Commutative Algebra (2015–16)

Modules.

Exercise 1. Let A be a ring and $R \subseteq A$ a subring. Show that the set

$$(R : A) = \{x \in R : \; xa \in R \;\; \forall \; a \in A\}$$

is the largest ideal in R that is also an ideal in A.

Exercise 2. Let $R = \mathbb{Z}/12\mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z}$. Show that M is a projective R-module and that it is not a free R-module.

Exercise 3. Let R be PID and $M \subseteq R^n$ a submodule. Show that M is projective.
[Hint: induction on n, using $\pi : R^n \to R^{n-1}$, $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$.]

Exercise 4. Let R be a ring, $\mathbf{Alg}_R$ the category of R-algebras. Show that the following functors are representable:

$$\begin{array}{rcl} \mathbb{G}_a : \mathbf{Alg}_R & \longrightarrow & \mathbf{Sets} \\ A & \longmapsto & A \end{array} \; ; \quad \begin{array}{rcl} \mathbb{G}_m : \mathbf{Alg}_R & \longrightarrow & \mathbf{Sets} \\ A & \longmapsto & A^\times \end{array} \; ; \quad \begin{array}{rcl} \boldsymbol{\mu}_n : \mathbf{Alg}_R & \longrightarrow & \mathbf{Sets} \\ A & \longmapsto & \{a \in A \,|\, a^n = 1\} \end{array}$$

Exercise 5. Let M be a torsion module, finitely generated over a ring R. Assume that for every $m \in M$ the ideal, $R/\mathrm{Ann}\,(m)$ is a field. Show that there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset R$ such that $M \simeq \bigoplus_{i=1}^n M/\mathfrak{m}_i M$.

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