Commutative Algebra Exam

Padova, 24/06/2016

Exercise 1. Let k be an algebraically closed field and consider $R = k[X] \subset A = k[X,Y]/(XY)$.

a) Does $R \subset A$ satisfy the Going Up property?

b) Does $R \subset A$ satisfy the Going Down property?

Exercise 2. Let $L = \mathbb{Q}(\alpha)$, where $\alpha^3 = 13$. Let $A = \mathbb{Z}[\alpha]$ and write $\mathcal{O}_L$ for the integral closure of $\mathbb{Z}$ in L.

a) Compute $\Delta(1, \alpha, \alpha^2)$.

b) Describe the decompostion of 2, 7 and 19 in L and compute the inertia and ramification degree above these primes.

c) Let $A_3 = (\mathbb{Z} - 3\mathbb{Z})^{-1}A$ and $\mathcal{O}_{L,3} = (\mathbb{Z} - 3\mathbb{Z})^{-1}\mathcal{O}_L$. Show that $(\alpha - 1)A_3$ is a maximal ideal.

d) Show that $A_3 = \mathcal{O}_{L,3}$ and compute the decomposition of 3 in L.

e) Let $A_{13} = (\mathbb{Z} - 13\mathbb{Z})^{-1}A$ and $\mathcal{O}_{L,13} = (\mathbb{Z} - 13\mathbb{Z})^{-1}\mathcal{O}_L$. Show that αA_{13} is a maximal ideal.

f) Show that $A_{13} = \mathcal{O}_{L,13}$ and compute the decomposition of 13 in L.

g) Find generators for every prime ideal in L dividing 10374.

Exercise 3. Show that $IJ = (I+J)(I \cap J)$ for any pair of fractional ideals I and J in a Dedekind domain.

Exercise 4. Let A be a domain, of finite type over a field. Let $\mathfrak{p} \subset A$ be a nonzero prime ideal and $\mathfrak{m} \supseteq \mathfrak{p}$ a maximal ideal. Put $R = A/\mathfrak{p}$ and $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$.

a) Assume that $\mathfrak{p}$ is a principal ideal and that $R_{\overline{\mathfrak{m}}}$ is regular. Show that $A_{\mathfrak{m}}$ is regular.

b) Show by an example that $A_{\mathfrak{m}}$ may be regular even if $R_{\overline{\mathfrak{m}}}$ is not. Can you find an example with R integrally closed?

c) Show by an example that, if $\mathfrak{p}$ is not principal, then $A_{\mathfrak{m}}$ may be singular even if $R_{\overline{\mathfrak{m}}}$ is regular. Can you find an example with A integrally closed?

[To find the examples in b) and c), it is recommended to adopt a geometric point of view.]

You may submit your answers in english, french or italian.

Solutions[1]

Exercise 1. R is a Dedekind domain, so every nonzero prime is maximal and by Weak Nulstellensatz maximal ideals are of the form $(X-\alpha)$ for $\alpha \in k$. The prime ideals of A are in bijection with the primes in $k[X,Y]$ containing XY. Therefore the minimal primes of A are (X) and (Y). Clearly $(X) \cap R = XR$ (maximal) and $(Y) \cap R = \{0\}$ (minimal). Hence (Y) is the only prime in A whose intersection with R is minimal. Every maximal ideal $(X-\alpha) \subset R$ is of the form $(X-\alpha, Y) \cap R$ (notice that $(X-\alpha, Y)$ is maximal: it is the kernel of the map $A \to k$ evaluating $f(X,Y) \in A$ at $(\alpha, 0)$). Finally, every chain $\{0\} \subset (X-\alpha)$ in R is obtained by intersection with the chain $(Y) \subset (X-\alpha, Y)$ in A, so the Going Up property holds.

On the other hand, the Going Down property fails: consider the chain $\mathfrak{p}_0 = \{0\} \subset (X) = \mathfrak{p}_1$ in R and take $\mathfrak{q}_1 = (X)$ in A (we have seen above $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$). Since $\mathfrak{q}_1$ is minimal, there is no $\mathfrak{q}_0 \subset \mathfrak{q}_1$ such that $\mathfrak{q}_0 \cap R = \mathfrak{p}_0$.
The failure of the Going Down can also be seen geometrically: $\operatorname{Spec} A \to \operatorname{Spec} R$ is the projection from the union of the two coordinate axes to the X-axis, and is therefore not open: the image of the open subset $\operatorname{Spec} A[\frac{1}{Y}]$ (the Y-axis minus the origin) is just the origin, a closed point.

Exercise 2. The minimal polynomial of α if X^3-13, hence that of α^2 is Y^3-169. Therefore $\Delta(1,\alpha,\alpha^2) = -N_{L/\mathbb{Q}}(3\alpha^2) = -3^3 N_{L/\mathbb{Q}}(\alpha^2) = -3^3 13^2$. Only 3 and 13 may ramify, and 3 ramifies for sure, since it appears with odd exponent. Moreover, $A\left[\frac{1}{39}\right] = \mathcal{O}_L\left[\frac{1}{39}\right]$ (see exercise 5.8).

As usual, it is convenient to compute the decompositions using Kummer's lemma (exercise 1.11): $X^3-13 \equiv X^3-1 \equiv (X-1)(X^2+X+1) \bmod 2$, and the quadratic factor is irreducible. Therefore $2\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$, with $e_1 = e_2 = f_1 = 1$ and $f_2 = 2$.

$X^3-13 \equiv X^3+1 \equiv (X+1)(X^2-X+1) = (X+1)(X-3)(X-5) \bmod 7$. Therefore 7 splits completely in L.

X^3-13 has no root mod 19 and is thus irreducible. Therefore 19 is inert in L.

The minimal polynomial of $\alpha-1$ is $(Y+1)^3-13 = Y^3+3Y^2+3Y-12$, Eisenstein with respect to 3. The reduction mod 3 of this polynomial is Y^3, so by Kummer's lemma we have that A_3 is local with maximal ideal $\mathfrak{m} = (\alpha-1, 3)$, and in fact $\mathfrak{m} = (\alpha-1)A_3$ because $3 = \frac{1}{4}\left((\alpha-1)^3+6(\alpha-1)^2+12(\alpha-1)\right) \in (\alpha-1)A_3$. So A_3 is a DVR, because it is local noetherian with maximal ideal generated by a non-nilpotent element. Hence $A_3 = \mathcal{O}_{L,3}$. We know that 3 ramifies and there is only one prime above it, so it is totally ramified.

X^3-13 is an Eisenstein polynomial with respect to 13. We have $\alpha^3\mathcal{O}_{L,13} = 13\mathcal{O}_{L,13}$. Let's decompose $\alpha\mathcal{O}_{L,13} = \mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_s^{n_s}$. Then $\alpha^3\mathcal{O}_{L,13} = \mathfrak{m}_1^{3n_1}\cdots\mathfrak{m}_s^{3n_s} = 13\mathcal{O}_{L,13}$. We get an equation $3n_1+\ldots+3n_s = 3$, whose only possible solutions in natural numbers force $s = 1 = n_1$. Thus $\alpha\mathcal{O}_{L,13} = \mathfrak{m}_1$ is maximal. Therefore αA_{13} is maximal ($\mathcal{O}_L$ is integral over A).

The computations above already show that 13 is totally ramified in $\mathcal{O}_L$. Moreover, we have established that A_{13} is a local noetherian domain whose maximal ideal is principal: it is a DVR. Therefore $A_{13} = \mathcal{O}_{L,13}$. Now $A \subseteq \mathcal{O}_L$ is an inclusion of finitely generated $\mathbb{Z}$-modules and $A_p = \mathcal{O}_{L,p}$ for every prime p, hence $A = \mathcal{O}_L$.

$10374 = 2\cdot 3\cdot 7\cdot 13\cdot 19$. We have already established that $19\mathcal{O}_L$ is the only prime above 19 and we have seen that there is only one prime above 13, with α a uniformiser; since $N_{L/\mathbb{Q}}(\alpha) = 13$, no other prime divides $\alpha\mathcal{O}_L$, so this ideal is itself prime. We have also seen that the only prime above 3 is $\mathfrak{m} = (\alpha-1, 3)$. The generators for the primes above 2 and 7 are similarly computed via Kummer's lemma:

$2\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1 = (2, \alpha-1)$ and $\mathfrak{p}_2 = (2, \alpha^2+\alpha+1)$;

$7\mathcal{O}_L = \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$ with $\mathfrak{q}_1 = (7, \alpha+1)$, $\mathfrak{q}_2 = (7, \alpha-3)$ and $\mathfrak{q}_3 = (7, \alpha-5)$.

[1] References to the lecture notes are numbered as in the version of 21/01/2016.

Exercise 3. Easy consequence of lemma 5.3.11: for every nonzero prime $\mathfrak{p}$ we have

$$\begin{aligned} v_\mathfrak{p}(IJ) &= v_\mathfrak{p}(I) + v_\mathfrak{p}(J) = \min\{v_\mathfrak{p}(I), v_\mathfrak{p}(J)\} + \max\{v_\mathfrak{p}(I), v_\mathfrak{p}(J)\} \\ &= v_\mathfrak{p}(I+J) + v_\mathfrak{p}(I \cap J) = v_\mathfrak{p}\left((I+J)(I\cap J)\right). \end{aligned}$$

Exercise 4. From proposition 6.1.35 we know that $\dim R = \dim A - \operatorname{ht}\mathfrak{p}$. Since $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$, we have $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \mathfrak{m}/(\mathfrak{p}+\mathfrak{m}^2)$. To compare with $\mathfrak{m}/\mathfrak{m}^2$ we use the sequence

$$0 \longrightarrow (\mathfrak{p}+\mathfrak{m}^2)/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/(\mathfrak{p}+\mathfrak{m}^2) \longrightarrow 0.$$

If $\mathfrak{p}$ is principal, $\dim(\mathfrak{p}+\mathfrak{m}^2)/\mathfrak{m}^2 \leq 1$, hence

$$\dim \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \geq \dim \mathfrak{m}/\mathfrak{m}^2 - 1 \geq \dim A_\mathfrak{m} - 1 = \dim R_{\overline{\mathfrak{m}}}$$

the last equality because $\operatorname{ht}\mathfrak{p} = 1$. If $R_{\overline{\mathfrak{m}}}$ is regular, $\dim \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \dim R_{\overline{\mathfrak{m}}}$, which forces all the inequalities to be equalities.

We may take $A = \mathbb{C}[X,Y]$, $\mathfrak{m} = (X,Y)$ and $\mathfrak{p} = (X^3+X^2-Y^2)$, so $R = \mathbb{C}[X,Y]/(X^3+X^2-Y^2)$. Then $\operatorname{ht}\mathfrak{p} = 1$ (it is principal) so $\dim R = 1$ and from example 5.1.7 we know that $R_{\overline{\mathfrak{m}}}$ is not a DVR, hence not regular by proposition 6.2.3. On the other hand, $A_\mathfrak{m}$ is regular ($\dim A = 2$ and $\mathfrak{m}/\mathfrak{m}^2$ is generated by the classes of X and Y). Geometrically this is an irreducible plane curve with a singularity.
Taking $A = \mathbb{C}[X,Y,Z]$, $\mathfrak{m} = (X,Y,Z)$ and $\mathfrak{p} = (Z^2 - XY)$ on the other hand, we have that $R = \mathbb{C}[X,Y,Z]/(Z^2-XY)$ is integrally closed and $R_{\overline{\mathfrak{m}}}$ is not regular, as seen in example 5.2.31 ($\dim \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = 3$ while $\dim R = 2$). Geometrically this is a line on a cone.

We only need to change notation in the example above: $A = \mathbb{C}[X,Y,Z]/(Z^2-XY)$, $\mathfrak{m} = (X,Y,Z)$ and $\mathfrak{p} = (Y,Z)$, so A is integrally closed with $A_\mathfrak{m}$ not regular, while $R = \mathbb{C}[X]$ is a Dedekind domain, hence regular.