# Covering Permutation Groups

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### Remark

*No finite group is the union of one conjugacy class of proper subgroups.*

However, if $G$ is infinite this is no longer true, for example $GL_n(\mathbb{C})$ is the union of the conjugates of the Borel subgroup (each complex matrix can be taken to upper triangular form).

### Proposition

*Let $f(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree $n > 1$. If $f(X)$ has a root modulo p for all primes p then $f(X)$ is reducible.*

The idea is the following: since the factorization patterns modulo (unramified) primes correspond to the cyclic structures of the elements of the Galois group acting on the roots, **the Galois group is the union of the point stabilizers**, therefore **it cannot act transitively on the roots (otherwise the point stabilizers would form one conjugacy class)**, i.e. $f(X)$ cannot be irreducible.

Given a finite group $G$, we call a "cover" of $G$ a family $\mathcal{H}$ of proper subgroups of $G$ such that $\bigcup_{H \in \mathcal{H}} H = G$. We say that the cover $\mathcal{H}$ of $G$ is a "normal cover" if $gHg^{-1} \in \mathcal{H}$ for every $H \in \mathcal{H}$, $g \in G$.

### Definition

*$\sigma(G)$, the covering number of $G$, will denote the smallest number of subgroups in a cover of $G$. $\gamma(G)$, the normal covering number of $G$, will denote the smallest number of conjugacy classes of subgroups in a normal cover of $G$. If $G$ is cyclic set $\sigma(G) = \gamma(G) = \infty$.*

Note that if $\mathcal{H}$ is any (resp. normal) cover of $G$ then the number of (resp. conjugacy classes of) elements of $\mathcal{H}$ is an upper bound for $\sigma(G)$ (resp. $\gamma(G)$). In particular, if $N$ is any normal subgroup of $G$ then since any (normal) cover of $G/N$ can be lifted to a (normal) cover of $G$ we obtain that

$$\sigma(G) \leq \sigma(G/N) \qquad \text{and} \qquad \gamma(G) \leq \gamma(G/N).$$

Let $G$ be a finite group.

### $\sigma \geq 3$

Since no finite group is the union of two proper subgroups, $\sigma(G) \geq 3$. For example $\sigma(C_2 \times C_2) = 3$. According to a theorem of Scorza, a group $G$ verifies $\sigma(G) = 3$ if and only if there exists $N \trianglelefteq G$ such that $G/N \cong C_2 \times C_2$.

### $\gamma \geq 2$

Since no finite group is the union of one single conjugacy class of proper subgroups, $\gamma(G) \geq 2$. For example $\gamma(S_3) = 2$. More in general, if $G$ is any solvable group such that $G/G'$ is cyclic then $\gamma(G) = 2$.

### Theorem (Bubboloni, Praeger, Spiga)

*Let $n \geq 5$ be an integer, and let $G$ be $Sym(n)$ or $Alt(n)$. There are positive constants $a, b$ such that $an \leq \gamma(G) \leq bn$.*

### Theorem (Bubboloni)

*Let $n$ be a positive integer.*

- $\gamma(S_n) = 2$ *if and only if* $n \in \{3, 4, 5, 6\}$;
- $\gamma(A_n) = 2$ *if and only if* $n \in \{4, 5, 6, 7, 8\}$.

### Theorem (Maróti, Britnell 2012)

*Let $G \in \{(P)SL(n, q), (P)GL(n, q)\}$. Then $n/\pi^2 \leq \gamma(G) \leq (n+1)/2$.*

### Theorem (E. Crestani, A. Lucchini [7])

*For every integer $n \geq 2$ there exists a finite solvable group $G$ with $\gamma(G) = n$.*

Clearly $\gamma(G) \leq \sigma(G)$ for every finite group $G$.

Let $G$ be a nilpotent group. Since every maximal subgroup of $G$ is normal, and since there always exist minimal (normal) covers consisting of maximal subgroups, we deduce $\sigma(G) = \gamma(G)$. It is possible to prove that

$$\sigma(G) = \gamma(G) = p + 1$$

where $p$ is the smallest prime divisor of $|G|$ such that the Sylow $p$-subgroup of $G$ is not cyclic.

Tomkinson proved that if $G$ is a finite solvable group then $\sigma(G) = q + 1$ where $q$ is the smallest order of a chief factor of $G$ with more than one complement. No such result is known for $\gamma(G)$ yet.

Here we address the following question: what can be said about the groups $G$ with (*) $\sigma(G) = \gamma(G)$ ? Remember that nilpotent groups verify (*). There are non-nilpotent groups for which $\sigma(G) = \gamma(G)$, here are a couple of examples.

- $G = C_2 \times S_n$ for $n \geq 7$ – in this case $\sigma(G) = \gamma(G) = 3$;
- $G = C_p \times C_p \times S_n$ for $p$ an odd prime and $n$ larger than a suitable function of $p$ – in this case $\sigma(G) = \gamma(G) = p + 1$.

We remark that

$$\gamma(G) = \gamma(G/G') \Rightarrow \sigma(G) = \gamma(G).$$

Indeed,

$$\gamma(G) \leq \sigma(G) \leq \sigma(G/G') = \gamma(G/G') = \gamma(G).$$

It is natural to ask whether $\sigma(G) = \gamma(G)$ implies $\gamma(G) = \gamma(G/G')$.

### Theorem (A. Lucchini, G.)

*$\sigma(G) = \gamma(G)$ if and only if $\gamma(G) = \gamma(G/G')$.*

The idea is to provide an invariant of $G$ which is an upper bound for $\gamma(G)$ and a lower bound for $\sigma(G)$.

### Definition

*Let $G$ be a noncyclic group. Define $\mu(G)$ to be the smallest positive integer $m$ such that $G$ has two maximal subgroups of index $m$.*

Cohn proved that $\mu(G) + 1 \leq \sigma(G)$. We proved the following result, which of course implies the theorem above.

### Proposition

*Let $G$ be a noncyclic group. Then $\gamma(G) \leq \mu(G) + 1$. Moreover equality holds if and only if $\mu(G) = p$ is a prime, $G$ contains two normal subgroups of index $p$ and $\gamma(G) = \gamma(G/G')$.*

## Proposition

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Let us see how the proof of this goes. There are two possibilities:

1. $G$ contains two maximal subgroups $A, B$ which are normal of index $m = \mu(G)$. Then $m$ is prime, $G/A \cap B \cong C_m \times C_m$ and hence $\gamma(G) \leq \gamma(C_m \times C_m) = m + 1$.
2. There exists a maximal subgroup $M$ of $G$, not normal, of index $m = \mu(G)$. Then $G/M_G$ is a noncyclic subgroup of $\mathrm{Sym}(m)$ and $\gamma(G) \leq \gamma(G/M_G)$.
   The result follows if we can prove that $\gamma(G/M_G) \leq m$.

It follows that what we need is a way to compare the normal covering number of a (primitive) permutation group with its degree as a permutation group.

**Theorem (A. Lucchini, G.)**

*Let $G$ be a noncyclic subgroup of $Sym(n)$.*
*Then $\gamma(G) \leq (n+2)/2$.*

*In other words, $G$ is the union of at most $(n+2)/2$ conjugacy classes of proper subgroups of $G$.*

Observe that this upper bound is achieved infinitely often: if $p$ is a prime, $C_p \times C_p < \mathrm{Sym}(2p)$ and $\gamma(C_p \times C_p) = p+1 = (2p+2)/2$.

Actually it is possible to prove that this is not a coincidence: if $G \leq \mathrm{Sym}(n)$ then $\gamma(G) = (n+2)/2$ if and only if $n/2 = p$ is a prime and $G \cong C_p \times C_p$, generated by two disjoint $p$-cycles.

## Theorem

*Let G be a noncyclic subgroup of Sym(n). Then* $\gamma(G) \leq (n+2)/2$*.*

The proof goes through the following steps:

1. Reduction to $G/G'$ cyclic. [If $G$ surjects onto $C_p \times C_p$ then $p^2|n!$ and $\gamma(G) \leq \gamma(C_p \times C_p) = p+1 \leq n/2+1 = (n+2)/2$.]
2. Reduction to the nonsolvable case. [If $H$ is a solvable noncyclic group such that $H/H'$ is cyclic then $\gamma(H) = 2$.]
3. Reduction to the transitive case. [$G$ is a subdirect product of its transitive components: use induction on the degree $n$.]
4. Reduction to the primitive case.
5. Reduction to the almost-simple case.

As it turns out, the primitive case is the crucial one.

Recall that $G \leq \mathrm{Sym}(n)$ is said to be imprimitive if there exists $B \subseteq \{1, \ldots, n\}$ with $|B| \neq 1, n$ and $B^g \cap B$ equals either $B$ or $\emptyset$ for every $g \in G$. If $G$ is not imprimitive, it is called primitive.

The key fact which allows to deal with primitive groups is the following.

## Proposition

*Let $L$ be a group with a unique minimal normal subgroup $N$. Suppose that $N$ is nonabelian and $L/N$ is cyclic. Write $N = S_1 \times \cdots \times S_t = S^t$ with $S$ a nonabelian simple group. Then $\gamma(L) < t \cdot m(S)/2$, where $m(S)$ is the smallest index of a proper subgroup of $S$.*

Note that this includes almost-simple groups (case $t = 1$). The proof consists of a reduction to the case $t = 1$.

Here it goes: let $X := N_L(S_1)/C_G(S_1)$. Then $X$ is an almost-simple group with socle isomorphic to $S$. Let $\mathcal{M}$ be a normal cover of $X$. One proves that $L$ is the union of the maximal subgroups of $L$ containing $N$ and the conjugates of the subgroups $N_L(M^t)$ for $M \in \mathcal{M}$. This implies that (here $\omega(x)$ is the number of prime divisors of $x$)

$$\gamma(L) \leq \omega(t \cdot |X/S|) + \gamma(X)$$

and basically reduces the problem to $\gamma(X)$.

**Proposition (A. Lucchini, G.)**

*Let $X$ be an almost simple group. If $X \neq Aut(Alt(6))$ then $\gamma(X) < m(\mathrm{soc}(X))/2$. Moreover $\gamma(Aut(Alt(6)) = 3$.*

In order to prove this we used the following tools.

- The Aschbacher classes for the linear almost-simple groups with socle $PSL(n, q)$.
- Upper bounds by Fulman and Guralnick [9] for $k(S)$, the number of conjugacy classes of the simple group $S$, for the other groups of Lie type.
- The results of Praeger and Bubboloni [6] about $\gamma(\mathrm{Alt}(n))$ and $\gamma(\mathrm{Sym}(n))$.

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