Commutative Algebra Test
20/01/2015

Exercise 1. Let R be a noetherian UFD and $S \subseteq R - \{0\}$ a multiplicative subset. Show that $S^{-1}R$ is a UFD. Is it true that $S^{-1}R$ UFD $\Longrightarrow$ R UFD? Prove it, or provide a counterexample.

Exercise 2. Let R be a ring and $S \subseteq R - \{0\}$ a multiplicative subset, $\varphi : R \to S^{-1}R$ the canonical map $\varphi(x) = \frac{x}{1}$. Let $I \subset R$ be an ideal such that $I \cap S = \varnothing$.

a) Show that if I is prime, then $\varphi^{-1}(S^{-1}I) = I$.

b) For an arbitrary ideal, is $\varphi^{-1}(S^{-1}I) = I$? Prove it, or provide a counterexample.

c) Same as b), but under the assumption that R is a domain.

d) Suppose that R is a noetherian domain, $S = \{1, f, f^2, \ldots\}$. Assume that both fR and I are prime ideals. Show that I is principal if and only if $S^{-1}I$ is principal.

Exercise 3.

a) Let R be a noetherian domain, $f \in R$ a prime element. Show that R is a UFD if and only if R_f is a UFD.

b) Let k be an algebraically closed field and $R = k[X, Y, Z, U, V]/(X^2 + Y^2 + Z^2 - UV)$. Assume that the characteristic of k is not 2. Show that R is a UFD.

c) Show that R is not regular.

Exercise 4. Let k be an algebraically closed field and $A = k[X, Y]/(X^3 + X - Y^2)$. Assume that the characteristic of k is not 2. As usual, let x, y be the images of X, Y in A.

a) Show that A is a Dedekind domain.

b) Describe the splitting of prime ideals in the extension $k[X] \subset A$.

c) Using the norm, show that $A^\times = k^\times$.

d) Show that y is irreducible.

e) Let $\mathfrak{m} = (x, y)$. Compute a uniformiser for $\mathfrak{m}A_\mathfrak{m}$.

f) Show that $\mathrm{Pic}(A) \neq 0$. Conclude that A is not a UFD.

g) Find an element in A with two non-equivalent factorisations.

You can submit your answers in english, french or italian.