# An introduction to Representation Theory of groups

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May 20, 2014

**Abstract**

Label the faces of a cube with the numbers from 1 to 6 in some order, then perform the following operation: replace the number labeling each given face with the arithmetic mean of the numbers labeling the adjacent faces. What numbers will appear on the faces of the cube after this operation is iterated many times? This is a sample problem whose solution is a model of the application of the theory of representations of groups to diverse problems of mathematics, mechanics, and physics that possess symmetry of one kind or another. In this introductory talk I will present the tools from representation theory needed to solve this problem. I will also point out the connection with harmonic analysis by expressing Fourier analysis as an instance of representation theory of the circle group (the multiplicative group of complex numbers with absolute value 1) and by stating a version of Heisenberg's uncertainty principle for finite cyclic groups.

## 1 A sample problem

I will start by stating a sample problem which I found in [2].

Take a cube and write the numbers $1, 2, 3, 4, 5, 6$ on its faces, in any way you like. Then perform the following operation on the cube: substitute to the number on each face the arithmetic mean of the numbers written on the (four) adjacent faces. Iterate this. The question is: what do the numbers on the faces of the cube look like after $n$ iterations, where $n$ is a large number? For example in the case of a die, from the first iteration onward the value on each face is constantly 3.5, because in a die the sum of the numbers labeling two opposite faces is always 7.

The idea to solve this problem is the following. Let $F$ be the set of faces of the cube, and let $W_F$ be the set of functions $F \to \mathbb{C}$. $W_F$ is a $\mathbb{C}$-vector space of dimension $|F| = 6$ spanned by $\{\delta_x \ : \ x \in F\}$ where $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ if $x \neq y$.

Call $L$ the operator $W_F \to W_F$ that takes a face label to the arithmetic mean of the four adjacent face labels: $L(f)(x) := \frac{1}{4} \sum_{y \in A_x} f(y)$.

$L$ is a linear operator whose matrix in the base $\{\delta_x \ : \ x \in F\}$ is

$$L = \begin{pmatrix} 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \end{pmatrix}$$

It obviously has rank 3 (opposite faces take the same value).

We are interested in the powers $L^n$. So our aim is to **diagonalize** $L$ (obviously the powers of a diagonal matrix are easy to compute). Of course, this can be done computationally, but what we want to do is to look for some geometrical way to do it, i.e. by means of some **group action on the space**.

The solution of this problem is in section 7.

# 2 Representations

Suppose we want to understand a set $X$ which has some symmetries. The idea is to consider the vector space

$$V_X := \{functions\ X \to \mathbb{C}\} = \{vectors\ (c_x)_{x\in X}\ x \in X, c_x \in \mathbb{C}\}.$$

This is a $\mathbb{C}$-vector space of dimension $|X|$. Consider the group $G$ of the symmetries of $X$ you are interested in. In other words, $G$ is some subgroup of the group $\mathrm{Sym}(X) = \{bijections\ X \to X\}$. Denote by $GL(V_X)$ the group of the linear isomorphisms $V_X \to V_X$ (it is a group with respect to composition of functions). Then we have a group homomorphism

$$\pi : G \to GL(V_X),\ g \mapsto \pi_g :\ v = (c_x)_{x\in X} \mapsto \pi_g(v) = (c_{g^{-1}(x)})_{x\in X}.$$

In other words, $\pi$ sends $g \in G$ to $\pi_g$, which is the linear isomorphism $V_X \to V_X$ that sends a vector $v = (c_x)_{x\in X}$ to the vector obtained by permuting the coordinates according to $g^{-1}$, $\pi_g(v) = (c_{g^{-1}(x)})_{x\in X}$. This is the object we want to study. The reason why it is a homomorphism is the following.

$$\pi_{gh}((c_x)_{x\in X}) = (c_{(gh)^{-1}(x)})_{x\in X} = (c_{h^{-1}(g^{-1}(x))})_{x\in X},$$

$$\pi_g(\pi_h((c_x)_{x\in X})) = \pi_g((c_{h^{-1}(x)})_{x\in X}) = (c_{h^{-1}(g^{-1}(x))})_{x\in X}.$$

Hence $\pi_{gh} = \pi_g \circ \pi_h$.

**Definition 1.** *A (complex, linear) representation of the group $G$ is a $\mathbb{C}$-vector space $V$ endowed with a group homomorphism*

$$\pi : G \to GL(V), \qquad g \mapsto \pi_g$$

*where $GL(V)$ is the group of the linear isomorphisms $V \to V$ with the operation given by composition of functions. This is the data of $V$ and $\pi$, so we will also write $(V, \pi)$ to denote this representation. The dimension of $V$ is called the dimension of the representation $(V, \pi)$.*

Let us give some examples.

- If $V$ is a $\mathbb{C}$-vector space, the group $GL(V)$ itself admits a representation given by the identity $GL(V) \to GL(V)$.
- If $G$ is a group of bijections $X \to X$ where $X$ is a finite set of cardinality $n$ then $G$ admits the $n$-dimensional representation $G \to GL(V_X)$ described above.
- The group $(\mathbb{Z}, +)$ (i.e. the set $\mathbb{Z}$ endowed with the operation $+$) admits a representation $\mathbb{Z} \to GL(\mathbb{C}^2)$ given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. This is indeed a group homomorphism:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}.$$

  The same is true if we replace $\mathbb{Z}$ with any additive subgroup of $\mathbb{C}$, for example $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ itself.

## 2.1 Permutation matrices

We give now an explicit instance of the second example given above, $G \to GL(V_X)$. Consider the group $S_4$ of bijections $X \to X$ where $X = \{1, 2, 3, 4\}$. The representation $\pi : S_4 \to GL(\mathbb{C}^4)$ described above sends a permutation $\sigma$ to the corresponding "**permutation matrix**", that is, the 1-0 matrix whose 1-entries are in the $(\sigma(i), i)$ positions, for $i \in \{1, 2, 3, 4\}$.

So for example for the permutation $(123) \in S_4$ (i.e. the permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$, $4 \mapsto 4$) we have

$$\pi_{(123)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(\mathbb{C}^4).$$

Note that this is precisely the linear operator that permutes the four canonical basis vectors $e_1, e_2, e_3, e_4$ the same way $(123)$ moves $1, 2, 3, 4$: it takes $e_1$ to $e_2$, $e_2$ to $e_3$, $e_3$ to $e_1$ and $e_4$ to $e_4$.

It is worth noting that the **trace** of $\pi_\sigma$ (i.e. the sum of its diagonal entries) is the number of **fixed points** of $\sigma$.

The map $\sigma \mapsto \mathrm{Tr}(\pi_\sigma)$ will be called the "**character**" of $\pi$.

## 2.2 Invariant subspaces

Fix a representation $(V, \pi)$ of $G$. The group $G$ "acts" on the vector space $V$, in the sense that $g \in G$ "moves the vectors" by sending $v \in V$ to $\pi_g(v)$. In this setting, the notion of "subspace" is weak: we are much more interested in "$G$-invariant" subspaces! What does this mean?

**Definition 2.** *A subspace $W$ of $V$ is called $G$-invariant (or simply, "invariant") if whenever $w \in W$ and $g \in G$, $\pi_g(w) \in W$.*

## 2.3 A 2-dimensional representation of $(\mathbb{R}, +)$

For example the additive group $G = \mathbb{R}$ has a 2-dimensional representation given by

$$\mathbb{R} \to GL(\mathbb{C}^2), \qquad a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

and the subspace $W$ of $V = \mathbb{C}^2$ given by $W = \{\begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{C}\}$, is NOT $G$-invariant. Indeed for example

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W.$$

Instead, the subspace $L = \{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{C}\}$ is $G$-invariant, being the eigenspace of 1 for $\pi_a$, for all $a \neq 0$.

## 2.4 Invariant homomorphisms

Also the familiar notion of linear homomorphism between to spaces is too weak for us. We need the notion of $G$-invariant homomorphism! What does this mean?

Suppose $(V, \pi)$ and $(W, \nu)$ are two representations of $G$ and $f : V \to W$ is a linear map. $f$ is called **$G$-invariant** if it satisfies

$$f(\pi_g(v)) = \nu_g(f(v)) \qquad \forall g \in G, v \in V, w \in W.$$

Such map is also called "**intertwining operator**", since it "intertwines" the two representations $\pi$ and $\nu$.

We give an example which constitutes the archetipe of intertwining operator. Consider the case $V = W$, choose a basis of $V$ and think of $\pi_g$ and $\nu_g$ as matrices. Suppose there is an invertible matrix $A$ such that $\nu_g = A\pi_g A^{-1}$ for all $g \in G$, in other words $\nu_g$ is obtained by $\pi_g$ via a **change of basis**. Then the map

$$f : V \to V \qquad v \mapsto Av$$

is $G$-invariant (i.e. it is an intertwining operator), indeed $\nu_g = A\pi_g A^{-1}$ means that $A\pi_g = \nu_g A$.

## 2.5 Irreducible subspaces

Usually when dealing with a big space what we want to do is try to decompose it in smaller pieces that cannot be further decomposed.

**Definition 3.** *A representation $(V, \pi)$ of $G$ is called irreducible if the only $G$-invariant subspaces of $V$ are $\{0\}$ and $V$.*

For example the representation of the additive group $G = \mathbb{R}$ considered above,

$$\mathbb{R} \to GL(\mathbb{C}^2), \qquad a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

is NOT irreducible, having the invariant subspace $L = \{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{C}\}$. Indeed we have, for $a \in \mathbb{R}$,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in L.$$

# 3 Looking for decompositions

Our aim in life is now the following. Let $(V, \pi)$ be a representation of a group $G$. If possible, we want to write $V$ as a direct sum of irreducible $G$-invariant subspaces (irreducible representations), i.e. we want an expression of the form

$$V = \bigoplus_{i=1}^{n} W_i = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

where $W_i$ is an irreducible $G$-invariant subspace of $V$ for $i \in \{1, \ldots, n\}$. In order to do this, we need first of all to understand better irreducible subspaces.

For $(V, \pi)$ and $(W, \nu)$ two representations of $G$, denote by $\mathrm{Hom}_G(V, W)$ the set of $G$-invariant homomorphisms $V \to W$.

**Lemma 1.** *If $f \in Hom_G(V, W)$ then both its kernel and its image are $G$-invariant subspaces, of $V$ and $W$ respectively.*

*Proof.* Let $f \in \mathrm{Hom}_G(V, W)$.

We prove that $\ker(f)$ is a $G$-invariant subspace of $V$. Let $v \in \ker(f)$, $g \in G$. We need to prove that $\pi_g(v) \in \ker(f)$, i.e. that $f(\pi_g(v)) = 0$. Since $f$ is $G$-invariant, $f(\pi_g(v)) = \nu_g(f(v)) = \nu_g(0) = 0$, where $f(v) = 0$ being $v \in \ker(f)$.

We prove that $\mathrm{Im}(f) = f(V)$ is a $G$-invariant subspace of $W$. Let $w = f(v) \in \mathrm{Im}(f)$, with $v \in V$, and let $g \in G$. We need to prove that $\nu_g(w) \in \mathrm{Im}(f)$, i.e. that there is some $v' \in V$ with $f(v') = \nu_g(w)$. Since $f$ is $G$-invariant, $\nu_g(w) = \nu_g(f(v)) = f(\pi_g(v))$. Choose $v' = \pi_g(v)$. □

This easily implies a fundamental fact, Schur's lemma, which is the starting point of representation theory.

**Theorem 1** (Schur's lemma)**.** *Let $(V, \pi)$ and $(W, \nu)$ be two irreducible representations of the group $G$. Then any nonzero $G$-invariant map $V \to W$ is an isomorphism.*

*Proof.* Let $f : V \to W$ be a nonzero $G$-invariant map. Since $f$ is nonzero, $\ker(f) \neq V$ and $\mathrm{Im}(f) \neq \{0\}$. On the other hand, $\ker(f)$ and $\mathrm{Im}(f)$ are $G$-invariant subspaces of $V$ and $W$ respectively (by Lemma 1), and $V$, $W$ do not have nontrivial $G$-invariant subspaces (they are irreducible!), hence it must be $\ker(f) = \{0\}$ and $\mathrm{Im}(f) = W$, in other words $f$ is an isomorphism. $\square$

Observe that $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V)$ has the structure of ring with respect to (SUM) pointwise sum and (PRODUCT) composition of functions. This allows to restate Schur's Lemma in the following form:

**Theorem 2** (Schur's Lemma)**.** *Let $(V, \pi)$ be an irreducible representation of $G$. Then in the ring $End_G(V)$ every nonzero element is invertible. In other words, $End_G(V)$ is a* ***skew field****.*

Now suppose $V$ is irreducible and finite dimensional, say $\dim_{\mathbb{C}}(V) = n$. Then also $\mathrm{End}_G(V)$ is a finite dimensional $\mathbb{C}$-vector space (it is a vector subspace of $\mathrm{End}(V)$, which has dimension $n^2$: it is isomorphic to the space of $n \times n$ matrices). Also, by Schur's lemma it is a skew field. Moreover, it contains a copy of $\mathbb{C}$ given by the scalar operators

$$V \to V,\ v \mapsto \lambda v, \qquad \lambda \in \mathbb{C}.$$

From the fact that $\mathbb{C}$ is **algebraically closed** (equivalently, it does not admit finite dimensional field extensions) and finite dimensionality it follows that $\mathrm{End}_G(V) \cong \mathbb{C}$. In other words, every $G$-invariant map $V \to V$ is scalar! Let us re-state Schur's Lemma accordingly.

**Theorem 3** (Schur's Lemma)**.** *Let $(V, \pi)$ be a finite dimensional irreducible representation of the group $G$, and let $f : V \to V$ be a $G$-invariant homomorphism. Then there exists $\lambda \in \mathbb{C}$ such that $f(v) = \lambda v$ for all $v \in V$.*

Let us see what this means in the case $G$ is **abelian**. In this case for $g, h \in G$ we have $gh = hg$, so that

$$\pi_g \pi_h = \pi_{gh} = \pi_{hg} = \pi_h \pi_g$$

($\pi : G \to GL(V)$ is a homomorphism!). This, by the very definition of intertwining operator, implies that $\pi_h$ is $G$-invariant (i.e. it is an intertwining operator) for all $h \in G$ ! So Schur's Lemma implies that $\pi_h$ is a scalar operator, for all $h \in G$. Hence irreducibility forces the dimension of $V$ to be 1: an irreducible representation that sends every group element to a scalar operator must be one-dimensional, because the scalar operators stabilize all subspaces (actually they are the only operators that stabilize all subspaces). We conclude that:

**Corollary 1.** *Let $(V, \pi)$ be an irreducible finite dimensional representation of the* ***abelian*** *group $G$. Then* $\dim(V) = 1$.

Let us go back to the 2-dimensional representation of the additive group $G = \mathbb{R}$

$$\pi : \mathbb{R} \to GL(\mathbb{C}^2), \qquad a \mapsto \pi_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

We know that $L = \{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{C}\}$ is a $G$-invariant subspace of $V = \mathbb{C}^2$. Now we ask, **is $V$ the direct sum of two 1-dimensional $G$-invariant**

**subspaces?** This would be great, because the way of understanding a space is by writing it as a direct sum of irreducible subspaces.

But what is a subspace invariant under $\pi_a$? It is just an **eigenspace** of $\pi_a$. Hence being able to write $V$ as direct sum of two 1-dimensional $G$-invariant subspaces would mean, in particular, being able to **diagonalize** $\pi_a$ (simultaneously, i.e. uniformly with respect to $a$ !).

The problem is that $\pi_a$ is not diagonalizable if $a \neq 0$. Hence $V$ is NOT the direct sum of two 1-dimensional $G$-invariant subspaces.

## 3.1 Unitarisability

So, we have a problem. We might find a $G$-invariant subspace $W$ of the finite dimensional space $V$ without a $G$-invariant complement (a complement of $W$ is a subspace $U$ of $V$ such that $V = U \oplus W$). Note that decomposing into direct sums is indeed equivalent to finding invariant subspaces complementing each other.

**UNITARISABILITY** (Weyl's unitary trick). Suppose that $G$ is a **finite** group. The formula

$$B(u,v) := \frac{1}{|G|} \sum_{g \in G} \pi_g(u) \cdot \overline{\pi_g(v)}$$

defines a hermitian inner product on $V$, which has the property of being **$G$-invariant** :

$$B(\pi_g(u), \pi_g(v)) = B(u,v) \qquad \forall u, v \in V.$$

If $U$ is a $G$-invariant subspace of $V$ then $U^\perp$, the space of vectors $v$ such that $B(u,v) = 0$ for all $u \in U$, is a $G$-invariant complement of $U$.

We deduce that if $G$ is finite then we can indeed decompose the space as direct sum of $G$-invariant irreducible subspaces:

**Theorem 4** (Maschke Theorem)**.** *Suppose that $G$ is finite. Then any finite dimensional representation of $G$ is completely reducible, i.e. it is a direct sum of irreducible $G$-invariant subspaces.*

# 4 Characters

Now for $g \in G$ define $\chi_\pi(g) := Tr(\pi_g)$, the **trace** of the (matrix!) operator $\pi_g$ (i.e. the sum of its diagonal entries).

For example, in the case of permutation matrix representations $\chi_\pi(g)$ is the number of fixed points of the permutation $g$.

The map $\chi_\pi : G \to \mathbb{C}$, $g \mapsto \chi_\pi(g)$ is called the "**character**" of the representation $\pi$.

**Theorem 5** (Frobenius)**.** *Let $\pi_1, \pi_2$ be two representations of the finite group $G$ and let $\chi_1, \chi_2$ be their characters. Then $\pi_1 \cong \pi_2$ if and only if $\chi_1 = \chi_2$.*

In other words, it is enough to know the traces (!) of the matrices $\pi_g$ to recover the whole representation $\pi$.

The idea to prove this is the following. Let $\chi_1$, $\chi_2$ be two functions $G \to \mathbb{C}$ (for example, two characters of $G$), i.e. elements of $\mathbb{C}^G$. Set

$$B(\chi_1, \chi_2) := \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}.$$

This defines a hermitian inner product on $\mathbb{C}^G = \{functions\ G \to \mathbb{C}\}$. Now suppose $\chi_i$ is the character of the representation $\pi_i$ for $i = 1, 2$, and suppose $\pi_1$ is irreducible. Then $B(\chi_1, \chi_2)$ equals the multiplicity of $\pi_1$ in the decomposition of $\pi_2$ into irreducibles. Let us state this explicitly.

**Proposition 1** (Orthogonality of Characters)**.** *Let $\chi$ be the character of an irreducible representation $V$ of the finite group $G$, and let $\theta$ be a representation of $G$. Write $\theta = \bigoplus_{i=1}^{n} \theta_i^{\oplus m_i}$ where $\theta_1, \ldots, \theta_n$ are irreducible representations of $G$ and $\theta_i^{\oplus m_i}$ means $\theta_i \oplus \ldots \oplus \theta_i$, $m_i$ times, where $m_i$ (the multiplicity of $\theta_i$ in $\theta$) is a positive integer. Denote by $\chi_\pi$ and $\chi_\theta$ the character of $\pi$, $\chi$ respectively. Then $B(\chi_\theta, \chi_\pi) = 0$ unless $\pi \cong \theta_i$ for some $i \in \{1, \ldots, n\}$ and in this case $B(\chi_\theta, \chi_\pi) = m_i$.*

The proof of this is a bit technical and we omit it. It can be found in any textbook of representation theory, cf. [1]. For example, suppose $\pi = \alpha \oplus \beta \oplus \beta$ with $\alpha$, $\beta$ irreducible. Then $B(\chi_\pi, \chi_\alpha) = 1$ and $B(\chi_\pi, \chi_\beta) = 2$. If $\gamma$ is an irreducible representation not isomorphic to $\alpha$ or $\beta$ then $B(\chi_\pi, \chi_\gamma) = 0$.

**Corollary 2.** *Let $\chi$ be a character of $G$. Then $\chi$ is irreducible if and only if $B(\chi, \chi) = 1$. Moreover if $\chi$ is irreducible and $\psi$ is an irreducible character of $G$ then $B(\chi, \psi) = 1$ if $\chi = \psi$ and $B(\chi, \psi) = 0$ if $\chi \neq \psi$. In other words, distinct irreducible characters are orthogonal to each other.*

We include the following corollary because it is beautiful.

**Corollary 3** (n-th Burnside Theorem)**.** *Let $n_1, \ldots, n_t$ be the degrees of the irreducible characters of the finite group $G$. Then $n_1^2 + \ldots + n_t^2 = |G|$.*

*Proof.* For every $g \in G$ consider the function $\gamma_g : G \to G$ given by $\gamma_g(x) = xg$. This is a bijective map whose inverse is $\gamma_{g^{-1}}$. The map $G \to \mathrm{Sym}(G)$ that sends $g$ to $\gamma_g$ is a group homomorphism, hence by considering permutation matrices we can associate to it a representation $\pi : G \to GL(V_G)$ whose character $\chi$ takes $g \in G$ to the number of fixed points of $\gamma_g$. But $\gamma_g$ is fixed-point-free for all $g \neq 1$, indeed if $xg = x$ then multiplying by $x^{-1}$ on the left we find $g = 1$. Clearly $\gamma_1 = 1$ has $|G|$ fixed points. It follows that $\chi(1) = |G|$ and $\chi(g) = 0$ if $g \neq 1$.

With this information we can now determine the decomposition of $\pi$ as direct sum of irreducible representations. Let $\pi_1, \ldots, \pi_t$ be the irreducible representations of $G$ and let $\chi_1, \ldots, \chi_t$ be their characters.

Using Maschke's Theorem we can write $\pi = \bigoplus_{i=1}^{t} \pi_i^{\oplus m_i}$, so that $\chi = m_1\chi_1 + \ldots + m_t\chi_t$. We want to compute the multiplicities $m_i$. For $i = 1, \ldots, t$ we have

$$m_i = B(\chi, \chi_i) = \frac{1}{|G|}\sum_{x\in G}\chi(x)\overline{\chi_i(x)} = \frac{1}{|G|}|G|\chi_i(1) = \chi_i(1) = n_i.$$

It follows by bilinearity of $B$ and orthogonality of irreducible characters that

$$\begin{aligned} n_1^2 + \ldots + n_t^2 &= B(\sum_{i=1}^{t} n_i\chi_i, \sum_{i=1}^{t} n_i\chi_i) = B(\chi,\chi) \\ &= \frac{1}{|G|}\sum_{x\in G}\chi(x)\overline{\chi(x)} = \frac{1}{|G|}|G|^2 = |G|. \end{aligned}$$

This concludes the proof. $\square$

## 4.1 The character table of $S_3$

Consider $S_3$, the group of bijections $\{1,2,3\} \to \{1,2,3\}$. $S_3$ permutes naturally the three basis vectors $e_1, e_2, e_3$ of $\mathbb{C}^3$. Thinking of the elements of $S_3$ as permutation matrices we can imagine that $S_3 \le GL(\mathbb{C}^3)$. It turns out that every element is conjugated to one of the matrices displayed below.

| $S_3$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ |
|---|---|---|---|
| $\chi_1 = 1$ | 1 | 1 | 1 |
| $\chi_2 = \det$ | 1 | $-1$ | 1 |
| $\chi_3$ | 2 | 0 | $-1$ |

$\chi_3$ is the character of the following representation: $S_3$ acts on $W := \{(x_1, x_2, x_3) \in \mathbb{C}^3 \ : \ x_1 + x_2 + x_3 = 0\}$ (2-dimensional) by permuting the canonical basis vectors (indeed, the equation $x_1 + x_2 + x_3 = 0$ is invariant under any permutation of the three indices $1, 2, 3$).

How does $V = \mathbb{C}^3$ decompose via this representation? Let $U$ be the space of constant vectors: $U := \{(a, a, a) \ : \ a \in \mathbb{C}\}$. Then $U$ and $W$ are irreducible $S_3$-invariant subspaces of $V$ and

$$V = U \oplus W.$$

Also, note that the column of the character table with the identity matrix on top gives precisely the degrees of the irreducible representations of $S_3$ (indeed, the trace of the identity matrix is just the dimension of the space!) and this fits with the $n$-th Burnside Theorem (Corollary 3) because

$$1^2 + 1^2 + 2^2 = 6 = 3! = |S_3|.$$

# 5 Fourier analysis

Let us relax the condition of finiteness of $G$ and consider **compactness**. Consider the **circle** of center the origin and radius 1:

$$G := S^1 = \{e^{i\theta} \ : \ 0 \leq \theta \leq 2\pi\}.$$

Why did I call it $G$? Because it is a group with respect to multiplication:

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}, \qquad (e^{i\theta})^{-1} = e^{-i\theta} = e^{i(2\pi-\theta)}.$$

Algebraically it can be viewed as the quotient $\mathbb{R}/2\pi\mathbb{Z}$.

- To make the theory of representations meaningful in this setting **we must take the topology into account**. In other words, we will have a topology on the space, and the notion of "$G$-invariant subspace" will be substituted by "**closed** $G$-invariant subspace". Also, we require morphisms to be **continuous**. Moreover, the notion of direct sum will be substituted with the notion of **orthogonal** direct sum (cf. below).
- What are the irreducible representations of $S^1$? Since $S^1$ is abelian (and compact: cf. Section 6), they are all 1-dimensional (by Schur's lemma!), i.e. they are continuous homomorphisms $S^1 \to \mathbb{C}^\times$. This forces them to be of the form $e^{i\theta} \mapsto e^{in\theta}$ where $n \in \mathbb{Z}$.

Now we need a vector space representing our group $G = S^1$.

- Let $V := L^2(G) = \{f : G \to \mathbb{C} \ : \ \int_G |f(e^{i\theta})|^2 d\theta < \infty\}$. It is a **Hilbert space** with the hermitian inner product given by

$$B(u,v) := \frac{1}{2\pi}\int_G u(e^{i\theta})\overline{v(e^{i\theta})}d\theta.$$

Note that $2\pi = \int_G d\theta$, hence it substitutes $|G|$, which was used in the finite case. Note that a function $S^1 \to \mathbb{C}$ can be thought of as a periodic function $\mathbb{R} \to \mathbb{C}$ of period $2\pi$. Orthogonality of characters now amounts to the following easy computation. Suppose $n, m$ are distinct integers. Then

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\overline{e^{im\theta}}d\theta &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}e^{-im\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta}d\theta \\
&= \frac{1}{2\pi}\frac{1}{i(n-m)}[e^{i(n-m)\theta}]_0^{2\pi} \\
&= \frac{1}{2\pi i(n-m)}(e^{i(n-m)2\pi} - e^{i(n-m)0}) = 0.
\end{aligned}$$

Instead, if $n = m$ then we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\overline{e^{in\theta}}d\theta &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}e^{-in\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1.
\end{aligned}$$

- Consider the following representation of $G$:

$$\pi : G \to GL(V), \qquad \pi_{e^{i\theta}}(f)(e^{i\theta_0}) := f(e^{i(\theta_0+\theta)}).$$

  It is **unitary**, i.e. $B(\pi_g(f_1), \pi_g(f_2)) = B(f_1, f_2)$ (this follows easily by the change of variables $\tau = \theta_0 + \theta$).
  – So Maschke Theorem holds!

- The irreducible invariant subspaces of $V$ are, for $n \in \mathbb{Z}$,

$$V_n := \{f \in V \ : \ f(e^{i(\theta_0+\theta)}) = e^{in\theta} f(e^{i\theta_0})\} = \mathbb{C}\{e^{i\theta} \mapsto e^{in\theta}\}.$$

  In other words, an element of $V_n$ has the form $e^{i\theta} \mapsto \lambda e^{in\theta}$ for some $\lambda \in \mathbb{C}$.

We want to **decompose our $V$ as direct sum of irreducible subspaces**. Now $V$ is a Hilbert space, and in this setting the right notion of direct sum to use is the notion of **orthogonal** direct sum. If $V$ is a Hilbert space and $\{H_i \ : \ i \in I\}$ is a family of subspaces, the orthogonal direct sum $\hat{\bigoplus} H_i$ is the set of elements $(h_i)_{i\in I} \in \prod_{i\in I} H_i$ such that $\sum_{i\in I} ||h_i||^2 < \infty$. It turns out that the orthogonal direct sum of the subspaces $H_i$ is the closure in $V$ of the algebraic direct sum $\bigoplus_{i\in I} H_i$, which is by definition the set of elements $(h_i)_{i\in I} \in \prod_{i\in I} H_i$ such that the set $\{i \in I \ : \ h_i \neq 0\}$ is finite.

The decomposition of $V = L^2(S^1)$ into irreducible $G$-invariant subspaces is the following:

$$\begin{aligned} V &= \widehat{\bigoplus}_{n\in\mathbb{Z}} V_n = \overline{\{\sum_{finite} v_n \ : \ v_n \in V_n\}} \\ &= \{\theta \mapsto \sum_{n\in\mathbb{Z}} c_n e^{in\theta} \ : \ c_n \in \mathbb{C}\ \forall n \in \mathbb{Z},\ \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty\}. \end{aligned}$$

This says that any function $S^1 \to \mathbb{C}$ (i.e. any periodic function of period $2\pi$ !) which is square-integrable admits an espression of the form

$$f(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

for some $c_n \in \mathbb{C}$ with $\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty$ (this ensures that $f$ is square-integrable). This is the Fourier series of $f$.

Now we want to compute the Fourier coefficients $c_m$ (what in the discrete case we were calling "multiplicities"!). Using orthogonality of characters we find

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-im\theta}d\theta = B(f(\theta), e^{im\theta}) = B(\sum_{n\in\mathbb{Z}} c_n e^{in\theta}, e^{im\theta}) = c_m.$$

Thus **Fourier analysis is the representation theory of the circle group** $S^1$. This is related to the representation theory of $G = SL(2, \mathbb{R})$, the group of $2 \times 2$ matrices with real coefficients and determinant 1: the matrices of the form $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ form a maximal compact subgroup $K$ of $G$ isomorphic to $S^1$. Using Harish-Chandra modules, it is possible to study the representations of the Lie group $G$ only using $K$ and the Lie algebra of $G$. A good source of material about this can be found in the videos of the $SL(2,\mathbb{R})$ Summer School in Utah in June 2006 [5].

# 6 Finite dimensionality of irreducible representations of compact groups

Let us spend some more words on why the irreducible representations of a compact group are finite dimensional. A Hilbert space is a vector space $H$ endowed with a positive definite Hermitian inner product $\langle\cdot,\cdot\rangle$ which makes it a complete metric space. Let $H$ be a Hilbert space and let $U(H)$ be the group of all bounded unitary operators on $H$. Recall that a linear operator $U : H \to H$ is said to be bounded if there exists a constant $M$ such that $||Uh||/||h|| \leq M$ for all $h \in H - \{0\}$. A bounded linear operator $U : H \to H$ is said to be unitary if $U$ is surjective and $\langle Ux, Uy\rangle_H = \langle x, y\rangle_H$ for all $x, y \in H$.

Let $G$ be a compact group, and let $\pi : G \to U(H)$ be a group homomorphism such that for all $v \in H$, the map $G \to H$, $g \mapsto \pi(g)v$ is continuous. This $\pi$ is what we call a unitary representation of $G$. $\pi$ is irreducible if $H$ has no closed invariant subspaces except for $\{0\}$ and $H$. Let $\pi : G \to U(H)$ be a unitary representation of $G$ and suppose that it is irreducible. We want to show that then $H$ must be finite-dimensional.

**Theorem 6** (Schur's Lemma)**.** *If $T$ is a bounded linear operator on $H$ such that $T\pi(g) = \pi(g)T$ for all $g \in G$ then $T$ is the multiplication by a scalar $\lambda \in \mathbb{C}$.*

Since $G$ is compact it admits a measure, called left (normalized) Haar measure. It is characterized as follows. A Borel set in $G$ is an element of the $\sigma$-algebra of $G$ generated by the open subsets of $G$. There is, up to a multiplicative constant, a unique countably additive, nontrivial measure $\mu$ on the Borel subsets of $G$ satisfying the following properties:

- $\mu$ is left-translation-invariant: $\mu(gE) = \mu(E)$ for every $g \in G$ and Borel set $E$.
- $\mu$ is finite on every compact set: $\mu(K) < \infty$ if $K$ is compact.
- $\mu$ is outer regular on Borel sets $E$:

$$\mu(E) = \inf\{\mu(U) \ : \ E \subseteq U,\ U \text{ open}\}.$$

- $\mu$ is inner regular on open sets $E$:

$$\mu(E) = \sup\{\mu(K) \ : \ K \subseteq E,\ K \text{ compact}\}.$$

It is called a "left Haar measure". The normalized left Haar measure on $G$ is the unique left Haar measure $\mu$ on $G$ such that $\mu(G) = 1$.

Using the Haar measure we can compute integrals. The map $G \to \mathbb{C}$, $g \mapsto \langle \pi(g)u, v\rangle$ is continuous on $G$ for all $u, v \in H$. For $v, v', w, w' \in H$ consider

$$I(v, v', w, w') := \int_G \langle \pi(g)v, w\rangle \cdot \overline{\langle \pi(g)v', w'\rangle} dg.$$

Think of $w, w'$ as fixed. It follows from the Riesz representation theorem that there is a bounded linear operator $T_{w,w'} : H \to H$ such that $I(v, v', w, w') = \langle T_{w,w'}v, v'\rangle$ for all $v, v' \in H$. Now we prove that for every $g \in G$ we have $\pi(g)T_{w,w'} = T_{w,w'}\pi(g)$. We have

$$\begin{aligned}
\langle T_{w,w'}\pi(g)v, v'\rangle &= \int_G \langle \pi(h)\pi(g)v, w\rangle \overline{\langle \pi(h)v', w'\rangle} dh \\
&= \int_G \langle \pi(h)v, w\rangle \overline{\langle \pi(hg^{-1})v', w'\rangle} dh \\
&= \langle T_{w,w'}v, \pi(g^{-1})v'\rangle = \langle \pi(g)T_{w,w'}v, v'\rangle.
\end{aligned}$$

The second equality follows from the fact that the measure $dg$ is $G$-invariant, the fourth equality follows from unitarity of $\pi$ (apply $\pi(g)$ to both the arguments of $\langle \cdot, \cdot\rangle$). This proves that $\pi(g)T_{w,w'} = T_{w,w'}\pi(g)$. By Schur's Lemma we deduce that $T_{w,w'}$ is the multiplication by the scalar $\lambda(w, w') \in \mathbb{C}$. We obtain that

$$\int_G \langle \pi(g)v, w\rangle \cdot \overline{\langle \pi(g)v', w'\rangle} dg = \lambda(w, w')\langle v, v'\rangle.$$

Repeating the same argument thinking of $v, v'$ as fixed we find that there is a function $\mu(v, v')$ such that

$$\int_G \langle \pi(g)v|w\rangle \cdot \overline{\langle \pi(g)v', w'\rangle} dg = \mu(v, v')\overline{\langle w, w'\rangle}.$$

It follows that for all $v, v', w, w' \in H$ we have $\lambda(w, w')\langle v, v'\rangle = \mu(v, v')\overline{\langle w, w'\rangle}$. Choosing $v = v' = v_0$ of norm 1 we find $\lambda(w, w') = \mu(v_0, v_0)\overline{\langle w, w'\rangle}$. Call $C := \mu(v_0, v_0)$. Then we find

$$\int_G \langle \pi(g)v, w\rangle \cdot \overline{\langle \pi(g)v', w'\rangle} dg = C\langle v, v'\rangle \overline{\langle w, w'\rangle}.$$

Suppose $v = v'$, $w = w'$ and $||v|| = ||w|| = 1$. Then

$$\int_G |\langle \pi(g)v, w\rangle|^2 dg = C > 0. \qquad (*)$$

We want to show that $H$ is finite dimensional. Suppose by contradiction that $H$ is infinite dimensional. Then for every positive integer $n$ we can find $e_1, \ldots, e_n \in H$ mutually orthogonal of length 1. Since $\pi$ is unitary, $\pi(g)e_1, \ldots, \pi(g)e_n$, for $g \in G$, are also mutually orthogonal of length 1. Using Bessel inequality for $v \in H$ we find

$$\sum_{i=1}^{n} |\langle v, \pi(g)e_i\rangle|^2 \leq ||v||^2. \qquad (**)$$

Now integrating over $G$ and using $(*)$ and $(**)$ we find

$$\begin{aligned} nC||v||^2 &= \sum_{i=1}^{n} C||v||^2 = ||v||^2 \sum_{i=1}^{n} \int_G |\langle \pi(g)e_i, \frac{v}{||v||}\rangle|^2 dg \\ &= \int_G \sum_{i=1}^{n} |\langle v, \pi(g)e_i\rangle|^2 \leq \int_G ||v||^2 dg = ||v||^2 \end{aligned}$$

hence $n \leq 1/C$. This cannot hold for every positive integer $n$, thus we found a contradiction. So $H$ is finite dimensional. Let now $n$ be its dimension. Then $\sum_{i=1}^{n} |\langle v, \pi(g)e_i\rangle|^2 = ||v||^2$ (the equality in $(**)$ becomes an equality). The above computation then shows that $C = 1/n$.

## 6.1 Heisenberg's Uncertainty Principle

Let $G = \mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$ (cyclic group of order $N$) and let

$$L(G) := \mathbb{C}^G = \{functions\ G \to \mathbb{C}\}.$$

It is a $\mathbb{C}$-vector space of dimension $|G|$.

Let $\hat{G} := \{group\ homomorphisms\ \chi : G \to \mathbb{C}^\times\}$.

- It is a group isomorphic to $G$, generated by $1 \mapsto e^{i2\pi/N}$.
- It is the set of linear (1-dimensional) characters of $G$.
- It is a basis of $L(G)$.

**Fourier transform**:

$$\mathcal{F} : L(G) \to L(\hat{G}), \qquad \mathcal{F}(f)(\chi) := B(f, \chi) = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{\chi(x)},$$

the coefficient of $\chi$ in the expression of $f$ in the base $\hat{G}$.

In other words, if $f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$ then $\mathcal{F}(f)(\chi) = \hat{f}(\chi)$.

For $f \in L(G)$ let $Supp(f) := \{x \in G \ : \ f(x) \neq 0\}$.

**Theorem 7** (Heisenberg's Uncertainty Principle)**.** *If $f \in L(G)$ then*

$$|Supp(f)| \cdot |Supp(\mathcal{F}(f))| \geq |G|.$$

*Proof.* For $f, h \in L(G)$ define

$$\langle f, h\rangle_{L(G)} := \frac{1}{|G|} \sum_{x \in G} f(x)\overline{h(x)}$$

and

$$||f|| := \sqrt{\langle f, f\rangle_{L(G)}}.$$

This defines a positive definite hermitian inner product in $L(G)$. For $f, h \in L(\hat{G})$ define

$$\langle f, h\rangle_{L(\hat{G})} := \sum_{\chi \in \hat{G}} f(\chi)\overline{h(\chi)}$$

and

$$||f|| := \sqrt{\langle f, f\rangle_{L(\hat{G})}}.$$

This defines a positive definite hermitian inner product in $L(\hat{G})$. We now prove that the Fourier transform $\mathcal{F}$ is unitary (this is known as **Parseval formula**): if $f, h \in L(G)$ then $\langle f, h\rangle_{L(G)} = \langle \hat{f}, \hat{h}\rangle_{L(\hat{G})}$. Indeed, by orthogonality of characters we have

$$\begin{aligned}
\langle f, h\rangle_{L^2(G)} &= \frac{1}{|G|}\sum_{x\in G} f(x)\overline{h(x)} = \frac{1}{|G|}\sum_{x\in G}\sum_{\chi,\eta\in\hat{G}} \hat{f}(\chi)\chi(x)\overline{\hat{h}(\eta)}\overline{\eta(x)} \\
&= \frac{1}{|G|}\sum_{\chi,\eta\in\hat{G}} \hat{f}(\chi)\overline{\hat{h}(\eta)}\sum_{x\in G}\chi(x)\overline{\eta(x)} = \sum_{\chi\in\hat{G}} \hat{f}(\chi)\overline{\hat{h}(\chi)} \\
&= \langle \hat{f}, \hat{h}\rangle_{L^2(\hat{G})}.
\end{aligned}$$

Recall that in both spaces $L(G)$, $L(\hat{G})$ we have the **Cauchy-Schwarz inequality**

$$|\langle f, h\rangle| \le ||f|| \cdot ||h||$$

for any $f, h$ in the space. With these ingredients we can proceed to our computation. For a function $f$ defined on a set $X$ define $||f||_\infty := \max_{x\in X} |f(x)|$. For $f \in L(G)$, $A = \mathrm{Supp}(f)$, $B = \mathrm{Supp}(\hat{f})$ we then have

$$\begin{aligned}
||\hat{f}||_\infty &= \max_{\chi\in\hat{G}} |f(\chi)| = \max_{\chi\in\hat{G}} |\frac{1}{|G|}\sum_{x\in G} f(x)\overline{\chi(x)}| \\
&= \max_{\chi\in\hat{G}} |\frac{1}{|G|}\sum_{x\in G} 1_A(x)f(x)\overline{\chi(x)}| \\
&= \max_{\chi\in\hat{G}} |\langle f, 1_A\chi\rangle| \le \max_{\chi\in\hat{G}} ||f||_{L^2(G)}||1_A\chi||_{L^2(G)}
\end{aligned}$$

where the last inequality is a consequence of the Cauchy-Schwarz inequality. Now since the elements of $\hat{G}$ are linear characters, they are homomorphisms $G \to \mathbb{C}^\times$ hence $\chi(x)$ is a root of unity for all $\chi \in \hat{G}$, $x \in G$ hence $|\chi(x)| = 1$. We deduce that

$$||1_A\chi||_{L^2(G)} = \sqrt{|\frac{1}{|G|}\sum_{x\in G} 1_A(x)\overline{\chi(x)}|} \le \sqrt{\frac{1}{|G|}\sum_{x\in G} 1_A(x)|\overline{\chi(x)}|} = \sqrt{|A|/|G|}.$$

We can now proceed with our estimation recalling that $\mathcal{F}$ is unitary.

$$\begin{aligned}
||\hat{f}||_\infty &\le \max_{\chi\in\hat{G}} ||f||_{L^2(G)}||1_A\chi||_{L^2(G)} \le \sqrt{|A|/|G|} \cdot ||f||_{L^2(G)} \\
&= \sqrt{|A|/|G|} \cdot ||\hat{f}||_{L^2(\hat{G})} = \sqrt{|A|/|G|}\sqrt{\sum_{\chi\in\hat{G}} 1_B(\chi)|\hat{f}(\chi)|^2} \\
&\le \sqrt{|A|/|G|} \cdot ||\hat{f}||_\infty\sqrt{\sum_{\chi\in\hat{G}} 1_B(\chi)} = \sqrt{|A||B|/|G|} \cdot ||\hat{f}||_\infty.
\end{aligned}$$

In conclusion $||\hat{f}||_\infty \le \sqrt{|A||B|/|G|} \cdot ||\hat{f}||_\infty$ hence $|A||B| \ge |G|$. □

# 7 The solution of the cube problem

Now we go back to our original problem. Let $G$ be **the group of rotations of the cube**. Then $|G| = 24$ (if you place a cube on a table, you can put each of the 6 faces up, and rotate that face in 4 ways).

It turns out that $G \cong S_4$ (the idea is to observe that $G$ permutes the four diagonals of the cube in any possible way!).

$G$ permutes the six faces of the cube. Let $F$ be the set of faces of the cube. This gives a permutation matrix representation

$$\pi : G \to GL(W_F) \qquad \text{where} \qquad W_F := \mathbb{C}^F = \{functions\ F \to \mathbb{C}\} \cong \mathbb{C}^6.$$

By computing the fixed points of the elements of $G$ we can compute the character of this representation. Call it $\chi$.

It turns out that $B(\chi, \chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = 3$.

Now we know by Maschke Theorem that $\pi$ is a direct sum of irreducible representations, so we can write $\pi = \bigoplus_{i=1}^n \pi_i^{\oplus m_i}$ with $m_1, \ldots, m_n$ positive integers ($m_i$ is the multiplicity of $\pi_i$ in $\pi$, i.e. the number of times $\pi_i$ appears in the decomposition of $\pi$: the notation $\pi_i^{\oplus m_i}$ means $\pi_i \oplus \ldots \oplus \pi_i$, $m_i$ times) and we deduce $\chi = \sum_{i=1}^n m_i \chi_i$ with $\chi_i$ the character of $\pi_i$ for $i = 1, \ldots, n$. On the other hand $B(\chi, \chi) = 3$ hence, by orthogonality of characters,

$$3 = B(\chi, \chi) = B(\sum_{i=1}^n m_i \chi_i, \sum_{i=1}^n m_i \chi_i) = m_1^2 + \ldots + m_n^2.$$

Since $m_1, \ldots, m_n$ are positive integers with the property that $m_1^2 + \ldots + m_n^2 = 3$, we deduce that $n = 3$ and $m_1 = m_2 = m_3 = 1$. In other words $\pi = \pi_1 \oplus \pi_2 \oplus \pi_3$ where $\pi_i$ is irreducible for $i = 1, 2, 3$ and $\chi = \chi_1 + \chi_2 + \chi_3$.

Hence $W_F = \mathbb{C}^F = \{functions\ F \to \mathbb{C}\} \cong \mathbb{C}^6$ **is the direct sum of three $G$-invariant irreducible subspaces**. We are left to find them. This is where the geometry comes in: our problem is now reduced to find $G$-invariant subspaces.

But we know the characters of the corresponding representations! This means that we have a lot of information about them, which leads us close to determining them explicitly. This is the use of representation theory: collect as much information as possible about the decomposition.

For a face $x$ let $-x$ denote the face opposite to $x$. Consider

- $W_1 := \{constant\ functions\ F \to \mathbb{C}\}$. This is clearly one-dimensional: $\dim_{\mathbb{C}}(W_1) = 1$.
- $W_2 := \{f : F \to \mathbb{C} \ : \ f(-x) = f(x)\ \forall x \in F,\ \sum_{x \in F} f(x) = 0\}$. This is given by $3 + 1 = 4$ equations so $\dim_{\mathbb{C}}(W_2) = 6 - 4 = 2$.

- $W_3 := \{f : F \to \mathbb{C} \;:\; f(-x) = -f(x)\ \forall x \in F\}$. This is given by 3 equations so $\dim_{\mathbb{C}}(W_3) = 6 - 3 = 3$.

The decomposition of $W_F$ into irreducible subspaces is

$$W_F = W_1 \oplus W_2 \oplus W_3.$$

The operator we are concerned with is

$$L : W_F \to W_F, \qquad L(f)(x) := \frac{1}{4} \sum_{y \in A_x} f(y)$$

where $A_x$ denotes the set of faces adjacent to the face $x$. It turns out that $L$ is $G$-invariant! This is because *rotating after averaging is the same as averaging after rotating.*

Since $W_1, W_2, W_3$ are irreducible, by Schur's lemma $L|_{W_i}$ is a scalar operator. Using $W_F = W_1 \oplus W_2 \oplus W_3$ it turns out that

$$L \sim \left(\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The operator $L^n$ has eigenvalues 1, $(-1/2)^n$ and 0, hence if $n$ is large then $L^n(f)$ is approximately equal to the projection of $f$ onto $W_1$. The projection of any vector $f$ whose entries are $1, 2, 3, 4, 5, 6$ in some order onto $W_1$ is always $(3.5, 3.5, 3.5, 3.5, 3.5, 3.5)$. Hence no matter what is the initial configuration, i.e. the initial position of the numbers from 1 to 6 labelling the faces of the cube, after many iterations the value on each face gets arbitrarily close to 3.5.

This phenomenon can be interpreted using ergodic theory. To give an idea of this fact let us state a version of Von Neumann ergodic theorem taken from [4] (the original source is [3]). Note however that the following result does not apply right away to our case because $L$ is not invertible.

**Theorem 8** (Von Neumann Ergodic Theorem)**.** *Let $U : H \to H$ be a unitary operator on a separable Hilbert space $H$. Then for every $v \in H$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n v = \pi(v)$$

*where $\pi : H \to H^U$ is the orthogonal projection from $H$ to the closed subspace $H^U := \{v \in H \;:\; Uv = v\}$ consisting of the $U$-invariant vectors.*

In our problem we actually dealt with the powers of the operator $L$ and not the arithmetic mean of the powers. But note that if a sequence $a_n$ converges to $a$ in a normed vector space then also the arithmetic mean $\frac{1}{N}\sum_{n=0}^{N-1} a_n$ converges to $a$ as $N \to \infty$.

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