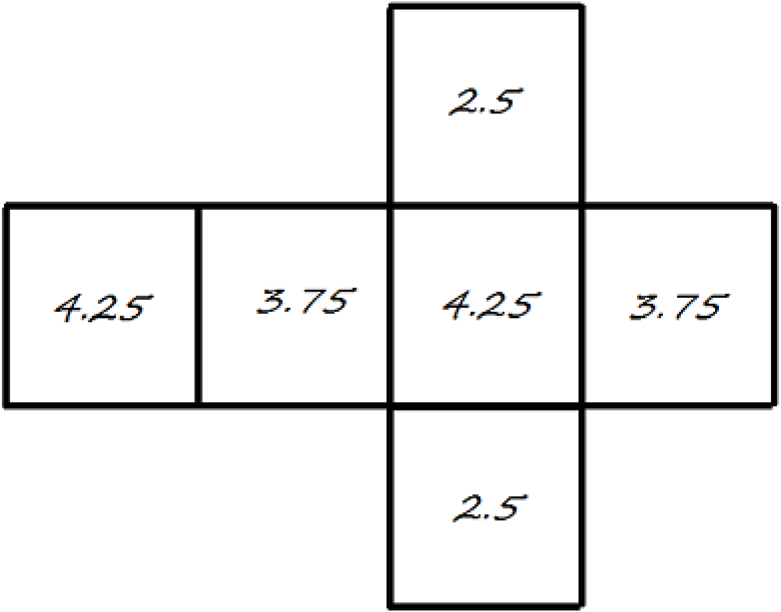
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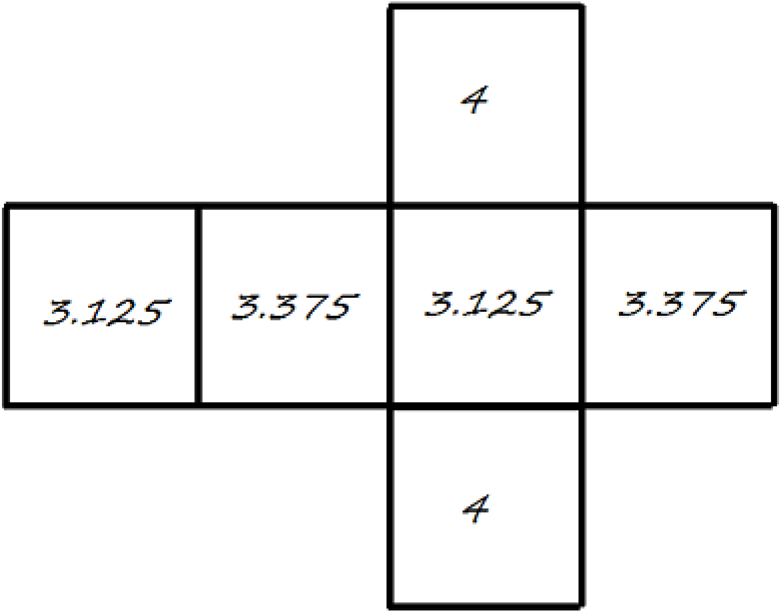
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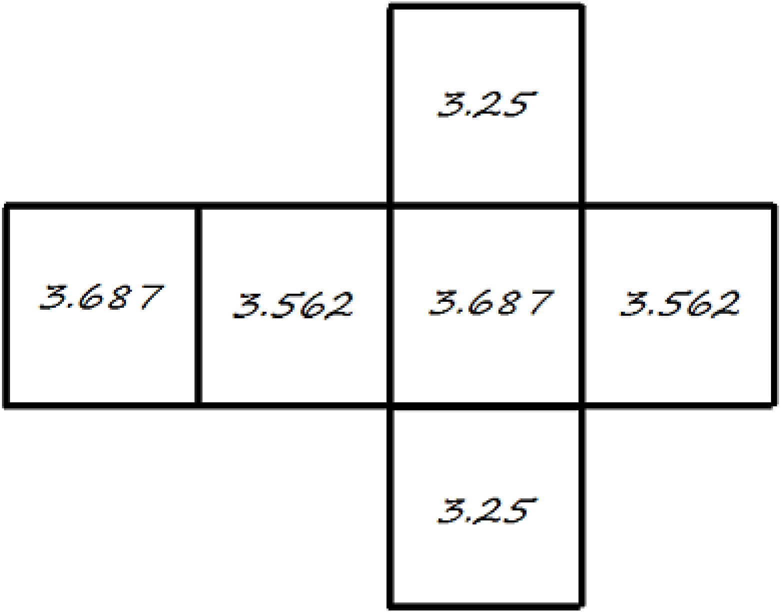
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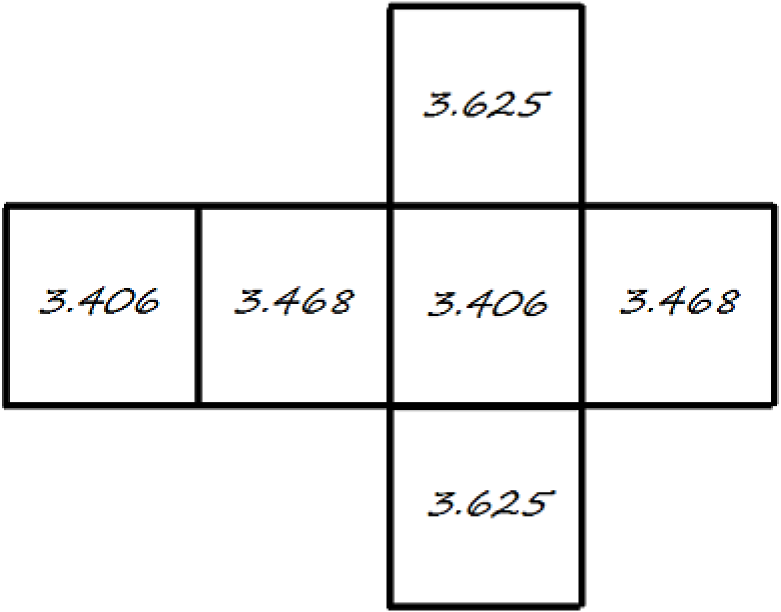
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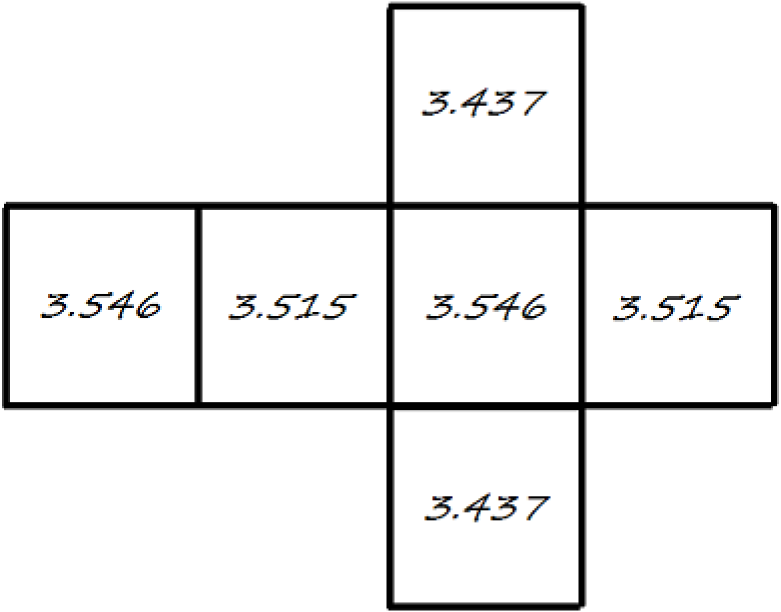
3.437
3.546
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Let F be the set of faces of the cube, and let W_F the set of functions $F \to \mathbb{C}$. W_F is a $\mathbb{C}$-vector space of dimension $|F| = 6$ spanned by $\{\delta_x \; : \; x \in F\}$ where $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ if $x \neq y$.

Call L the operator $W_F \to W_F$ that takes a face label to the arithmetic mean of the four adjacent face labels: $L(f)(x) := \frac{1}{4} \sum_{y \in A_x} f(y)$.

L is a linear operator whose matrix in the base $\{\delta_x \; : \; x \in F\}$ is

$$L = \begin{pmatrix} 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \end{pmatrix}$$

It obviously has rank 3 (opposite faces take the same value).

We are interested in the powers L^n. So our aim is to **diagonalize** L. Of course, this can be done computationally, but what we want to do is to look for some geometrical ways to do it, i.e. by means of some **group action on the space**.

Suppose we want to understand a set X which has some symmetries. The idea is to consider the vector space

$$V_X := \{functions\ X \to \mathbb{C}\} = \{vectors\ (c_x)_{x \in X}\ x \in X, c_x \in \mathbb{C}\}.$$

This is a $\mathbb{C}$-vector space of dimension $|X|$. Consider the group G of the symmetries of X you are interested in. In other words, G is some subgroup of the group $\mathrm{Sym}(X) = \{bijections\ X \to X\}$. Then we have a group homomorphism

$$G \to GL(V_X),\ g \mapsto \pi_g :\ v = (c_x)_{x \in X} \mapsto \pi_g(v) = (c_{g^{-1}(x)})_{x \in X}.$$

This is the object we want to study.

A (complex, linear) representation of the group G is a $\mathbb{C}$-vector space V endowed with a group homomorphism

$$\pi : G \to GL(V), \qquad g \mapsto \pi_g.$$

This is the data of V and π, so we will also write (V, π) to denote this representation. The dimension of V is called the dimension of the representation (V, π).

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Permutation matrices

Consider the group S_4 of bijections $X \to X$ where $X = \{1, 2, 3, 4\}$. The representation $\pi : S_4 \to GL(\mathbb{C}^4)$ described above sends a permutation σ to the corresponding "**permutation matrix**", that is, the 1-0 matrix whose 1-entries are in the $(\sigma(i), i)$ positions, for $i \in \{1, 2, 3, 4\}$.

So for example for the permutation $(123) \in S_4$ (i.e. the permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$, $4 \mapsto 4$) we have

$$\pi_{(123)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(\mathbb{C}^4).$$

It is worth noting that the **trace** of $\pi(\sigma)$ (i.e. the sum of its diagonal entries) is the number of **fixed points** of σ.

The map $\sigma \mapsto \mathrm{Tr}(\pi(\sigma))$ will be called the "**character**" of π.

So we have a group G, a vector space V and some homomorphism $G \to GL(V)$, in other words we have a way of associating to every element $g \in G$ a linear isomorphism $\pi(g) = \pi_g : V \to V$ (up to choosing a basis of V, you can think of π_g as an invertible matrix!).

Some examples

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- The group $GL(V)$ itself admits a representation given by the identity $GL(V) \to GL(V)$.
- If G is a group of bijections $X \to X$ where X is a finite set of cardinality n then G admits the n-dimensional representation $G \to GL(V_X)$ described above.
- The group $\mathbb{Z}$ admits a representation $\mathbb{Z} \to GL(\mathbb{C}^2)$ given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

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So G "acts" on the vector space V, in the sense that $g \in G$ "moves the vectors" by sending $v \in V$ to $\pi_g(v)$. In this setting, the notion of "subspace" is weak: we are much more interested in "G-invariant" subspaces! What does this mean?

A subspace W of V is called G-invariant (or simply, "invariant") if whenever $w \in W$ and $g \in G$, $\pi_g(w) \in W$.

A 2-dimensional representation of $(\mathbb{R}, +)$

For example the additive group $G = \mathbb{R}$ has a 2-dimensional representation given by

$$\mathbb{R} \to GL(\mathbb{C}^2), \qquad a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

and the subspace W of $V = \mathbb{C}^2$ given by $W = \{\begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{C}\}$, is NOT G-invariant. Indeed for example

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W.$$

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Also the familiar notion of linear homomorphism between to spaces is too weak for us. We need the notion of G-invariant homomorphism! What does this mean?

Suppose (V, π) and (W, ν) are two representations of G and $f : V \to W$ is a linear map. f is called G-**invariant** if it satisfies

$$f(\pi_g(v)) = \nu_g(f(v)) \qquad \forall g \in G, v \in V, w \in W.$$

Such map is also called "**intertwining operator**", since it "intertwines" the two representations π and ν.

THE ARCHETIPE OF INTERTWINING OPERATOR

Consider the case $V = W$, choose a basis of V and think of π_g and ν_g as matrices. Suppose there is an invertible matrix A such that $\nu_g = A\pi_g A^{-1}$ for all $g \in G$, in other words ν_g is obtained by π_g via a **change of basis**. Then the map

$$f : V \to V \qquad v \mapsto Av$$

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A representation (V, π) *of* G *is called IRREDUCIBLE if the only* G*-invariant subspaces of* V *are* $\{0\}$ *and* V*.*

BACK TO $(\mathbb{R}, +)$

For example the representation of the additive group $G = \mathbb{R}$ considered above,

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Our aim in life is now the following.

Let (V, π) be a representation of a group G.

If possible, we want to write V as a direct sum of irreducible G-invariant subspaces (irreducible representations), i.e. we want an expression of the form

$$V = \bigoplus_{i=1}^{n} W_i = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

where W_i is an irreducible G-invariant subspace of V for $i \in \{1, \ldots, n\}$.

Let (V, π) and (W, ν) be two representations of G. Denote by $\mathrm{Hom}_G(V, W)$ the set of G-invariant homomorphisms $V \to W$. If $f : V \to W$ is such a homomorphism then both its kernel and its image are G-invariant subspaces, of V and W respectively.

This easily implies a fundamental fact, Schur's lemma, which is the starting point of representation theory.

Theorem (Schur's lemma)

Let (V, π) and (W, ν) be two irreducible representations of the group G. Then any nonzero G-invariant map $V \to W$ is an isomorphism.

Observe that $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$ has the structure of ring with respect to (SUM) pointwise sum and (PRODUCT) composition of functions.

Theorem (Schur's Lemma)

Let (V, π) be an irreducible representation of G. Then in the ring $\mathrm{End}_G(V)$ every nonzero element is invertible. In other words, $\mathrm{End}_G(V)$ is a ***skew field****.*

Now suppose V is irreducible and finite dimensional. Then also $\mathrm{End}_G(V)$ is a finite dimensional $\mathbb{C}$-vector space. Also, by Schur's lemma it is a skew field. Moreover, it contains a copy of $\mathbb{C}$ given by the scalar operators

$$V \to V, \; v \mapsto \lambda v, \qquad \lambda \in \mathbb{C}.$$

From the fact that $\mathbb{C}$ is **algebraically closed** and finite dimensionality it follows that $\mathrm{End}_G(V) \cong \mathbb{C}$.

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Theorem (Schur's Lemma)

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Let us see what this means in the case G is **abelian**. In this case for $g, h \in G$ we have $gh = hg$, so that

$$\pi_g \pi_h = \pi_{gh} = \pi_{hg} = \pi_h \pi_g$$

($\pi : G \to GL(V)$ is a homomorphism!). This implies that π_h is G-invariant for all $h \in G$! So Schur's Lemma implies that π_h is a scalar operator, for all $h \in G$. Since scalar operators stabilize all subspaces, irreducibility forces the dimension of V to be 1.

Corollary

Let (V, π) be an irreducible finite dimensional representation of the abelian group G. Then $\dim(V) = 1$.

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Let (V, π) be an irreducible finite dimensional representation of the abelian group G. Then $\dim(V) = 1$.

Let us go back to the 2-dimensional representation of the additive group $G = \mathbb{R}$

$$\pi : \mathbb{R} \to GL(\mathbb{C}^2), \qquad a \mapsto \pi_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

We know that $L = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{C} \}$ is a G-invariant subspace of $V = \mathbb{C}^2$. Now we ask, **is V the direct sum of two 1-dimensional G-invariant subspaces?** This would be great, because the way of understanding a space is by writing it as a direct sum of irreducible subspaces.

But what is a subspace invariant under π_a? It is just an **eigenspace** of π_a. Hence being able to write V as direct sum of two 1-dimensional G-invariant subspaces would mean, in particular, being able to **diagonalize** π_a (simultaneously, i.e. uniformly with respect to a !).

The problem is that π_a is not diagonalizable if $a \neq 0$. Hence V is NOT the direct sum of two 1-dimensional G-invariant subspaces.

So, we have a problem. We might find a G-invariant subspace W of the finite dimensional space V without a G-invariant complement (a complement of W is a subspace U of V such that $V = U \oplus W$).

UNITARISABILITY (Weyl's unitary trick). Suppose that G is a **finite** group. The formula

$$B(u,v) := \frac{1}{|G|} \sum_{g \in G} \pi_g(u) \cdot \overline{\pi_g(v)}$$

defines a hermitian inner product on V, which has the property of being G**-invariant** :

$$B(\pi_g(u), \pi_g(v)) = B(u,v) \qquad \forall u,v \in V.$$

If U is a G-invariant subspace of V then $U^\perp$, the space of vectors v such that $B(u,v) = 0$ for all $u \in U$, is a G-invariant complement of U.

Theorem (Maschke Theorem)

Suppose that G is finite. Then any finite dimensional representation of G is completely reducible, i.e. it is a direct sum of irreducible G-invariant subspaces.

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Now for $g \in G$ define $\chi_\pi(g) := Tr(\pi_g)$, the **trace** of the (matrix!) operator π_g (i.e. the sum of its diagonal entries).

For example, in the case of permutation matrix representations $\chi_\pi(g)$ is the number of fixed points of the permutation g.

The map $\chi_\pi : G \to \mathbb{C}$, $g \mapsto \chi_\pi(g)$ is called the "**character**" of the representation π.

Theorem (Frobenius)

Let π_1, π_2 be two representations of the group G and let χ_1, χ_2 be their characters. Then $\pi_1 \cong \pi_2$ if and only if $\chi_1 = \chi_2$.

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Now for $g \in G$ define $\chi_\pi(g) := Tr(\pi_g)$, the **trace** of the (matrix!) operator π_g (i.e. the sum of its diagonal entries).

For example, in the case of permutation matrix representations $\chi_\pi(g)$ is the number of fixed points of the permutation g.

The map $\chi_\pi : G \to \mathbb{C}$, $g \mapsto \chi_\pi(g)$ is called the "**character**" of the representation π.

THEOREM (FROBENIUS)

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Theorem (Frobenius)

Let π_1, π_2 be two representations of the group G and let χ_1, χ_2 be their characters. Then $\pi_1 \cong \pi_2$ if and only if $\chi_1 = \chi_2$.

The idea is the following.

Let χ_1, χ_2 be two functions $G \to \mathbb{C}$ (for example, two characters of G), i.e. elements of $\mathbb{C}^G$. Set

$$B(\chi_1, \chi_2) := \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}.$$

This defines a hermitian inner product on $\mathbb{C}^G = \{\textit{functions } G \to \mathbb{C}\}$. Now suppose χ_i is the character of the representation π_i for $i = 1, 2$, and suppose π_1 is irreducible. Then $B(\chi_1, \chi_2)$ equals the multiplicity of π_1 in the decomposition of π_2 into irreducibles.

An example

Suppose $\pi = \alpha \oplus \beta \oplus \beta$ with α, β irreducible. Then $B(\chi_\pi, \chi_\alpha) = 1$ and $B(\chi_\pi, \beta) = 2$. If γ is an irreducible representation not isomorphic to α or β then $B(\chi_\pi, \chi_\gamma) = 0$.

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THE CHARACTER TABLE OF S_3

Consider S_3, the group of bijections $\{1,2,3\} \to \{1,2,3\}$. S_3 permutes naturally the three basis vectors e_1, e_2, e_3 of $\mathbb{C}^3$. Thinking of the elements of S_3 as permutation matrices we can imagine that $S_3 \leq GL(\mathbb{C}^3)$. It turns out that every element is conjugated to one of the matrices displayed below.

S_3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\chi_1 = 1$	1	1	1
$\chi_2 = \det$	1	-1	1
χ_3	2	0	-1

χ_3: act on $W := \{(x_1, x_2, x_3) \in \mathbb{C}^3 \ : \ x_1 + x_2 + x_3 = 0\}$ $(\dim(W) = 2)$.

How does $V = \mathbb{C}^3$ decompose via this representation? Let U be the space of constant vectors: $U := \{(a, a, a) \ : \ a \in \mathbb{C}\}$. Then U and W are irreducible S_3-invariant subspaces of V and

$$V = U \oplus W.$$

Let us relax the condition of finiteness of G and consider compactness. Consider the **circle** of center the origin and radius 1:

$$G := S^1 = \{e^{i\theta} \ : \ 0 \leq \theta \leq 2\pi\}.$$

Why did I call it G? Because it is a group with respect to multiplication:

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}, \qquad (e^{i\theta})^{-1} = e^{-i\theta} = e^{i(2\pi-\theta)}.$$

Algebraically it can be viewed as the quotient $\mathbb{R}/2\pi\mathbb{Z}$.

- To make the theory of representations meaningful in this setting we must take the topology into account. In other words, we will have a topology on the space, and the notion of "G-invariant subspace" will be substituted by "**closed** G-invariant subspace". Also, we require morphisms to be **continuous**.
- What are the irreducible representations of S^1? Since S^1 is abelian (and compact), they are all 1-dimensional (by Schur's lemma!), i.e. they are continuous homomorphisms $S^1 \to \mathbb{C}^\times$. This forces them to be of the form $e^{i\theta} \mapsto e^{in\theta}$ where $n \in \mathbb{Z}$.

Now we need a vector space representing our group $G = S^1$.

- Let $V := L^2(G) = \{f : G \to \mathbb{C} \ : \ \int_G |f(e^{i\theta})|^2 d\theta < \infty\}$. It is a **Hilbert space** with the hermitian inner product given by

$$B(u, v) := \frac{1}{2\pi} \int_G u(e^{i\theta})\overline{v(e^{i\theta})} d\theta.$$

Note that $2\pi = \int_G d\theta$, hence it substitutes $|G|$, which was used in the finite case. Note that a function $S^1 \to \mathbb{C}$ can be thought of as a periodic function $\mathbb{R} \to \mathbb{C}$ of period 2π.

- Consider the following representation of G:

$$\pi : G \to GL(V), \qquad \pi_{e^{i\theta}}(f)(e^{i\theta_0}) := f(e^{i(\theta_0+\theta)}).$$

It is **unitary**, i.e. $B(\pi_g(f_1), \pi_g(f_2)) = B(f_1, f_2)$.
– So Maschke Theorem holds!

- The irreducible invariant subspaces of V are, for $n \in \mathbb{Z}$, $V_n := \{f \in V \ : \ f(e^{i(\theta_0+\theta)}) = e^{in\theta} f(e^{i\theta_0})\} = \mathbb{C}\{e^{i\theta} \mapsto e^{in\theta}\}$.

We want to **decompose our V as (topological) direct sum of irreducible subspaces**.

The decomposition of $V = L^2(S^1)$ into irreducible G-invariant subspaces is the following:

$$V = \widehat{\bigoplus_{n \in \mathbb{Z}} V_n} = \overline{\{\sum_{finite} v_n \ : \ v_n \in V_n\}} = \{\theta \mapsto \sum_{n \in \mathbb{Z}} c_n e^{in\theta} \ : \ \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty\}.$$

This says that any function $S^1 \to \mathbb{C}$ (i.e. any periodic function of period 2π !) which is square-integrable admits an espression of the form

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$$

for some $c_n \in \mathbb{C}$ with $\sum_{n \in \mathbb{Z}} |c_n|^2 < \infty$. This is the Fourier series of f.

Using orthogonality of characters we find

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-im\theta} d\theta = B(f(\theta), e^{im\theta}) = B(\sum_{n \in \mathbb{Z}} c_n e^{in\theta}, e^{im\theta}) = c_m.$$

Thus **Fourier analysis is the representation theory of the circle group S^1.**

Let $G = \mathbb{Z}/N\mathbb{Z} = \{0, 1, \dots, N-1\}$ (cyclic group of order N) and let

$$L(G) := \mathbb{C}^G = \{\mathit{functions}\ G \to \mathbb{C}\}.$$

It is a $\mathbb{C}$-vector space of dimension $|G|$.
Let $\hat{G} := \{\mathit{group\ homomorphisms}\ \chi : G \to \mathbb{C}^\times\}$.

- It is a group isomorphic to G, generated by $1 \mapsto e^{i2\pi/N}$.
- It is the set of linear (1-dimensional) characters of G.
- It is a basis of $L(G)$.

Fourier transform:

$$\mathcal{F} : L(G) \to L(\hat{G}), \qquad \mathcal{F}(f)(\chi) := B(f,\chi) = \frac{1}{|G|}\sum_{x \in G} f(x)\overline{\chi(x)},$$

the coefficient of χ in the expression of f in the base $\hat{G}$.
In other words, if $f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$ then $\mathcal{F}(f)(\chi) = \hat{f}(\chi)$.
For $f \in L(G)$ let $Supp(f) := \{x \in G \ : \ f(x) \neq 0\}$.

Theorem (Heisenberg's Uncertainty Principle)

If $f \in L(G)$ then $|Supp(f)| \cdot |Supp(\mathcal{F}(f))| \geq |G|$.

Now we go back to our original problem. Let G be **the group of rotations of the cube**. Then $|G| = 24$ (if you place a cube on a table, you can put each of the 6 faces up, and rotate that face in 4 ways).

It turns out that $G \cong S_4$ (the idea is to observe that G permutes the four diagonals of the cube in any possible way!).

G permutes the six faces of the cube. Let F be the set of faces of the cube. This gives a permutation matrix representation

$$\pi : G \to GL(W_F) \quad \text{where} \quad W_F := \mathbb{C}^F = \{\textit{functions } F \to \mathbb{C}\} \cong \mathbb{C}^6.$$

By computing the fixed points of the elements of G we can compute the character of this representation. Call it χ.

It turns out that $B(\chi, \chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = 3$.

Using this information we can deduce that π is the direct sum of three non-isomorphic irreducible representations.

Hence $W_F = \mathbb{C}^F = \{functions\ F \to \mathbb{C}\} \cong \mathbb{C}^6$ **is the direct sum of three G-invariant irreducible subspaces**. We are left to find them. This is where the geometry comes in: our problem is now reduced to find G-invariant subspaces.

But we know the characters of the corresponding representations! This means that we have a lot of information about them, which leads us close to determining them explicitly.

For a face x let $-x$ denote the face opposite to x. Consider

- $W_1 := \{constant\ functions\ F \to \mathbb{C}\}$. This is clearly one-dimensional: $\dim_{\mathbb{C}}(W_1) = 1$.
- $W_2 := \{f : F \to \mathbb{C}\ :\ f(-x) = f(x)\ \forall x \in F,\ \sum_{x \in F} f(x) = 0\}$. This is given by $3 + 1 = 4$ equations so $\dim_{\mathbb{C}}(W_2) = 6 - 4 = 2$.
- $W_3 := \{f : F \to \mathbb{C}\ :\ f(-x) = -f(x)\ \forall x \in F\}$. This is given by 3 equations so $\dim_{\mathbb{C}}(W_3) = 6 - 3 = 3$.

The decomposition of W_F into irreducible subspaces is

$$W_F = W_1 \oplus W_2 \oplus W_3.$$

The operator we are concerned with is

$$L : W_F \to W_F, \qquad L(f)(x) := \frac{1}{4} \sum_{y \in A_x} f(y)$$

where A_x denotes the set of faces adjacent to the face x. It turns out that L is G-invariant! This is because *rotating after averaging is the same as averaging after rotating*.

Since W_1, W_2, W_3 are irreducible, by Schur's lemma $L|_{W_i}$ is a scalar operator. Using $W_F = W_1 \oplus W_2 \oplus W_3$ it turns out that

$$L \sim \left(\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The operator L^n has eigenvalues 1, $(-1/2)^n$ and 0, hence if n is large then $L^n(f)$ is approximately equal to the projection of f onto W_1. The projection of $f = (1,2,3,4,5,6)$ onto W_1 is $(3.5, 3.5, 3.5, 3.5, 3.5, 3.5)$. Hence the value on each face gets arbitrarily close to 3.5.