TENSOR PRODUCTS, LOCALISATION AND FLATNESS.

Exercise 1.

a) Compute $\mathbb{C}[X,Y]/\left(XY-1\right)\otimes_{\mathbb{C}}\mathbb{C}[X,Y]/\left(X^2-Y\right)$.

b) Compute $\mathbb{C}[X,Y]/\left(XY-1\right)\otimes_{\mathbb{C}[X,Y]}\mathbb{C}[X,Y]/\left(X^2-Y\right)$.

b) Find the dimension of $\mathbb{C}[X,Y]/\left(XY-1\right)\otimes_{\mathbb{C}[X,Y]}\mathbb{C}[X,Y]/\left(X^2-Y\right)$ as a $\mathbb{C}$-vector space. [Hint: use the weak Nullstellensatz: any maximal ideal $\mathfrak{m}\subset\mathbb{C}[X,Y]$ is of the form $\mathfrak{m}=(X-\alpha,Y-\beta)$ for suitable $\alpha,\beta\in\mathbb{C}$.]

Exercise 2. Let R be a ring, $f,g\in R$. Show that $R\left[\frac{1}{f}\right]\otimes_R R\left[\frac{1}{g}\right]\cong R\left[\frac{1}{fg}\right]$.

Exercise 3. Show that $\mathbb{Z}/m\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/n\mathbb{Z}=0$ if and only if m and n are coprime.

Exercise 4. Let R be a ring and M an R-module. Define the **support** of M as

$$\operatorname{supp}(M)=\{\mathfrak{p}\in\operatorname{Spec}R\mid M_{\mathfrak{p}}\neq 0\}.$$

Show that, if M is finitely generated, $\operatorname{supp}(M)=\mathcal{Z}(\operatorname{Ann}(M))$.

Exercise 5. Let R be a ring and $0\longrightarrow M'\xrightarrow{f}M\xrightarrow{g}M''\longrightarrow 0$ a short exact sequence of R-modules. Assume that M'' is a flat R-module. Show that M is flat if and only if M' is flat. [Hint: show that the sequence $0\longrightarrow M'\otimes_R N\overset{f\otimes id_N}{\longrightarrow}M\otimes_R N\overset{g\otimes id_N}{\longrightarrow}M''\otimes_R N\longrightarrow 0$ is exact for every R-module N.]

SOLUTIONS

Exercise 1. Let us first notice that $R[X]\otimes_R R[X]\cong R[X,Y]$ for any ring R. Indeed, the bilinear map

$$\begin{aligned} R[X]\times R[X] &\longrightarrow R[X,Y]\\ (f(X),g(X)) &\longmapsto f(X)g(Y)\end{aligned}$$

induces a map $\phi: R[X]\otimes_R R[X]\to R[X,Y]$, which is an isomorphism because both modules are free with bases $X^i\otimes X^j$ and $X^iY^j=\phi(X^i\otimes X^j)$ respectively.
Clearly the map $\mathbb{C}[Z]\to\mathbb{C}[X,Y]/\left(X^2-Y\right)$ sending $Z\mapsto X$ is an isomorphism (with inverse induced by $\mathbb{C}[X,Y]\to\mathbb{C}[Z]$ given by $X\mapsto Z$ and $Y\mapsto Z^2$). Tensoring the sequence

$$0\longrightarrow (XY-1)\,\mathbb{C}[X,Y]\longrightarrow\mathbb{C}[X,Y]\longrightarrow\mathbb{C}[X,Y]/\left(XY-1\right)\longrightarrow 0$$

by the free (hence flat) module $\mathbb{C}[Z]$, since $\mathbb{C}[X,Y]\otimes\mathbb{C}[Z]\cong\mathbb{C}[X,Y,Z]$ we get an exact sequence

$$0\longrightarrow (XY-1)\,\mathbb{C}[X,Y,Z]\longrightarrow\mathbb{C}[X,Y,Z]\longrightarrow\mathbb{C}[X,Y]/\left(XY-1\right)\otimes_{\mathbb{C}}\mathbb{C}[Z]\longrightarrow 0.$$

Therefore $\mathbb{C}[X,Y]/\left(XY-1\right)\otimes_{\mathbb{C}}\mathbb{C}[X,Y]/\left(X^2-Y\right)\cong\mathbb{C}[X,Y,Z]/\left(XY-1\right)$.

Let R be a ring, $I, J \subseteq R$ ideals and M and R-module. We have already used several times that $M \otimes_R R/I \cong M/IM$ (proof: tensoring $0 \longrightarrow I \stackrel{\iota}{\longrightarrow} R \stackrel{\pi}{\longrightarrow} R/I \longrightarrow 0$ by M we get the exact sequence $I \otimes M \stackrel{id_M \otimes \iota}{\longrightarrow} M \stackrel{id_M \otimes \pi}{\longrightarrow} M \otimes_R R/I \longrightarrow 0$ and we conclude because $\operatorname{im}(id_M \otimes \iota) = IM$). For $M = R/J$ we get

$$R/J \otimes_R R/I \cong (R/J)\,/I\,(R/J) \cong (R/J)\,/\,(I + J/J) \cong R/(I + J). \tag{1}$$

In particular $\mathbb{C}[X,Y]/\,(XY-1) \otimes_{\mathbb{C}[X,Y]} \mathbb{C}[X,Y]/\,\left(X^2 - Y\right) \cong \mathbb{C}[X,Y]/\,\left(XY-1, X^2-Y\right)$.
Geometrically, $\mathcal{Z}(X^2 - Y)$ and $\mathcal{Z}(XY-1)$ represent a parabola and a hyperbola in the affine plane and $\mathcal{Z}(X^2-Y, XY-1) = \mathcal{Z}(X^2-Y) \cap \mathcal{Z}(XY-1)$ is their intersection, which consists of the three points with coordinates $(1,1)$, (ζ, ζ^2) and (ζ^2, ζ), where $\zeta = \exp \frac{2\pi i}{3}$.
Put $\mathfrak{m}_0 = (X-1, Y-1)$, $\mathfrak{m}_1 = (X-\zeta, Y-\zeta^2)$ and $\mathfrak{m}_2 = (X-\zeta^2, Y-\zeta)$ and notice that $\left(XY-1, X^2-Y\right) \subseteq \mathfrak{m}_j$ for $j = 0,1,2$, hence $\left(XY-1, X^2-Y\right) \subseteq \mathfrak{m}_0 \cap \mathfrak{m}_1 \cap \mathfrak{m}_2$ (geometrically: $\mathcal{Z}(\mathfrak{m}_0) \cup \mathcal{Z}(\mathfrak{m}_1) \cup \mathcal{Z}(\mathfrak{m}_2) \subseteq \mathcal{Z}\left(XY-1, X^2-Y\right) = \mathcal{Z}(X^2-Y) \cap \mathcal{Z}(XY-1)$). We shall prove that these inclusions are equalities. By the weak Nullstellensatz, these are the only maximal ideals containing $\left(XY-1, X^2-Y\right)$: for $\mathfrak{m} = (X-\alpha, Y-\beta)$ to contain it, (α, β) must be a root of both $X^2 - Y$ and $XY - 1$.
Consider the ring homomorphism

$$\begin{aligned} \widetilde{\varphi} : \mathbb{C}[X,Y] &\longrightarrow \mathbb{C} \times \mathbb{C} \times \mathbb{C} \\ f(X,Y) &\longmapsto \left(f(1,1), f(\zeta,\zeta^2), f(\zeta^2,\zeta)\right) \end{aligned}$$

It is clearly surjective (the j-th basis vector is the image of the polynomial $\frac{X^3-1}{X-\zeta^j}$) and obviously $\ker \widetilde{\varphi} = \mathfrak{m}_0 \cap \mathfrak{m}_1 \cap \mathfrak{m}_2$. Since this kernel contains $\left(XY-1, X^2-Y\right)$, the map $\widetilde{\varphi}$ factors through $\varphi : \mathbb{C}[X,Y]/\left(XY-1, X^2-Y\right) \to \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. We shall prove that it is an isomorphism, whence $\dim_{\mathbb{C}} \mathbb{C}[X,Y]/\left(XY-1, X^2-Y\right) = 3$. Viewing φ as a map of $\mathbb{C}[X,Y]$-modules, it suffices to show that $\varphi_{\mathfrak{m}}$ is an isomorphism for every maximal ideal $\mathfrak{m} \subset \mathbb{C}[X,Y]$.
Let R be a ring, $I \subseteq R$ an ideal and $\mathfrak{p} \subset R$ a prime ideal. We know that $(R/I)_{\mathfrak{p}} = R_{\mathfrak{p}}/I_{\mathfrak{p}}$. Moreover, if $I \nsubseteq \mathfrak{p}$ then $(R/I)_{\mathfrak{p}} = 0$: for any $\bar{x} \in R/I$, the class $\frac{\bar{x}}{1} = 0$ because $u \cdot \bar{x} = 0$ for any $u \in (R - \mathfrak{p}) \cap I$.
Applying this to $\mathbb{C} \times \mathbb{C} \times \mathbb{C} = \prod_{j=0}^{2} \mathbb{C}[X,Y]/\mathfrak{m}_j$ we get that $(\mathbb{C} \times \mathbb{C} \times \mathbb{C})_{\mathfrak{m}} = 0$ for $\mathfrak{m} \notin \{\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2\}$ and $(\mathbb{C} \times \mathbb{C} \times \mathbb{C})_{\mathfrak{m}_j} \cong \mathbb{C}$.
Similarly, $\left(\mathbb{C}[X,Y]/\left(XY-1, X^2-Y\right)\right)_{\mathfrak{m}} = 0$ for $\mathfrak{m} \notin \{\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2\}$. To conclude that $\varphi_{\mathfrak{m}_j}$ is an isomorphism, we should prove that $\left(XY-1, X^2-Y\right)\mathbb{C}[X,Y]_{\mathfrak{m}_j} = \mathfrak{m}_j \mathbb{C}[X,Y]_{\mathfrak{m}_j}$. We will do so using Nakayama's lemma. In fact, we have the equality $\left(XY-1, X^2-Y\right) + \mathfrak{m}_j^2 = \mathfrak{m}_j$ in $\mathbb{C}[X,Y]$. We do it for $j = 0$, the cases $j = 1,2$ are similar. The trick is to expand the generators at the point $(1,1)$:

$$XY - 1 = (X-1+1)(Y-1+1) - 1 = (X-1)(Y-1) + (X-1) + (Y-1)$$

$$X^2 - Y = (X-1+1)^2 - (Y-1+1) = (X-1)^2 + 2(X-1) - (Y-1)$$

whence

$$X - 1 = \frac{1}{3}(XY-1) + \frac{1}{3}(X^2-Y) - \frac{1}{3}\left[(X-1)(Y-1) - (X-1)^2\right] \in \left(XY-1, X^2-Y\right) + \mathfrak{m}_0^2$$

$$Y - 1 = \frac{2}{3}(XY-1) - \frac{1}{3}(X^2-Y) - \frac{1}{3}\left[2(X-1)(Y-1) + (X-1)^2\right] \in \left(XY-1, X^2-Y\right) + \mathfrak{m}_0^2.$$

Exercise 2. Recalling that $R\left[\frac{1}{f}\right] \cong R[X]/(Xf-1)$, the isomorphism (1) yields $R\left[\frac{1}{f}\right] \otimes_R R\left[\frac{1}{g}\right] \cong R[X]/(Xf-1) \otimes_R R[X]/(Xg-1) \cong R[X,Y]/(Xf-1, Yg-1)$. On the other hand $R\left[\frac{1}{fg}\right] \cong$

$R[Z]/(Zfg-1)$. Define

$$\begin{array}{rl} \varphi : R[X,Y] & \longrightarrow R[Z]/(Zfg-1); \\ X & \longmapsto Zg \\ Y & \longmapsto Zf \end{array} \qquad \begin{array}{rl} \psi : R[Z] & \longrightarrow R[X,Y]/(Xf-1, Yg-1). \\ Z & \longmapsto XY \end{array}$$

Clearly $(Xf-1, Yg-1) \subseteq \ker\varphi$ and $(Zfg-1) \subseteq \ker\psi$, whence morphisms

$\overline{\varphi} : R[X,Y]/(Xf-1, Yg-1) \to R[Z]/(Zfg-1)$, $\overline{\psi} : R[Z]/(Zfg-1) \to R[X,Y]/(Xf-1, Yg-1)$.

It is immediate to check that $\overline{\varphi}$ and $\overline{\psi}$ are inverse to each other.

Notice the geometric meaning of this isomorphism: the two rings correspond to the open subsets $\operatorname{Spec} R - \mathcal{Z}(f)$ and $\operatorname{Spec} R - \mathcal{Z}(g)$; the tensor product of the rings corresponds to the intersection of the open subsets: $\operatorname{Spec} R - \mathcal{Z}(fg) = (\operatorname{Spec} R - \mathcal{Z}(f)) \cap (\operatorname{Spec} R - \mathcal{Z}(g))$.

Exercise 3. Recalling that $n\mathbb{Z} + m\mathbb{Z} = \gcd(n,m)\mathbb{Z}$, the result follows immediately from isomorphism (1) which says that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$.

Exercise 4. Let $\mathfrak{p} \in \operatorname{supp}(M)$. Recall that an element $\frac{m}{s} \in M_{\mathfrak{p}}$ is zero if and only if there exists some $u \in R - \mathfrak{p}$ such that $um = 0$. Therefore $\frac{m}{s} \neq 0$ if and only if $um \neq 0$ for every $u \in R - \mathfrak{p}$. Hence $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(m) \subseteq \mathfrak{p}$. Thus $\operatorname{supp}(M) \subseteq \mathcal{Z}(\operatorname{Ann}(M))$.

Conversely, suppose that $\operatorname{Ann}(M) \subseteq \mathfrak{p}$. Let $m_1, \ldots, m_r$ be generators for M. If $M_{\mathfrak{p}} = 0$ then $\frac{m_i}{1} = 0$: there exists an element $u_i \in R - \mathfrak{p}$ such that $u_i m_i = 0$. Taking $u = \prod_{i=1}^r u_i \in R - \mathfrak{p}$ we have that $um_i = 0$ for all $i = 1, \ldots, r$. This implies that $um = 0$ for every $m = \sum_{i=1}^r \in M$, hence $u \in \operatorname{Ann}(M)$. But $u \notin \mathfrak{p}$, contradiction.

Exercise 5. Assume that the sequence

$$0 \longrightarrow M' \otimes_R N \xrightarrow{f \otimes id_N} M \otimes_R N \xrightarrow{g \otimes id_N} M'' \otimes_R N \longrightarrow 0 \tag{2}$$

is exact for every R-module N as suggested. For any injection of R-modules $h : N' \to N$ we can build the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' \otimes_R N' & \xrightarrow{f \otimes id_{N'}} & M \otimes_R N' & \xrightarrow{g \otimes id_{N'}} & M'' \otimes_R N' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_{M'} \otimes h} & & \downarrow{\scriptstyle id_M \otimes h} & & \downarrow{\scriptstyle id_{M''} \otimes h} & & \\ 0 & \longrightarrow & M' \otimes_R N & \xrightarrow{f \otimes id_N} & M \otimes_R N & \xrightarrow{g \otimes id_N} & M'' \otimes_R N & \longrightarrow & 0 \end{array}$$

Since M'' is flat, $id_{M''} \otimes h$ is injective. The snake lemma now implies that $id_M \otimes h$ is injective if and only if $id_{M'} \otimes h$ is.

To show that sequence (2) is exact, we only need to show that $f \otimes id_N$ is injective. Pick a presentation $0 \longrightarrow S \xrightarrow{\iota} F \xrightarrow{\pi} N \longrightarrow 0$ where F is a free R-module and consider the commutative diagram with exact rows and columns (recall that a free module is flat)

$$\begin{array}{ccccccccc} & & M' \otimes_R S & \xrightarrow{f \otimes id_S} & M \otimes_R S & \xrightarrow{g \otimes id_S} & M'' \otimes_R S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_{M'} \otimes \iota} & & \downarrow{\scriptstyle id_M \otimes \iota} & & \downarrow{\scriptstyle id_{M''} \otimes \iota} & & \\ 0 & \longrightarrow & M' \otimes_R F & \xrightarrow{f \otimes id_F} & M \otimes_R F & \xrightarrow{g \otimes id_F} & M'' \otimes_R F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_{M'} \otimes \pi} & & \downarrow{\scriptstyle id_M \otimes \pi} & & \downarrow{\scriptstyle id_{M''} \otimes \pi} & & \\ & & M' \otimes_R N & \xrightarrow{f \otimes id_N} & M \otimes_R N & \xrightarrow{g \otimes id_N} & M'' \otimes_R N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3}$$

Since M'' is flat, $\ker(id_{M''} \otimes \iota) = 0$. The bottom row in diagram (3) identifies with the exact sequence $\operatorname{coker}(id_{M'} \otimes \pi) \longrightarrow \operatorname{coker}(id_M \otimes \pi) \longrightarrow \operatorname{coker}(id_{M''} \otimes \pi) \longrightarrow 0$. The snake lemma applied to the top two rows in diagram (3) allows now to conclude that $f \otimes id_N$ is injective.